AP Calculus BC Formulas

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January 28, 2025

Average Rate of Changing over $[a, b]$: $\frac{f(b)-f(a)}{b-a}$

Limits at a point:

If L, M, c, and k are real numbers and $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$, then the following properties are true:

- Limit of a constant: $\lim_{x\to c} k = k$
- Limit of x: $\lim_{x\to c} x = c$
- Sum rule: $\lim_{x\to c}(f(x) + g(x)) = L + M$
- Difference rule: $\lim_{x\to c}(f(x) - g(x)) = L - M$
- Product rule: $\lim_{x\to c}(f(x) \cdot g(x)) = L \cdot M$
- Constant multiple rule: $\lim_{x\to c}(k(f(x))) = k \cdot L$
- Quotient rule: $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{M}, M \neq 0$
- Power rule: $\lim_{x\to c}(f(x))^{r/s} = L^{r/s}$, if r and s are integers, and $s \neq 0$
- Limit of a composite function: $\lim_{x\to c} f(g(x)) = f(\lim_{x\to c} g(x))$, if f is a continuous function

Properties of limits as $x \to \pm\infty$

If L, M, c, and k are real numbers and $\lim_{x\to\pm\infty} f(x) = L$ and $\lim_{x\to\pm\infty} g(x) = M$, then the following properties are true:

- Constant rule: $\lim_{x\to\pm\infty} c = c$
- Sum rule: $\lim_{x\to\pm\infty}(f(x) + g(x)) = L + M$
- Difference rule: $\lim_{x\to\pm\infty}(f(x) - g(x)) = L - M$
- Product rule: $\lim_{x\to\pm\infty}(f(x) \cdot g(x)) = L \cdot M$
- Constant multiple rule: $\lim_{x\to\pm\infty}(k(f(x))) = k \cdot L$
- Quotient rule: $\lim_{x\to\pm\infty} \frac{f(x)}{g(x)} = \frac{L}{M}$, $M \neq 0$
- Power rule: $\lim_{x\to\pm\infty}(f(x))^{r/s} = L^{r/s}$, if r and s are integers and $s \neq 0$
- Limit of $\frac{c}{x^r}$: $\lim_{x\to\pm\infty} \frac{c}{x^r} = 0$

Squeeze Theorem

Conditions:

- $g(x) \leq f(x) \leq h(x)$ for $x \neq c$
- $\lim_{x\to c} g(x) = L$ and $\lim_{x\to c} h(x) = L$

Conclusion: $\lim_{x\to c} f(x) = L$

Definition of Continuity

A function $f(x)$ is continuous at $x = c$ if all of the following conditions are met:

- $f(c)$ is defined
- $\lim_{x\to c} f(x)$ exists
- $\lim_{x\to c} f(x) = f(c)$

The graph of a continuous function has no "gaps".

Intermediate Value Theorem

If a function f is continuous on the interval $[a, b]$ and k is a number between $f(a)$ and $f(b)$, then there is at least one x-value c between a and b such that $f(c) = k$.

Any continuous function connecting $(a, f(a))$ and $(b, f(b))$ must pass through every y-value between $f(a)$ and $f(b)$ at least once.

Limit Definitions of the Derivative Derivative of f at $x = a$: $f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$

Defintion of derivative of f at $x = a$: $f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$

Definition of Differentiability

f is differentiable at $x = c$: $\lim_{x \to c} \frac{f(x)-f(c)}{x-c}$ exists and is equal to $f'(c)$. $\frac{f(x)-f(c)}{x-c}$ is the difference quotient.

Derivative Rules Basic:

- Constant: $\frac{\mathrm{d}}{\mathrm{d}x}[c] = 0$
- Power: $\frac{\mathrm{d}}{\mathrm{d}x}[x^n] = nx^{n-1}$
- Natural exponential: $\frac{\mathrm{d}}{\mathrm{d}x}[e^x] = e^x$
- Exponential: $\frac{\mathrm{d}}{\mathrm{d}x}[a^x] = (\ln a)a^x$
- Natural log: $\frac{\mathrm{d}}{\mathrm{d}x}[\ln(x)] = \frac{1}{x}$
- Constant multiple: $\frac{\mathrm{d}}{\mathrm{d}x}[cf(x)] = cf'(x)$
- Sum and difference: $\frac{\mathrm{d}}{\mathrm{d}x}[f(x) \pm g(x)] = f'(x) \pm g'(x)$

