1 Vector Calculus

1.1 Vector Fields

Definition

Let f and g be defined on a region R of $\mathbb{R}^2$. A vector field in $\mathbb{R}^2$ is a function $\mathbf{F}$ that assigns to each point (x, y) in R a vector $\mathbf{F}(x, y)$ where

$$\mathbf{F}(x, y) = f(x, y)\mathbf{i} + g(x, y)\mathbf{j}$$

or

$$F(x, y) = \langle f(x, y), g(x, y)\rangle$$

The vector field $\mathbf{R}$ is continuous or differentiable on R is f and g are continuous or differentiable on R.

Note: A vector field is both a vector valued function and a function of several variables.

Definition

Let f, g, and h be defined on a region D of $\mathbb{R}^3$. A vector field in $\mathbb{R}^3$ is a function $\mathbf{F}$ that assigns to each point (x, y, z) in D a vector $\mathbf{F}(x, y, z)$ where

$$\mathbf{F}(x, y, z) = f(x, y, z)\mathbf{i} + g(x, y, z)\mathbf{j} + h(x, y, z)\mathbf{k}$$

or

$$\mathbf{F}(x, y, z) = \langle f(x, y, z), g(x, y, z), h(x, y, z)\rangle$$

The vector field $\mathbf{F}$ is continuous or differentiable on D is f, g, and h are continuous or differentiable on D.

Definition: Radial Vector Field in $\mathbb{R}^2$

Let $\mathbf{r} = \langle x, y\rangle$ and p is any real number, then

$$\mathbf{F}(x, y) = \frac{\mathbf{r}}{|\mathbf{r}|^p} = \frac{\langle x, y\rangle}{|\mathbf{r}|^p}$$

is a radial vector field.

Definition

Let φ be a differentiable region of $\mathbb{R}^2$ or $\mathbb{R}^3$. The vector field **F**$= \nabla\varphi$ is a gradient field and the function φ is a potential function for **F**.

Recall $\nabla\varphi = \langle \varphi_x, \varphi_y \rangle$ or $\langle \varphi_x, \varphi_y, \varphi_z \rangle$ and that the vector field **F** $= \nabla_\varphi$ is orthogonal to the level curves of φ at (x, y).

In $\mathbb{R}^3$ the gradient field will be orthogonal to level surfaces of φ.

Definition

Let φ be a potential function for a vector field in **F** in $\mathbb{R}^2$. That is, **F**$= \nabla_\varphi$.

The level curves of a potential function are called equipotential curves.

Also, the vector field may be visualized by drawing continuous flow curves or streamlines that are everywhere orthogonal to the equipotential curves.

These ideas can be extended to $\mathbb{R}^3$ in which case we will have equipotential surfaces.

1.2 Line Integrals

Definition

Suppose the scalar-valued function f is defined on the region containing the smooth curve C given by **r**$(t) = \langle x(t), y(t) \rangle$, for $a \leq t \leq b$.

The line integral of f over C is

$$\int_C f(x(t), y(t))\mathrm{d} = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x(t_k^*), y(t_k^*))\Delta s_k$$

provided this limit exists over all partitions of $[a, b]$.

If $f > 0$ then the line integral computes the area of the "curtain" under f and over C.

Scalar line integrals are independent of the orientation and parameterization of the curve C.

Evaluting Scalar Line integrals in $\mathbb{R}^2$.

$$\int_C f\mathrm{d}s$$

What is $\mathrm{d}s$?

Let C be given by **r**$(t) = \langle x(t), y(t) \rangle$ for $a \leq t \leq b$.

Recall: The length of the curve C over $[a, t]$ is given by $s(t) = \int_a^t |\mathbf{r}'(u)|\mathrm{d}u$.

By differentiating both sides, $s'(t) = |\mathbf{r}'(t)|$.
Thus, $\mathrm{d}s = s'(t)\mathrm{d}t = |\mathbf{r}'(t)|$.

