1 Multiple Integration

1.1 Double Integrals over Rectangular Regions

Definition

A function f defined on a rectangular region R in the xy-plane is integrable on R if

$$\lim_{\Delta \to 0} \sum_{k=1}^{n} f\left(x_k^*, y_k^*\right) \Delta A_k$$

exists for all partitions of R and for all choices of (x_k^*, y_k^*) within those partitions. The limit is the double integral of f over R, which we write

$$\iint_R f(x,y)\mathrm{d}A = \lim_{\Delta \to 0} \sum_{k=1}^{n} f\left(x_k^*, y_k^*\right) \Delta A_k$$

We usually use Fubini's theorem.

Theorem 1.1

Let f be continuous on the rectangular region $R = (x,y) : a \leq x \leq b, c \leq y \leq d$. The double integral of f over R may be evaluated by either of two iterated integrals:

$$\iint_R f(x,y)\mathrm{d}A = \int_c^d \int_a^b f(x,y)\mathrm{d}x\mathrm{d}y = \int_a^b \int_c^d f(x,y)\mathrm{d}y\mathrm{d}x$$

Definition

The average value of an integrable function f over a region R is

$$\bar{f} = \frac{1}{\text{area of } R} \iint_R f(x,y)\mathrm{d}A$$

The average height of $f(x,y)$ is $\frac{1}{Area(R)}$(Volume under $f(x,y)$).

1.2 Double Integrals over General Regions

In order to do a double integral over a general integral:

- Divide the plane into rectangles.
- In each rectangle R_k inside R, choose a point (x_k^*, y_k^*).
- Let $\Delta A_k =$ area(R_k), calculate $\sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k$
- Take the limit as $\Delta \to 0$, where Δ is the maximum length of diagonal of R_k.

We get

$$\iint_R f(x,y)\mathrm{d}A = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k$$

Given the graph of a surface $z = f(x,y)$ for (x,y) in a planar region R where $f(x,y) \geq 0$ for all (x,y) in R, the volume of the solid bounded by the surface $z = f(x, y)$ and the set R in the xy-plane is given by

$$\text{Volume} = \iint_R f(x, y)\mathrm{d}A$$

If R can be described as $a \leq x \leq b$ and $g(x) \leq y \leq h(x)$ then

$$\iint_R f(x, y)\mathrm{d}A = \int_b^a \int_{g(x)}^{h(x)} f(x, y)\mathrm{d}y\mathrm{d}x$$

If R can be described as $g(y) \leq x \leq h(y)$ and $c \leq y \leq d$ then

$$\iint_R f(x, y)\mathrm{d}A = \int_c^d \int_{g(y)}^{h(y)} f(x, y)\mathrm{d}x\mathrm{d}y$$

To evaluate a $\mathrm{d}y\mathrm{d}x$ double integral of the form

$$\int_a^b \int_{g(x)}^{f(x)} f(x, y)\mathrm{d}y\mathrm{d}$$

1. Integrate $f(x, y)$ with respect to y.
2. Substitute $y = h(x)$, $y = g(x)$ and subtract, resulting in a function of x (call it $A(x)$).
3. Evaluate the integral of the resulting function

$$\int_a^b A(x)\mathrm{d}x$$

To evaluate a $\mathrm{d}x\mathrm{d}y$ double integral of the form

$$\int_c^d \int_{g(y)}^{h(y)} f(x, y)\mathrm{d}x\mathrm{d}$$

1. Integrate $f(x, y)$ with respect to x.
2. Substitute $x = h(y)$, $x = g(y)$ and subtract, resulting in a function of y (call it $A(y)$).
3. Evaluate the integral of the resulting function

$$\int_c^d A(y)\mathrm{d}y$$

Definition

Let R be a region in the xy-plane. Then

$$\text{area of } R = \iint_R \mathrm{d}A$$

1.3 Double Integrals in Polar Coordinates

A cartesian rectangle can be described as:

$$R = (x, y) : a \leq x \leq b, c \leq y \leq d$$

A polar rectangle is:

$$R = (r, \theta) : a \leq r \leq b, \alpha \leq \theta \leq \beta$$

Approximation to polar double integral: Given $f(x, y) = f(r\cos\theta, r\sin\theta)$

