1 Integration with Data, Functions Defined by Integrals, and Natural Logs

1.1 Integration Using Data

Example

Water is flowing into a tank over a 24-hour period. The rate at which water is flowing into the tank at various times is measured, and the results are given in the table below, where $R(t)$ is measured in gallons per hour and t is measured in hours. The tank contains 150 gallons of water when $t = 0$.

t (hours)	0	4	8	12	16	20	24
$R(t)$ (gal/hr)	8	8.8	9.3	9.2	8.9	8.1	6.7

(a) Estimate the number of gallons of water in the tank at the end of 24 hours by using a midpoint Riemann sum with three subintervals and values from the table. Show the computations that lea dto your answer.

We are estimating $150 + \int_0^{24} R(t)dt = W(24) - W(0)$.

So $150 + [3(8.8) + 8(9.2) + 8(8.1)] = 358.8$ gallons

(b) Estimate the number of gallons of water in the tank at the end of 24 hours by using a trapezoidal sum with three subintervals and values from the table. Show the computations that lead to your answer.

This is $150 + \left[8\left(\frac{8+9.3}{2}\right) + 8\left(\frac{9.3+8.9}{2}\right) + 8\left(\frac{8.9+6.7}{2}\right)\right] = 354.4$ gallons.

(c) A model for this function is given by $W(t) = \frac{1}{75}(600 + 20t - t^2)$. Use the model to find the number of gallons of water in the tank at the end of 24 hours.

$\int_0^{24} w(t)dt = W(24) - W(0)$.

$150 + \int_0^{24} W(t)dt = 357.36$

(d) Use the model given in (c) to find the average rate of water flow over the 24-hour period.

$\frac{1}{24-0} \int_0^{24} W(t)dt = 8.64$ gallons/hr

1.2 Second Fundamental Theorem of Calculus

Let us investigate first.

Find $\frac{d}{dx} \int_1^x t^2 dt$. This is equal to x^2.

Find $\frac{d}{dx} \int_{\pi/6}^x \cos t dt$. This is equal to $\cos x$.

See a pattern?

Theorem 1.1: Second Fundamental Theorem of Calculus

$$\frac{d}{dx} \int_a^x f(t)dt = f(x)$$

Example

$$\frac{d}{dx}\int_x^4 t^2 dt$$

This is $-\frac{d}{dx}\int_4^x t^2 dt = -x^2$

In general, $\frac{d}{dx}\int_x^a f(t)dt = -f(x)$.

Example

$$\frac{d}{dx}\int_{\pi/6}^{x^2} \cos t dt$$

This is $\frac{d}{dx}[\sin t]$ with bounds $\pi/6$ to x^2.

We end up getting $\frac{d}{dx}[\sin(x^2) - \frac{1}{2}] = \cos(x^2)\cdot 2x$.

Theorem 1.2: Second Fundamental Theorem of Calculus

$$\frac{d}{dx}\int_a^{g(x)} f(t)dt = f(g(x))\cdot g'(x)$$

Example

Use the Second Fundamental Theorem to evaluate.

(a) $\frac{d}{dx}\int_3^x \sqrt{1+t^2}dt$

This is $\sqrt{1+x^2}$

(b) $\frac{d}{dx}\int_2^x \tan(t^3)dt$

This is $\tan(x^3)$.

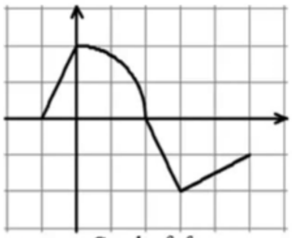
Exercise Same as above for $\frac{d}{dx}\int_{-1}^{x^3}\frac{1}{1+t}dt$

Exercise Same as above for $\frac{d}{dx}\int_2^{\sin x}\sqrt[3]{1+t^2}dt$

Example

The graph of a function f consists of a quarter circle and line segments. Let g be the function given by

$$g(x) = \int_0^x f(t)dt$$



Graph of f

(a) Find $g(0), g(-1), g(2), g(5)$

$g(0) = \int_0^0 f(t)dt = 0$

$g(-1) = \int_0^{-1} f(t)dt = -\int_{-1}^0 f(t)dt = -1$

$g(2) = \int_0^2 f(t)dt = \pi$

$g(5) = \int_0^5 f(t)dt = \pi - 4$

(b) Find all values of x on the open interval $(-1, 5)$ at which g has a relative maximum. Justify your answer.