Trig:

- $\frac{\mathrm{d}}{\mathrm{d}x}[\sin x] = \cos x$
- $\frac{\mathrm{d}}{\mathrm{d}x}[\cos x] = -\sin x$
- $\frac{\mathrm{d}}{\mathrm{d}x}[\tan x] = \sec^2 x$
- $\frac{\mathrm{d}}{\mathrm{d}x}[\cot x] = -\csc^2 x$
- $\frac{\mathrm{d}}{\mathrm{d}x}[\csc x] = -\csc(x)\cot(x)$
- $\frac{\mathrm{d}}{\mathrm{d}x}[\sec x] = \sec(x)\tan(x)$

Product Rule: $\frac{\mathrm{d}}{\mathrm{d}x}[uv] = uv' + vu'$

Quotient Rule: $\frac{\mathrm{d}}{\mathrm{d}x}[\frac{u}{v}] = \frac{vu'-uv'}{v^2}$

Chain Rule: $\frac{\mathrm{d}}{\mathrm{d}x}[f(g(x))] = f'(g(x)) \cdot g'(x)$

Derivatives of Inverse Functions: $(f^{-1})'(a) = \frac{1}{f'(b)}$

PVA Derivatives:

- Position: $x(t)$
- Velocity: $v(t) = x'(t)$
- Acceleration: $a(t) = x''(t)$

Integrals:

- Integrate $a(t)$ to get $v(t)$
- Integrate $v(t)$ to get $s(t)$

L'Hospital's Rule

Use L'Hospital's Rule to find the limit of the ratio of two differentiable functions $\frac{f(x)}{g(x)}$ as x approaches c. If direct substitution produces one of the indeterminate forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then differentiate the numerator f and the denominator g independently.

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

L'Hospital's Rule also applies to limits such as $x \to \infty$ or $x \to -\infty$

Mean Value Theorem

Conditions:

- f is continuous on $[a, b]$
- f is differentiable on (a, b)

Conclusion: For some c in (a, b): $f'(c) = \frac{f(b)-f(a)}{b-a}$. $f'(c)$ is the instantaneous rate of change at $x = c$ and $\frac{f(b)-f(a)}{b-a}$ is the average rate of change on $[a, b]$.

Rolle's Theorem

If a function f satisfies each of the following conditions:

- continuous on the closed interval $[a, b]$
- differentiable on the open interval (a, b)
- $f(a) = f(b)$

then there is at least one number c in (a, b) such that $f'(c) = 0$

Graphically, the slope of the secant line on $[a, b]$ and the slope of the tangent line at $x = c$ both equal zero for at least one value of c in (a, b).

Rolle's Theorem is a special case of the Mean Value Theorem in which the average rate of change is 0:

$$f'(c) = \frac{f(b) - f(a)}{b - a} = 0$$

Extreme Value Theorem

If a function f is continuous on the closed interval $[a, b]$, then f is guaranteed to attain an absolute minimum and absolute maximum value on $[a, b]$.

First Derivative Test

If $f'(c) = 0$ or undefined, there is a local maximum if $f'(x)$ changes from positive to negative and a local minimum when $f'(x)$ changes from negative to positive.

Second Derivative Test

If $f''(c) < 0$, $f(c)$ is a relative maximum. If $f''(c) = 0$ the test is inconclusive. If $f''(c) > 0$ then $f'(c)$ is a relative minimum.

Riemann Sums A left Riemann sum approximates the value of a definite integral $\int_a^b f(x)\mathrm{d}x$. The interval $[a, b]$ is divided into subintervals, and the area boudned by the graph of f and the x-axis on each subinterval is estimated with a rectangle.

The base length b_n of each rectangle is the distance between the endpoints of the subinterval, and the height h_n is the function value at the left endpoint.

$$\int_a^b f(x)\mathrm{d}x \approx b_1 h_1 + b_2 h_2 + \dots$$

A midpoint Riemann sum approximates the value of a definite integral $\int_a^b f(x)\mathrm{d}x$. The interval $[a, b]$ is divided into subintervals, and the area bounded by the graph of f and the x-axis on each subinterval is estimated with a rectangle.