If we let ΔA_k be the area of the kth polar rectangle, then the approximation is

$$\sum_{k=1}^{n} f(r_k^* \cos\theta_k^*, r_k^* \sin\theta_k^*)\Delta A_k$$

Which can be written as:

$$\iint_R f(x, y)\mathrm{d}A = \lim_{\Delta\to 0} \sum_{k=1}^{n} f(r_k^* \cos\theta_k^*, r_k^* \sin\theta_k^*)\Delta A_k$$

Over polar rectangles we have:

$$\iint_R f(x, y)\mathrm{d}A = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)r\mathrm{d}r\mathrm{d}\theta$$

$$R = (r, \theta) : a \leq r \leq b, \alpha \leq \theta \leq \beta$$

Theorem 1.2

Let f be continuous on the region R in the xy-plane expressed in polar coordinates as

$$R = (r, \theta) : 0 \leq g(\theta) \leq r \leq h(\theta), \alpha \leq \theta \leq \beta$$

where $0 < \beta - \alpha \leq 2\pi$. Then

$$\iint_R f(x, y)\mathrm{d}A = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)r\mathrm{d}r\mathrm{d}\theta$$

1.4 Triple Integrals

Consider $f(x, y, z)$ defined on D.

- Divide region containing D into rectangular boxes, numbered $k = 1, 2, \ldots, n$.
- Let ΔV_k be the volume of the kth box.
- For each k, choose a point (x_k^*, y_k^*, z_k^*) in the kth box.

  Approximation $= \sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\Delta V_k$.
- Set $\Delta =$ maximum length of a diagonal of a box.

$$\iiint_D f(x, y, z)\mathrm{d}V = \lim_{\Delta\to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\Delta V_k$$

Two applications of triple integrals:

1.

$$\iiint_D 1\mathrm{d}V = \text{Volume}(D)$$

2. If $\rho(x, y, z)$ represents the density of a solid at any point (x, y, z) of a solid then

$$\iiint_D \rho(x, y, z)\mathrm{d}V = \text{mass}(D)$$

Possible orders for integration:

1. $\mathrm{d}x\mathrm{d}y\mathrm{d}z$
2. $\mathrm{d}x\mathrm{d}z\mathrm{d}y$
3. $\mathrm{d}y\mathrm{d}x\mathrm{d}z$
4. $\mathrm{d}y\mathrm{d}z\mathrm{d}x$
5. $\mathrm{d}z\mathrm{d}x\mathrm{d}y$
6. $\mathrm{d}z\mathrm{d}y\mathrm{d}x$

Let's consider the $\mathrm{d}z\mathrm{d}y\mathrm{d}x$ order.

Theorem 1.3

Let f be continuous over the region

$$D = (x, y, z) : a \leq x \leq b, g(x) \leq y \leq h(x), G(x, y) \leq z \leq H(x, y)$$

, where g, h, G, and H are continuous functions. Then f is integrable over D and the triple integral is evaluated as the iterated integral

$$\iiint_D f(x, y, z)\mathrm{d}V = \int_a^b \int_{g(x)}^{h(x)} \int_{G(x,y)}^{H(x,y)} f(x, y, z)\mathrm{d}z\mathrm{d}y\mathrm{d}x$$

1.5 Triple Integrals in Cylindrical and Spherical Coordinates

A point P in three-dimensional space can be described in cylindrical coordinates $P(r, \theta, z)$.

- P^* is the projection of P into the xy-plane.
- (r, θ) is the polar coordinates of P^*.
- (r, θ, z) is the cylindrical coordinates of P.

