1 Techniques of Integration

1.1 Integration by Parts

Integration by parts is used to integrate a product, such as the product of an algebraic and a transcendental function:

For example, $\int xe^x dx, \int x \sin x dx, \int x \ln x dx$.

Recall the product rule is $\frac{d}{dx}[uv] = u\frac{dv}{dx} + v\frac{du}{dx}$.

Integrating both sides, we get $uv = \int udv + vdu$.

Rearranging we get the formula for integration by parts:

$$\int udv = uv - \int vdu$$

Example

$$\int x \sin x dx$$

Let $u = x$, $dv = \sin x dx$, then $du = dx$ and $v = -\cos x$.

We get $x \cos x + \int \cos dx$.

Simplifying, we get $-x \cos x + \sin x + C$

Exercise $\int x^2 e^x dx$

A tabular approach is helpful with these "repeated" integration by parts problems

Example

$$x^2 e^x dx$$

u	v
x^2	e^x
$2x$	e^x
2	e^x
0	e^x

Criss crossing gives you $x^2e^x - 2xe^x + 2e^x$.

If you have limits of integration, first integrate without them.

Example

$$\int_0^{\pi/2} x \sin x dx$$

Using integration by parts you get $[-x \cos x + \sin x]$.

Using the limits of integration, you get 1.

Exercise $\int e^x \cos x dx$

1.2 Integration by Partial Fractions

Fractions which have a denominator that can be factored can be decomposed into a sum or difference of fractions.

Fractions which have a denominator that can be factored into distinct linear factors

$$\frac{4x+1}{x^2-5x+6} = \frac{4x+1}{(x-3)(x-2)} = \frac{A}{x-3} + \frac{B}{x-2}$$

Solving for A and B results in $A = 13$ and $B = -9$, so that the above equals

$$\frac{13}{x-3} - \frac{9}{x-2}$$

Example

$$\int \frac{4x+41}{x^2+3x-10} dx$$

This is $\frac{4x+41}{(x-2)(x+5)} = \frac{A}{x-2} + \frac{B}{x+5}$ inside the integral.

$4x + 41 = A(x + 5) + B(x - 2)$. If we let $x = -5$, $B = -3$. Let $x = 2$, then $A = 7$.

We are now integrating $\int \frac{7}{x-2} - \frac{3}{x+5} dx$.

This is $7 \ln |x - 2| - 3 \ln |x + 5| + C$.

1.3 Logistic Growth

In exponential growth (or decay), we assume that the rate of increase (or decrease) of a population at any time t is directly proportional to the population P. In otehr words, $\frac{dP}{dt} = kP$. However, in many situations population growth levels off and approaches a limiting number L (the carrying capacity) because of limited resources. In this situation the rate of increase (or decrease) is directly proportional to both P and $L - P$. This type of growth is called logistic growth. It is modeleted by the differential equation $\frac{dP}{dt} = kPL - P$.

If we find $\frac{d^2P}{dt^2}$ we can find out an important fact about the time when P is growing the fastest. We will do this in the example below.

Example

The population $P(t)$ of fish in a lake satisfies the logistic differnetial equation $\frac{dP}{dt} = 3P - \frac{P^2}{6000}$ where t is measured in years and $P(0) = 4000$.

(a) $\lim\limits_{t\to\infty} P(t) =?$

18000

(b) What is the range of the solution curve?

$4000 \leq P(t) < 18000$

(c) For what values of P is the solution curve increasing? Decreasing? Justify your answer.

$P(t)$ is increasing because $\frac{dP}{dt} > 0$.

(d) For what values of P is the solution curve concave up? Concave down? Justify your answer.

Concave up from $(4000, 9000)$ and concave down $(9000, 18000)$.

(e) Does the solution curve have an inflection point? Justify your answer.

Yes because the second derivative changed signs.

Exercise The population $P(t)$ of fish in a lake satisfies the logistic differential equation $\frac{dP}{dt} = 3P - \frac{P^2}{6000}$ where t is measured in years and $P(0) = 10000$.

(a) $\lim_{t\to\infty} P(t)$

(b) What is the range of the solution cuve?

(c) For what values of P is the solution curve increasing? Decreasing? Justify your answer.

(d) For what values of P is the solution curve concave up? Concave down? Justify your answer.

(e) Does the solution curve have an inflection point? Justify your answer.

Exercise The population $P(t)$ of fish in a lake satisfies the logistic differential equation $\frac{dP}{dt} = 3P - \frac{P^2}{6000}$ where t is measured in years and $P(0) = 20000$.

(a) $\lim_{t\to\infty} P(t)$

(b) What is the range of the solution cuve?

(c) For what values of P is the solution curve increasing? Decreasing? Justify your answer.

(d) For what values of P is the solution curve concave up? Concave down? Justify your answer.

(e) Does the solution curve have an inflection point? Justify your answer.