The base length b_n of each rectangle is the distance between the endpoints of the subinterval, and the height h_n is the function value at the midpoint of the subinterval m_n.

$$\int_a^b f(x)\mathrm{d}x \approx b_1 h_1 + b_2 h_2 + \dots$$

A right Riemann sum approximates the value of a definite integral $\int_a^b f(x)\mathrm{d}x$. The interval $[a, b]$ is divided into subintervals, and the area boudned by the graph of f and the x-axis on each subinterval is estimated with a rectangle.

The base length b_n of each rectangle is the distance between the endpoints of the subinterval, and the height h_n is the function value at the right endpoint.

$$\int_a^b f(x)\mathrm{d}x \approx b_1h_1 + b_2h_2 + \ldots$$

A trapezoidal sum approximates the value of a definite integral $\int_a^b f(x)\mathrm{d}x$. The interval $[a, b]$ is divided into subintervals, and the area bounded by the graph of f and the x-axis on each subinterval is estimated with a trapezoid.

The height h_n of each trapezoid is the distance between the endpoints of the subinterval, and the bases b_n and b_{n+1} are the function values at the endpoints.

$$\int_a^b f(x)\mathrm{d}x \approx \frac{1}{2}h_1(b_1 + b_2) + \frac{1}{2}h_2(b_2 + b_3)$$

Limit of a Right Riemann Sum

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(a + \Delta xi)\Delta x = \int_a^b f(x)\mathrm{d}x$$

Fundamental Theorem of Calculus $\int_a^b f(x)\mathrm{d}x = F(b) - F(a)$

$\int_a^b f'(x)\mathrm{d}x = f(b) - f(a)$

$f(b) = f(a) + \int_a^b f'(t)\mathrm{d}t$, where $f(b)$ is the final quantity, $f(a)$ is the initial quantity and $\int_a^b f'(t)\mathrm{d}t$ is the net change.

Second FTC $\frac{\mathrm{d}}{\mathrm{d}x}[\int_a^x f(t)\mathrm{d}t] = f(x)$

Basic Integration Rules

- Constant: $\int c\mathrm{d}x = cx + C$
- Power: $\int x^n\mathrm{d}x = \frac{x^{n+1}}{n+1} + C$
- Constant multiple: $\int cf(x)\mathrm{d}x = c\int f(x)\mathrm{d}x$
- Sum and difference: $\int f(x) \pm g(x)\mathrm{d}x = \int f(x)\mathrm{d}x \pm \int g(x)\mathrm{d}x$
- Natural exponential: $\int e^x\mathrm{d}x = e^x + C$
- Natural log: $\int \frac{1}{x}\mathrm{d}x = \ln|x| + C$

Trig Integrals

- $\int \sin u\mathrm{d}u = -\cos u + C$
- $\int \cos u\mathrm{d}u = \sin u + C$
- $\int \sec^2 u\mathrm{d}u = \tan u + C$
- $\int \csc^2 u\mathrm{d}u = -\cot u + C$
- $\int(\sec u \tan u)\mathrm{d}u = \sec u + C$
- $\int(\csc u \cot u)\mathrm{d}u = -\csc u + C$
- $\int \tan u\mathrm{d}u = -\ln|\cos u| + C$
- $\int \cot u\mathrm{d}u = \ln|\sin u| + C$
- $\int \sec u\mathrm{d}u = \ln|\sec u + \tan u| + C$
- $\int \csc u\mathrm{d}u = -\ln|\csc u + \cot u| + C$

Properties of Definite Integrals The following are properties of definite integrals, where functions f and g are continuous on the closed interval $[a, b]$ and a, b, and k are constants.