$g'(x) = f(x)$ crosses the x-axis from positive to negative at $x = 2$.

Exercise Using the information above, (c) Find the absolute minimum value of g on $[-1, 5]$ and the value of x at which it occurs. Justify your answer.

Exercise Using the information above, (d) Find the x-coordinate of each point of inflection of the graph of g on $(-1, 5)$. Justify your answer.

1.3 Natural Logs and Differentiation

Definition: Natural Logarithmic Function

The natural logarithmic function is defined by $\ln x = \int_1^x \frac{1}{t}dt$ where $x > 0$.

The base of natural logs is the number e. e was named for a Swiss mathematician, Leonhard Euler.

By definition:

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n \approx 2.7183\ldots$$

$y = \ln x$ and $y = e^x$ are inverses.

Properties of Natural Logs:

1. Domain of $y = \ln x$ is $(0, \infty)$. Range of $y = \ln x$ is $(-\infty, \infty)$.
2. The graph of $y = \ln x$ is continuous, increasing, and one-to-one
3. The graph of $y = \ln x$ is concave down.

Other properties:

If a and b are positive numbers and n is rational, then:

1. $\ln 1 = 0$
2. $\ln e = 1$
3. $\ln ab = \ln a + \ln b$
4. $\ln \frac{a}{b} = \ln a - \ln b$
5. $\ln a^n = n \ln a$

Exercise Write as a sum, difference, or multiple of logs: $\ln \frac{(x^2+3)^2}{\sqrt[3]{x^2+1}}$

Exercise Write as a single log: $2\ln(x+3) + \frac{1}{2}\ln(x-2)$

Definition

$$\frac{d}{dx}[\ln u] = \frac{1}{u}\frac{du}{dx}, u > 0$$

Example

If $y = \ln(2x)$, what is y'.

$y' = \frac{1}{2x} \cdot 2 = \frac{1}{x}$.

Exercise Find $f'(x)$ if $f(x) = \ln(x^2+1)$

Exercise Find y' if $y = x \ln x$

Exercise Find $f'(x)$ if $f(x) = \ln \sqrt{x+1}$

Exercise Find y' if $y = \ln(\ln x)$

Exercise Find y' if $y = \ln(x^3)$

Exercise Find y' if $y = (\ln x)^3$

Example

Show that $y = x \ln x - 4x$ is a solution to the differential equation

$$x + y - xy' = 0$$

$y' = -3 + \ln x$, so plugging this in gives $x + (x\ln x - 4x) - x(-3 + \ln x) = 0$.

Everything cancels out and we see that $0 = 0$ which is true.

1.4 The Natural Log Function and Integration

We previously saw differentiation.

Now integration.

$$\int \frac{1}{u} du = \ln|u| + C$$

Example

$\int \frac{2}{x} dx$

Simple! This is $2\ln|x| + C$.

Exercise $\int_1^e \frac{2}{x} dx$

Exercise $\int \frac{1}{2x-1} dx$

Exercise $\int \frac{3x^2+1}{x^3+x} dx$

Exercise $\int_1^e \frac{(1+\ln x)^3}{x} dx$

Exercise $\int_e^{e^2} \frac{(\ln x)^4}{x} dx$

Exercise $\int_0^3 \frac{x^2-5}{x+2} dx$

If you are integrating a quotient and the power of the numerator is greater than or equal to the power of the denominator you must divide.

Four more integration formulas:

- $\int \tan u du = -\ln|\cos u| + C$
- $\int \cot u du = \ln|\sin u| + C$
- $\int \sec u du = \ln|\sec u + \tan u| + C$
- $\int \csc u du = -\ln|\csc u + \cot u| + C$

Example

Why is $\int \tan x dx = -\ln|cosx| + C$ true?

Let $\int \frac{\sin x}{\cos x} dx$ and this is equal to $-\ln|\cos x| + C$.

Exercise Show why $\int \sec x dx$ works.

