1 Parametrics, Vectors, and Polar

1.1 Parametric Equations: The Basics

It is a way to add a third variable into a two dimensional picture.

We let $x = f(t)$ and $y = f(t)$ and can introduce that third variable.

Let's say $x = t^2 - 4$ and $y = 1/2t$. We can eliminate the parameter by plugging in $2y = t$ into $x = t^2 - 4$. And then we get $x = 4y^2 - 4$.

Exercise Eliminate the parameter for $x = 3\cos t$ and $y = 4\sin t$. What do you get?

Slope of a parametric is $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$.

For example, if $x = \cos t$ and $y = \sin t$, then $\frac{dy}{dx} = \frac{\cos t}{-\sin t} = -\cot t$.

Exercise Find $\frac{dy}{dx}$ at $(2, 3)$ if $x = \sqrt{t}$ and $y = \frac{1}{4}(t^2 - 4)$.

The second derivative is $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$.

Arc length is

$$\int_{t=a}^{t=b} \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\, dx$$

1.2 Vectors and Motion along a Curve

Example

A particle moves in the xy-plane so that at any time t, the position of the particle is given by

$$x(t) = 2t^3 - 5t^2, y(t) = 2t^4 + t^3$$

(a) Find the velocity vector when $t = 1$.

$v(t) = \langle x'(t), y'(t)\rangle$, so $v(t) = \langle 6t^2 - 10t, 8t^3 + 3t^2\rangle$.

Therefore $v(1) = \langle -4, 1\rangle$.

(b) Find the acceleration vector when $t = 1$.

A similar process, $a(t) = \langle 12t - 10, 24t^2 + 6t\rangle$, so $a(1) = \langle 2, 30\rangle$.

The magnitude of the position vector is $\sqrt{(x(t))^2 + (y(t))^2}$

The magnitude of the velocity vector is $\sqrt{(x'(t))^2 + (y'(t))^2}$. The magnitude of the velocity vector is called the speed of the object moving along the curve.

The magnitude of the acceleration vector is $\sqrt{(x''(t))^2 + (y''(t))^2}$

Exercise A particle moves in the xy-plane so that any time t, $t \geq 0$, the position of the particle is given by $x(t) = t^2 + 5t, y(t) = \ln(t^2 + 4)$. Find the magnitude of the velocity vector when $t = 3$.

Exercise A particle moves in the xy-plane so that $x = \sqrt{3} - 4\cos t$ and $y = 1 - 2\sin t$, where $0 \leq t \leq 2\pi$. The path of the particle intersects the x-axis twice. Write an expression that represents the distance traveled by the particle between the two x-intercepts. Do not evaluate.

Exercise A particle moves in the xy-plane so that at any time t, the posiiton of the particle is given by $x(t) = 2t^3 - 15t^2 + 36t + 5, y(t) = t^3 - 3t^2 + 1$, where $t \geq 0$. For what value(s) of t is the particle at rest?

Exercise A particle moves in the xy-plane in such a way that its velocity vector is $\langle 3t^2 - 4t, 8t^3 + 5\rangle$. At $t = 0$, the position of the particle is $(7, -4)$. Find the position of the particle at $t = 1$.

Example

A particle is moving along a curve in the xy-plane has a position $\langle x(t), y(t)\rangle$ at time t with $\frac{dx}{dt} = \sin(t^3)$, $\frac{dy}{dt} = \cos(t^2)$. At time $t = 2$, the object is at the position $(7, 4)$.

(a) Write the equation of the tangent line to the curve at the point where $t = 2$.

Recall the derivative of a parametric function.

You should get $y - 4 = \frac{\cos 4}{\sin 8}(x - 7)$.

(b) Find the speed of the particle at $t = 2$.

The speed is $\sqrt{(\sin 8)^2 + (\cos 4)^2} = 1.186$.

(c) For what value of t, $0 < t < 1$, does the tangent line to the curve have a slope of 4? Find the acceleration vector at this time.

$\frac{dy}{dx} = 4$, $t - .616$, $a(.616) = \langle 1.107, -.456\rangle$.

(d) Find the position of the particle at time $t = 1$.

$\int_2^1 \sin(t^3)dt = x(1) - x(2)$.

$x(1) = 7 + \int_2^1 \sin(t^3)dt$.

$4 + \int_2^1 \cos(t^2)dt = y(1)$.

So $(6.7819, 4.44306)$ is the answer.

1.3 Polar Coordinates and Polar Graphs

Rectangular coordinates are in the form (x, y).

Polar coordinates are in the form (r, θ).