We can transform from Rectangular to cylindrical:

$$r^2 = x^2 + y^2 \qquad tan\theta = y/x \qquad z = z$$

To convert from cylindrical to rectangular:

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

Approximate volume given a cylindrical point: $(r_k^*, \theta_k^*, z_k^*)$ and rectangular point: (x_k^*, y_k^*, z_k^*), the approximate volume is $\Delta V_k = r_k^* \Delta r \Delta\theta \Delta z$.

$$\iiint_D f(x, y, z)\mathrm{d}V = \lim_{\Delta\to 0} \sum_{k+1}^{n} f(r_k^* \cos\theta_k^*, r_k^* \sin\theta_k^*, z_k^*) r_k^* \Delta r \Delta\theta \Delta z$$

Theorem 1.4

Let f be continuous over the region D, expressed in cylindrical coordinates as

$$D = (r, \theta, z) : 0 \leq g(\theta) \leq r \leq h(\theta), \alpha \leq \theta \leq \beta, G(x, y) \leq z \leq H(x, y)$$

Then f is integrable over D, and the triple integral of f over D is

$$\iiint_D f(x, y, z)\mathrm{d}V = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} \int_{G(r\cos\theta, r\sin\theta)}^{H(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\mathrm{d}z r\mathrm{d}r\mathrm{d}\theta$$

In a triple integral in spherical coordinates, the coordinate is described as $P(\rho, \phi, \theta)$.

- ρ is the distance from the origin to P.
- ϕ is the angle between the positive z-axis and the line from the origin to P.
- θ is the same angle as in cylindrical coordinates; measures rotation around the z-axis relative to x-axis.

Some relations:

- $x^2 + y^2 = r^2$.
- $\tan\theta = \frac{y}{x}$.
- $x = r\cos\theta$.
- $y = r\sin\theta$.
- $x^2 + y^2 + z^2 = \rho^2$.
- $r = \rho\sin\phi$.
- $z = \rho\cos\phi$.
- $\tan\phi = \frac{r}{z}$.
- $x = \rho\sin\phi\cos\theta$.
- $y = \rho\sin\phi\sin\theta$.

Given the spherical coordinate $(\rho_k^*, \phi_k^*, \theta_k^*)$ and the rectangular coordinate (x_k^*, y_k^*, z_k^*).

The approximate volume of would be $\Delta V_k = \rho_p^{*2}\sin\phi_k^*\Delta\rho\Delta\phi\Delta\theta$.

$$\iiint_D f(x, y, z)\mathrm{d}V = \lim_{\Delta\to 0}\sum_{k=1}^{n} f(\rho_k^*\sin\phi_k^*\cos\theta_k^*, \rho_k^*\sin\phi_k^*\sin\theta_k^*, \rho_k^*\cos\phi_k^*)\rho_k^{*2}\sin\phi_k^*\Delta\rho\Delta\phi\Delta\theta$$

Theorem 1.5

Let f be continuous over the region D, expressed in spherical coordinates as

$$D = (\rho, \phi, \theta) : 0 \le g(\phi, \theta) \le \rho \le h(\phi, \theta), a \le \phi \le b, \alpha \le \theta \le \beta$$

Then f is integrable over D and the triple integral of f over D is

$$\iiint_D f(x, y, z)\mathrm{d}V = \int_\alpha^\beta \int_a^b \int_{g(\phi,\theta)}^{h(\phi,\theta)} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2\sin\phi\mathrm{d}\rho\mathrm{d}\phi\mathrm{d}\theta$$

1.6 Integrals for Mass Calculations

If masses $m_1, m_2, \ldots, m_n$ are arranged on the x-axis at coordinates $x_1, x_2, \ldots, x_n$ respectively the masses will be balanced about the point $\bar{x}$ if

$$\sum_{k=1}^{n} m_k(x_k - \bar{x}) = 0 \implies \bar{x} = \frac{\sum_{k=1}^{n} m_k x_k}{\sum_{k=1}^{n} m_k}$$

We can obtain the center of mass in 1 dimension as

$$\bar{x} = \frac{\int_a^b x\rho(x)\mathrm{d}x}{\int_a^b \rho(x)\mathrm{d}x}$$

Definition

Let ρ be a integrable density function on the interval $[a, b]$ (which represents a thin rod or wire). The center of mass is located on the point $\bar{x} = \frac{M}{m}$, where the total moment M and mass m are

$$M = \int_a^b x\rho(x)\mathrm{d}x \qquad m = \int_a^b \rho(x)\mathrm{d}x$$

Definition

Let ρ be an integrable area density function defined over a closed bounded region R in $\mathbb{R}^2$. The coordinates of the center of mass of the object represented by R are