Exercise $\int \tan(3x) dx$

1.5 Derivatives of Inverse Functions

A function g is the inverse of a function f if and only if

$f(g(x)) = x$ for each x in the domain of g and $g(f(x)) = x$ for each x in the domain of f.

The inverse of f is denoted f^{-1}.

Properties of inverses:

- If g is the inverse of f, then f is the inverse of g.
- The domain of f^{-1} is equal to the range of f, and the range of f^{-1} is equal to the domain of f.
- Not every function has an inverse, but if a function does have an inverse, the inverse is unique.

Example

(a) Find the inverse function of f.

The inverse function is $x^2 + 1 = y$.

(b) State the domain and range of f and f^{-1}.

For $f(x)$ the domain is $x \geq 1$, range $y \geq 0$.

For $f^{-1}(x)$ the domain is $x \geq 0$, range is $y \geq 1$.

Example

Given $f(x) = x^3$ and $f^{-1}(x) = \sqrt[3]{x}$

(a) $f(2)$

8

(b) $f'(x)$

$3x^2$

(c) $f'(2)$

12

(d) $f^{-1}(8)$

2

(e) What is the derivative of $f^{-1}(x)$?

$\frac{1}{3}x^{-2/3}$

(f) $(f^{-1})'(8)$

$\frac{1}{12}$

In (c) and (f) notice they are reciprocals.

Theorem 1.3: Derivative of an Inverse Function

Let f be any function that is differentiable on an interval I. If f has an inverse function g, then g is differentiable at any x for which $f'(g(x)) \neq 0$ and $g'(x) = \frac{1}{f'(g(x))}$ so that $(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$.

Example

If $f(3) = 5$ and $f'(3) = \frac{7}{2}$, find $(f^{-1})'(5)$.

$(f^{-1})'(5) = \frac{1}{f'(3)} = \frac{1}{\frac{7}{2}} = \frac{2}{7}$.

Exercise Let $f(x) = x^3 + 2x - 1$. Find $(f^{-1})'(2)$.

Exercise Let $g(x) = \sqrt{x+1}$. Find $(g^{-1})'(2)$.

Exercise Let $f(x) = \cos x, 0 \leq x \leq \pi$. Find $(f^{-1})'\left(\frac{\sqrt{3}}{2}\right)$.

1.6 Exponential Functions

You learned in the past

$y = \log_b x$ means $x = b^y$ where $b > 0$ and $x > 0$

$y = \ln x$ means $x = e^y$ where $x > 0$.

Exercise Solve $e^{x+1} = 7$

Exercise Solve $\ln(2x - 3) = 5$.

Derivative of an exponential function: $\frac{d}{dx}[e^u] = e^u \frac{du}{dx}$.

Example

Find the derivative.

(a) $y = e^{3x^2}$

$y' = 6xe^{3x^2}$

(b) $y = \sin^2(e^x)$

$y' = 2\sin(e^x)\cos(e^x) \cdot e^x$

Exercise Find the derivative of $y = \ln(4 + e^{3x})$

Exercise Find the derivative of $y = \ln(e^{x^3})$

Exercise Find the derivative of $f(x) = \ln\left(\frac{3+e^x}{3-e^x}\right)$.

Exercise Find the derivative of $y = x^2e^{-x}$

Exercise Use implicit differentiation to find the derivative $\frac{dy}{dx}$ of $e^{xy} + x^2 - y^2 = 10$.

Example

Find the relative extrema and the points of inflection for

$$f(x) = xe^x$$

The first derivative of this is $f' = xe^x + e^x$.

We can see that -1 is a relative minimum.

The second derivative is $xe^x + e^x + e^x$.

We can see that -2 is a point of inflection.

The integral of an exponential function is $\int e^u du = e^u + C$.

Example

$$\int e^{3x+1} dx$$

Let $u = 3x + 1$ then $\frac{1}{3}du = dx$.

$\frac{1}{3}\int e^u du = \frac{1}{3}e^{3x+1} + C$.