In the past you learnt that $\cos\theta = \frac{x}{r}$, $\sin\theta = \frac{y}{r}$, $\tan\theta = \frac{y}{x}$.

So we have $x = r\cos\theta, y = r\sin\theta$ and $r = \pm\sqrt{x^2 + y^2}$ (from $x^2 + y^2 = r$).

Example

Convert $\left(2, \frac{5\pi}{6}\right)$ to rectangular coordinates.

Using the formulas above should give you $(-\sqrt{3}, 1)$.

Exercise Convert $(3, -3)$ to polar coordinates.

Example

Convert the following equation to polar form. $y = 4$.

$r\sin\theta = 4$, so $r = 4\csc\theta$.

Exercise Convert $x^2 + y^2 = 25$ to polar form.

Exercise Convert $r\sin\theta = 3$ to rectangular form and graph.

Exercise Convert $r = 2\cos\theta$ to rectangular form and graph.

Exercise Convert $\theta = \frac{2\pi}{3}$ to rectangular form and graph.

To find the slope of a tangent line to a polar graph $r = f(\theta)$, we can use the facts that $x = r\cos\theta$ and $y = r\sin\theta$ together with the product rule:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

Example

Find $\frac{dy}{dx}$ and the slope of the graph of the polar curve at the given value of θ.

$$r = 3 + 2\sin\theta, \theta = \frac{\pi}{6}$$

We have $x = (3 + 2\sin\theta)\cos\theta$ and $y = (3 + 2\sin\theta)\sin\theta$.

Using the formula above, we should get $-5\sqrt{3}$.

1.4 Area Bounded by a Polar Curve

Example

Find the area bounded by the graph $r = 2 + 2\sin\theta$.

A good idea is to draw this graph. You get a cardioid.

The area of a polar graph is $A = \frac{1}{2}\int_a^b r^2 d\theta$.

So this graph goes from 0 to 2π, so $\frac{1}{2}\int_0^{2\pi}(2 + 2\sin\theta)^2 d\theta = 18.8496$.

Exercise Sketch, and set up an integral expression to find the area of one petal of $r = 2\sin(3\theta)$. Do not evaluate.

Exercise Sketch, and set up an integral expression to find the area of one petal of $r = 4\cos(2\theta)$. Do not evaluate.

1.5 Notes on Polar

Example

Set up an integral expression to find the area inside the graph of $r = 3\sin\theta$ and outside the graph of $r = 2 - \sin\theta$. Do not evaluate.

We end up getting two polar graphs and we are finding the area where they do not have in common.

$3\sin\theta = 2 - \sin\theta$ gives $\theta = \pi/6, 5\pi/6$.

The integral is then

$$A = \int_{\pi/6}^{\pi/2} 9\sin^2\theta d\theta - \int_{\pi/6}^{\pi/2}(2 - \sin\theta)^2 d\theta$$

Exercise Sketch, and set up an integral expression to find the area of the common interior of $r = 3\cos\theta$ and $r = 1 + \cos\theta$.

1.6 More on Polar Graphs

Example

A curve is drawn in the xy-plane and is described by the equation in polar coordinates $r = 2 + \sin(2\theta)$ for $0 \leq \theta \leq \pi$, where r is measured in meters and θ is measured in radians.

(a) Find the area bounded by the curve and the x-axis.

$A = \frac{1}{2}\int_0^\pi r^2 d\theta = 7.069.$

(b) Find the angle θ that corresponds to the point on the curve with x-coordinate -1.

$x = r\cos\theta$. $x = \underbrace{(2+\sin(2\theta))\cos\theta}_{y_1} = \underbrace{-1}_{y_2}$.

Get $\theta = 2.63036$.

(c) Find the value of $\frac{dr}{d\theta}$ at the instant that $\theta = \frac{5\pi}{7}$. What does your answer tell you about r? What does it tell you about the curve?

$r = 2 + \sin(2\theta)$. $\frac{dr}{d\theta} = 2\cos(2\theta)$.

So $\frac{dr}{d\theta} = -.445$. This is less than 0, so r is decreasing, and the curve closes to the pole as a result.

(d) A particle is traveling along the polar curve given by $r = 2 + \sin(2\theta)$ so that its position at time t is $(x(t), y(t))$ and such that $\frac{d\theta}{dt} = 3$. Find the value of $\frac{dx}{dt}$ at the instant that $\theta = \frac{\pi}{6}$, and interpret the meaning of your answer in the context of the problem.

$\frac{dx}{dt} = (2 + \sin(2\theta)) - 3\sin\theta + \cos\theta(\cos(2\theta))$, so at $\pi/6$, this is -1.70096.

x is decreasing because $\frac{dx}{dt} < 0$.