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m}\iint_R x\rho(x,y)\mathrm{d}A \qquad \bar{y} = \frac{M_x}{m} = \frac{1}{m}\iint_R y\rho(x,y)\mathrm{d}A$$

where $m = \iint_R \rho(x,y)\mathrm{d}A$ is the mass, and M_y and M_x are the moments with respect to the y-axis and x-axis, respectively. If ρ is constant, the center of mass is called the centroid and is independent of the density.

$$M_y = \iint_R x\rho(x,y)\mathrm{d}A \qquad M_x = \iint_R y\rho(x,y)\mathrm{d}A$$

Definition

Let ρ be an integrable density function on a closed bounded region D in $\mathbb{R}^3$. The coordinates of the center of mass of the region are

$$\bar{x} = \frac{M_{yz}}{m} = \frac{1}{m}\iiint_D x\rho(x,y,z)\mathrm{d}V \quad \bar{y} = \frac{M_{xz}}{m} = \frac{1}{m}\iiint_D y\rho(x,y,z)\mathrm{d}V$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{1}{m}\iiint_D z\rho(x,y,z)\mathrm{d}V$$

$$M_{yz} = \iiint_D x\rho(x,y,z)\mathrm{d}V \quad M_{xz} = \iiint_D y\rho(x,y,z)\mathrm{d}V \quad M_{xy} = \iiint z\rho(x,y,z)\mathrm{d}V$$

1.7 Change of Variables in Multiple Integrals

We have substitution in single integrals as:

$$\int_a^b f(u(x))\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x = \int_{u(a)}^{u(b)} f(u)\mathrm{d}u$$

For double integrals, we could use polar coordiantes using the following we know:

- $x = r\cos\theta$
- $y = r\sin\theta$
- $r^2 = x^2 + y^2$
- $\tan\theta = \frac{y}{x}$.
- $\mathrm{d}A \to r\mathrm{d}r\mathrm{d}\theta$

Definition

A transformation T from a region S to a region R is one-to-one on S if $T(P) = T(Q)$ only when $P = Q$, where P and Q are points in S.

Definition

Given a transformation $T : x = g(u, v), y = h(u, v)$, where g and h are differentiable on a region of the uv-plane, the Jacobian determinant (or Jacobian) of T is

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial y} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Theorem 1.6

Let $T : x = g(u, v), y = h(u, v)$ be a transformation that maps a closed bounded region S in the uv-plane to a region R in the xy-plane. Assume T is one-to-one on the interior of S and g and h have continuous first partial derivatives there. If f is continuous on R, then

$$\iint_R f(x,y)\mathrm{d}A = \iint_S f(g(u,v), h(u,v)) \mid J(u,v) \mid \mathrm{d}A$$

Definition

Given a transformation $T : x = g(u, v, w), y = h(u, v, w)$, and $z = p(u, v, w)$, where g, h, and p are differentiable on a region of uvw-space, the Jacobian determinant (or Jacobian) of T is

$$J(u,v,w) = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

We can express $J(u,v,w) = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v}\frac{\partial z}{\partial w} + \frac{\partial x}{\partial v}\frac{\partial y}{\partial w}\frac{\partial z}{\partial u} + \frac{\partial x}{\partial w}\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial x}{\partial w}\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial x}{\partial u}\frac{\partial y}{\partial w}\frac{\partial z}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\frac{\partial z}{\partial w} - \frac{\partial x}{\partial w}\frac{\partial y}{\partial v}\frac{\partial z}{\partial u}$

Theorem 1.7

Let $T : x = g(u, v, w), y = h(u, v, w)$, and $z = p(u, v, w)$ be a transformation that maps a closed bounded region S in uvw-space to a region $D = T(S)$ in xyz-space. Assume T is one-to-one on the interior of S and g, h, and p have continuous first partial derivatives there. If f is continuous on D, then

$$\iiint_D f(x,y,z)\mathrm{d}V = \iiint_S f(g(u,v,w), h(u,v,w), p(u,v,w,)) \mid J(u,v,w) \mid \mathrm{d}V$$