Exercise $\int 5xe^{-x^2} dx$

Exercise $\int \frac{e^{1/x}}{x^2} dx$

Exercise $\int \sin xe^{\cos x} dx$

Exercise $\int_0^1 \frac{e^x}{1+e^x} dx$

Exercise $\int_{-1}^0 e^x \cos(e^x) dx$

Example

Solve the differential equation

$$\frac{dy}{dx} = (e^x - e^{-x})^2$$

We have $\int dy = \int (e^x - e^{-x})^2 dx$

This gives $y = \frac{1}{2}e^{2x} - 2x - \frac{1}{2}$.

Exercise Find the particular solution of the differential equation that satisfies the initial conditions.

$$f''(x) = \sin x + e^{2x}, f(0) = \frac{1}{4}, f'(0) = \frac{1}{2}$$

Example

The rate at which water is being pumped into a tank is $r(t) = 20e^{0.02t}$ where t is in minutes and $r(t)$ is in gallons per minute. How many gallons of water have been pumped into the tank in the first five minutes?

$$\int_0^5 r(t)dt = 105.171 \text{ gallons}$$

1.7 Bases other than e

Remember that $y = \log_b x$ means $x = b^y$ where $b > 0$ and $x > 0$.

Exercise Solve $2^{3x} = 45$

Exercise Solve $\log_5(x - 2) = 3$.

Formulas:

- $\frac{d}{dx}[\ln u] = \frac{1}{u}\frac{du}{dx}$
- $\frac{d}{dx}[e^u] = e^u \frac{du}{dx}$
- $\int e^u du = e^u + C$
- $\frac{d}{dx}[\log_a u] = \frac{1}{u \ln a}\frac{du}{dx}$
- $\frac{d}{dx}[a^u] = a^u \ln a \frac{du}{dx}$
- $\int a^u du = \frac{a^u}{\ln a} + C$

Example

Find the derivative of $y = 2^{x^3}$.

Let $u = x^3$ so $\frac{du}{dx} = 3x^2$.

So $y' = 2^{x^3} \cdot \ln 2 \cdot 3x^2$.

Exercise Differentiate $f(x) = \log_3(x^2 + 1)$

Exercise $\int 2^x dx$

Exercise $\int x^2 3^{x^3} dx$

If you are asked to differentiate a function that contains a variable raised to a power that contains a variable, we have no formula for this and must use a process called logarithmic differentiation.

Example

Find $\frac{dy}{dx}$ in terms of x.

$$y = (x+1)^{x-3}$$

Let $y = x^x$.

We can see logarithmic differentiation is needed.

$\frac{d}{dx}(\ln y = (x-3)\ln(x+1))$

We can see that $y' = \left(\frac{x-3}{x+1} + \ln(x+1)\right)$.

1.8 Inverse Trig Functions and Differentiation

In the past, you learned two notations for inverse trig functions. The inverse of cosine can be symbolized as $\arccos x$ or $\cos^{-1} x$. You were also taught restrictions for these.

Exercise $\arcsin\left(-\frac{1}{2}\right)$

Exercise $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

Exercise $\arctan(-0.3)$

We can derive the formulas for the derivatives of the inverse trig functions by using implicit differentiation.

Example

Let $y = \arcsin x$

$x = \sin y$

$1 = \cos y \cdot y'$

$y' = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$

A similar process can be done for $y = \arctan x$.

$$\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}[\arctan u] = \frac{u'}{1+u^2}$$

$$\frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$$

Exercise $f(x) = \arcsin(2x)$. What is $f'(x)$?

Exercise $f(x) = \tan^{-1}(3x)$. What is $f'(x)$?

Exercise $f(x) = \cos(\arcsin(3x))$. What is $f'(x)$?

1.9 Inverse Trig Integration

$$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$$
$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$$

Example

$$\int \frac{dx}{\sqrt{4-x^2}}$$

$a = 2$, $u = x$, $\frac{du}{dx} = 1$, $du = dx$.

Integrate to get $\sin^{-1}\left(\frac{x}{2}\right) + C$

Exercise $\int \frac{dx}{\sqrt{4-25x^2}}$

Exercise $\int_{\sqrt{3}}^{3} \frac{1}{9+x^2}dx$

Exercise $\int \frac{dx}{x^2-4x+7}$

Exercise $\int_0^{\sqrt{2}/2} \frac{\arccos x}{\sqrt{1-x^2}}dx$