- $\int_a^a f(x)\mathrm{d}x = 0$
- $\int_a^b f(x)\mathrm{d}x = -\int_b^a f(x)\mathrm{d}x$
- $\int_a^c f(x)\mathrm{d}x = \int_a^b f(x)\mathrm{d}x + \int_b^c f(x)\mathrm{d}x$
- $\int_a^b kf(x)\mathrm{d}x = k\int_a^b f(x)\mathrm{d}x$
- $\int_a^b [f(x) \pm g(x)]\mathrm{d}x = \int_a^b f(x)\mathrm{d}x \pm \int_a^b g(x)\mathrm{d}x$

Improper Integral

$$\int_a^\infty f(x)\mathrm{d}x = \lim_{t\to\infty}\int_a^t f(x)\mathrm{d}x$$

Integration by Parts $\int u\mathrm{d}v = uv - \int v\mathrm{d}u$

Euler's Method $y_{n+1} = y_n + f'(x_n)(\Delta x)$, where y_{n+1} is the next y-value, $f'(x_n)$ is the derivative at current x_n-value and Δx is the step size.

Exponential Growth and Decay Differential Equation: $\frac{\mathrm{d}y}{\mathrm{d}t} = k \cdot y$, where $\frac{\mathrm{d}y}{\mathrm{d}t}$ is the rate of change of y and k is the constant of proportionality.

General Solution: $y = C{\cdot}e^{k\cdot t}$, where C is the initial value of y (when $t = 0$), k is the constant of proportionality, and t is time.

Logistic Growth/Decay $\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{a}\right)$

$\frac{\mathrm{d}P}{\mathrm{d}t} = kP(a - P)$

Average Value $\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x$

Total Distance Traveled $\int_{t_1}^{t_2} |v(t)|\mathrm{d}t$

Area Between Curves In terms of x: $A = \int_{x_1}^{x_2}(\text{top} - \text{bottom})\mathrm{d}x$ is the area bounded by two functions on $[x_1, x_2]$.

In terms of y: $A = \int_{y_1}^{y_2}(\text{right} - \text{left})\mathrm{d}y$ is the area bounded by two functions on $[y_1, y_2]$.

Disk Method Use the disk method to determine the volume of a solid of revolution formed by rotating a region about a horizontal line $y = c$ (axis of revolution) over the interval $a < x < b$ when $y = c$ is a boundary of the region - there is no space between the region and $y = c$.

$$\pi\int_a^b r^2\mathrm{d}x$$

When a region is revolved about an axis of revolution, a perpendicular cross section of the solid is a disk where

- r is the distance from the axis of revolution to the closest function $f(x)$
- $\mathrm{d}x$ is the thickness of the disk

Use the disk method to determine the volume of a solid of revolution formed by rotating a region about a vertical line $x = k$ (axis of revolution) over the interval $c < y < d$ when $x = k$ is a boundary of the region - there is no space between the region and the line $x = k$.

$$\int_c^d (r(y))^2\mathrm{d}y$$

When a region is revolved about an axis of revolution, a perpendicular cross section of the solid is a disk where:

- r is the distance from the axis of revolution to the closest function $f(y)$
- $\mathrm{d}y$ is the thickness of the disk

Washer Method Use the washer method to determine the volume of a solid of revolution formed by rotating a region bounded by $f(x)$ and $g(x)$ about a horizontal line $y = c$ (axis of revolution) over the interval $a < x < b$ when $y = c$ is not a boundary of the region - there is space between the region and $y = c$.

$$\pi \int_a^b ((R(x))^2 - (r(x))^2)\mathrm{d}x$$

When a region is revolved about an axis of revolution, a perpendicular cross section of the resulting solid is a disk with a hole (washer) where:

- R is the distance from the axis of revolution to the farthest function $f(x)$
- r is the distance from the axis of revolution to the closest function $g(x)$
- $\mathrm{d}x$ is the thickness of the washer

Use the washer method to determine the volume of a solid of revolution formed by rotating a region bounded by $f(y)$ and $g(y)$ about a horizontal line $x = k$ (axis of revolution) over the interval $c < y < d$ when $x = k$ is not a boundary of the region - there is space between the region and $x = k$.

$$\pi \int_c^d ((R(y))^2 - (r(y))^2)\mathrm{d}y$$

When a region is revolved about an axis of revolution, a perpendicular cross section of the resulting solid is a disk with a hole (washer) where:

- R is the distance from the axis of revolution to the farthest function $f(y)$
- r is the distance from the axis of revolution to the closest function $g(y)$
- $\mathrm{d}y$ is the thickness of the washer

