1 Area between Curves, Volume, and Arc Length

1.1 Area Between Two Curves

To find the area bounded by two functions $y = f(x)$ and $y = g(x)$ on the interval $[a, b]$:

Area $= \int_a^b f(x) - g(x)dx$

To find the area bounded by two functions $x = f(y)$ and $x = g(y)$ on the interval $[a, b]$:

Area $= \int_a^b g(y) - f(y)dy$

Example

Find the area bounded by the graphs $y = 3 - x^2$ and $y = -x + 1$.

The intersections are when the two graphs are equal to each other.

Setting $3 - x^2 = -x + 1$ results in $x = -1$ and $x = 2$.

$A = \int_{-1}^{2} 3 - x^2 - (-x + 1)dx$

This is equal to $\int_{-1}^{2} 2 - x^2 + xdx$.

Simplifying this gives $A = 4.5$.

Exercise Find the area between the two graphs $x = 5 - y^2$ and $x = y - 1$.

1.2 Volume with Known Cross Sections

For this, we will find the volume of a solid whose cross sections are familiar geometric shapes, such as squares, rectangles, triangles, and semicircles.

For cross sections of area $A(x)$ taken perpendicular to the x-axis, the volume is $\int_a^b A(x)dx$

For cross sections of area $A(y)$ taken perpendicular to the y-axis, volume is $\int_{y=c}^{y=d} A(y)dy$

Example

Set up the integrals needed to find the volume of the solid whose base is the area bounded by the lines $y = x^2$ and $y = -2x + 3$ and whose cross sections perpendicular to the x-axis are the following shapes.

(a) Rectangles of height 4

$V = \int_{-3}^{1} -8x + 12 - 4x^2dx$

(b) Semicircles

$V = \frac{1}{2}\pi \int_{-3}^{1} \left(\frac{-2x+3-x^2}{2}\right)^2 dx$

Exercise Set up the integrals needed to find the volume of the solid whose base is the area bounded by the circle $x^2 + y^2 = 9$ and whose cross sections perpendicular to the x-axis are equilateral triangles. Note the area of an equilateral triangle is $\frac{s^2\sqrt{3}}{4}$ where s is a side of a triangle.

Exercise The base of a solid is bounded by $y = x^2$, $y = 0$, and $x = 2$. For this solid, each cross section perpendicular to the y-axis is square. Set up the integral needed to find the volume of this solid.

1.3 Volume: The Disc Method

If we revolve a figure around a line, a solid of revolution is formed. The line is called the axis of revolution. The simplest such solid is a right circular cylinder or disc.

To find the volume of the solid, we partition it into rectangles, which are revolved about the axis of revolution.

Each disc is a thin cylinder standing on its side. A volume of a cylinder is $\pi r^2 h$, a volume of a disc is $\pi(R(x))^2 \Delta x$.

Adding the volumes of all of the discs together, we get the volume of a solid to be approximately $\sum_{i=1}^{n} \pi[R(x_1)]^2 \Delta x_i$.

To get the exact volume, this is equal to

$$\lim_{n\to\infty} \sum_{i=1}^{n} \pi[R(x_i)]^2 \Delta x_i = \pi \int_a^b [R(x)]^2 dx$$

Volume about horizontal axis by discs: $V = \pi \int_a^b [R(x)]^2 dx$

Volume about vertical axis by discs: $V = \pi \int_c^d [R(y)]^2 dy$

The disc method can be extended to cover solids of revolutions with a hole in them. This is called the washer method.

If $R(x)$ is the outer radius and $r(x)$ is the inner radius:

Volume about horizontal axis by washers: $V = \pi \int_a^b [R(x)]^2 - [r(x)]^2 dx$

Volume about vertical axis by washers: $V = \pi \int_c^d [R(y)]^2 - [r(y)]^2 dy$

Things to remember: In the disc or washer method:

1. The representative rectangle is always perpendicular to the axis of revolution.
2. If the representative rectangle is vertical, you will work in x's. If the representative rectangle is horizontal, you will work in y's.

Example

Find the volume of the solid formed by revolving the region bounded by the graphs of the given equations about the indicated axis.

$$y = 9 - x^2, x = 0, y = 0$$

(a) about the x-axis.

$\pi \int_0^3 (9 - x^2)^2 dx$

(b) about the line $y = -2$

$V = \pi \int_0^3 (11 - x^2)^2 - 2^2 dx$

(c) about the y-axis

$V = \pi \int_0^9 (\sqrt{9 - y})^2 dy$

(d) about the line $x = -2$

$V = \pi \int_0^9 (\sqrt{9 - y} + 2)^2 - 2^2 dy$

Exercise Find the volume of the solid former by revolving the region bounded by the graphs of $y = 2x - x^2$ and $y = x^2$ about the line $y = 3$.

1.4 Arc Length

Arc length of $f(x)$ from $x = a$ to $x = b$:

$$s = \int_a^b \sqrt{1 + (f'(x))^2} dx$$

Arc length of $f(y)$ from $y = c$ to $y = d$:

$$s = \int_c^d \sqrt{1 + (f'(y))^2} dy$$

Example

Find the arc length of the graph of the given function over the indicated interval.

$$y = x^{3/2} - 1 \quad [0, 4]$$

$s = \int_0^4 \sqrt{1 + \left(\frac{3}{2}x^{1/2}\right)^2} dx$

Exercise Find the arc length of the graph of the function $y = 3x^{2/3} - 10$ on the interval $[8, 27]$.