Arc Length

$$\int_a^b \sqrt{1 + (f'(x))^2}\mathrm{d}x$$

Parametrics Parametric Slope: $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y'(t)}{x'(t)}$

Parametric Speed: $s(t) = \sqrt{(x'(t))^2 + (y'(t))^2}$

Parametric Arc Length: $\int_{t_1}^{t_2} \sqrt{(x'(t))^2 + (y'(t))^2}\mathrm{d}t$

Derivatives of Vector Valued Functions $f(t) = \langle x(t), y(t) \rangle$

$f'(t) = \langle x'(t), y'(t) \rangle$

$f''(t) = \langle x''(t), y''(t) \rangle$

Total Distance of Vectors $\int_{t_1}^{t_2} \sqrt{(x'(t))^2 + (y'(t))^2}\mathrm{d}t$

Polar to Rectangular Coordinates $x = r\cos\theta$

$y = r\sin\theta$

Slope of Polar Curve $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}}{\mathrm{d}\theta}[y]}{\frac{\mathrm{d}}{\mathrm{d}\theta}[x]}$

Sum of Geometric Series $S = \frac{a_1}{1-r}$

Convergence Tests The harmonic series is an infinite series given by

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

The harmonic series diverges by the p-series test.

p-series test.

- p-series of the form $\frac{1}{n^p}$ converges for $p > 1$

$$\sum_{n=1}^{\infty}\frac{1}{n} \implies \sum_{n=1}^{\infty}\frac{1}{n^1}, p=1$$

nth Term Test $\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n \neq 0$ and is inconclusive when $\lim_{n\to\infty} a_n = 0$

A series of the form $\sum_{n=1}^{\infty}\frac{1}{n^p} = \frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\dots$ is called a p-series.

- p-series converges if $p>1$
- p-series diverges if $0<p\leq 1$

If $p=1$, the resulting series $\sum_{n=1}^{\infty}\frac{1}{n} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots$ is called a harmonic series, which diverges.

Geometric Series

$$\sum_{n=0}^{\infty} ar^n$$

When $|r|<1$ the series converges to $S=\frac{a_1}{1-r}$, where a_1 is the first term of the series. If $|r|\geq 1$ the series diverges.

Integral Test

If f is continuous, positive, and eventually decreases as $x\to\infty$, and $\int_c^{\infty} f(x)$:

- converges then $\sum_{n=c}^{\infty} f(n)$ converges and $\sum_{n=c}^{\infty} f(n) > \int_c^{\infty} f(x)\mathrm{d}x$
- diverges: then, $\sum_{n=c}^{\infty} f(n)$ diverges

Direct Comparison Test

$$0<a_n<b_n$$

If the larger series $\sum_{n=1}^{\infty} b_n$ converges, the smaller series $\sum_{n=1}^{\infty} a_n$ converges

If the smaller series $\sum_{n=1}^{\infty} a_n$ diverges, the larger series $\sum_{n=1}^{\infty} b_n$ diverges.

Limit Comparison Test

If $\lim_{n\to\infty}\frac{a_n}{b_n} = L$, where L is finite and positive and $a_n>0$, $b_n>0$, then:

$\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} a_n$ converge or $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} a_n$ diverge.

Ratio Test

If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = k$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely if $k<1$ or diverges if $k>1$.

Alternating Series Test

$$\sum_{n=1}^{\infty}(-1)^n a_n$$

converges if:

- $\lim_{n\to\infty} a_n = 0$ and
- a_n is a positive, decreasing sequence

Taylor/Maclaurin Polynomials nth-degree Taylor polynomial of f about $x=c$

$$P_n(x) = f(c)+f'(c)(x-c)+\frac{f''(c)}{2!}(x-c)^2+\dots+\frac{f^(n)(c)}{n!}(x-c)^n$$

Maclaurin polynomial

$$P_n(x) = f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\dots+\frac{f^{(n)}(0)}{n!}x^n$$

Known Power Series

- $\frac{1}{1-x} = 1+x+x^2+x^3+\dots+x^n+\dots$
- $e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots+\frac{x^n}{n!}+\dots$

- $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \dots$
- $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + (-1)^n \frac{x^{2n}}{(2n)!} + \dots$