Theorem 1.1

Let f be continuous on a region containing a smooth curve C: $\mathbf{r}(t) = \langle x(t), y(t)\rangle$, for $a \leq t \leq b$. Then

$$\int_C f \mathrm{d}s = \int_a^b f(x(t), y(t))|\mathbf{r}'(t)|\mathrm{d}t = \int_a^b f(x(t), y(t))\sqrt{((x'(t))^2 + (y'(t))^2)}\mathrm{d}t$$

Theorem 1.2

Let f be continuous on a region containing a smooth curve C: $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, for $a \leq t \leq b$. Then

$$\int_C f \mathrm{d}s = \int_a^b f(x(t), y(t), z(t))|\mathbf{r}'(t)|\mathrm{d}t$$
$$= \int_a^b f(x(t), y(t), z(t))\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\mathrm{d}t$$

Definition: Line Integral of a Vector Field

Let $\mathbf{F}$ be a vector field that is continuous on a region containing a smooth oriented curve C parametrized by arc length. Let $\mathbf{T}$ be the unit tangent vector at each point of C consistent with the orientation. The line integral of $\mathbf{F}$ over C is

$$\int_C \mathbf{F} \cdot \mathbf{T} \mathrm{d}s$$

Observations

- $\mathbf{F}\cdot\mathbf{T}$=$|\mathbf{f}||\mathbf{T}|\cos\theta = |\mathbf{F}|\cos\theta$
- The line integral adds up these components
- The orientation of the curve matters! $\int_{-C} \mathbf{F} \cdot \mathbf{T}\mathrm{d}s = -\int_C \mathbf{f} \cdot \mathbf{T}\mathrm{d}s$

Evaluating The Line Integral of a Vector Field

$$\int_C \mathbf{F} \cdot \mathbf{T}\mathrm{d}s = \int_a^b \mathbf{F} \cdot \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}|\mathbf{r}'(t)|\mathrm{d}t = \int_a^b \mathbf{F} \cdot \mathbf{r}'(t)\mathrm{d}t$$

Different Forms of Line Integrals of Vector Fields: Given $\mathbf{F} = \langle f, g, h\rangle$ and C with parameterization $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ for $a \leq t \leq b$:

$$\int_C \mathbf{F} \cdot \mathbf{T}\mathrm{d}s = \int_a^b \mathbf{F} \cdot \mathbf{r}'(t)\mathrm{d}t = \int_a^b (f(t)x'(t) + g(t)y'(t) + h(t)z'(t))\mathrm{d}t$$
$$= \int_C (f\mathrm{d}x + g\mathrm{d}y + h\mathrm{d}z)$$
$$= \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

This works similarly for vector fields in $\mathbb{R}^2$.

Definition

Let $\mathbf{F}$ be a continuous force field in a region D of $\mathbb{R}^3$. Let

$$C : \mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle \text{for} a \leq t \leq b$$

be a smooth curve in D with a unit tangent vector $\mathbf{T}$ consistent with the orientation.

The work done (by the force field) in moving an object along C in the positive direction is

$$W = \int_C \mathbf{F} \cdot \mathbf{T} \mathrm{d}s = \int_a^b \mathbf{F} \cdot \mathbf{r}'(t) \mathrm{d}t$$

- $\mathbf{F} \cdot \mathbf{T} = |\mathbf{F}| \cos\theta$ is the tangential component of $\mathbf{F}$ along C (in direction of the motion).
- The vector line integral sums the work done at each point along C.

Definition

Definition: Circulation

Let $\mathbf{F}$ be a continuous vector field on a region R of $\mathbb{R}^2$, and let C be a closed smooth oriented curve in R. The circulation of $\mathbf{F}$ on C is $\int_C \mathbf{F} \cdot \mathbf{T} \mathrm{d}s$, where $\mathbf{T}$ is the unit vector tangent to C consistent with the orientation.

A curve C in $\mathbb{R}^2$ is closed if its initial and terminal points are the same.

Circulation is a measure of how much of the vector field points in the direction of C.

Definition

Definition: Flux

Let $\mathbf{F}$ be a continuous vector field on a region R of $\mathbb{R}^2$, and let C be a closed smooth oriented curve in R. The flux of the vector field $\mathbf{F}$ across C is $\int_C \mathbf{F} \cdot \mathbf{n} \mathrm{d}s$, where $\mathbf{n} = \mathbf{T} \times \mathbf{k}$ is the unit normal vector and $\mathbf{T}$ is the vector tangent to C consistent with the orientation.

Flux is a measure of how much the vector field points orthogonally to C.

In practice, use $\mathbf{n} = \mathbf{T} \times \mathbf{k}$ and $\mathrm{d}s = |\mathbf{r}'(t)| \mathrm{d}t$.

Given $\mathbf{F} = \langle f, g \rangle$ and $C : \mathbf{r}(t) = \langle x(t), y(t) \rangle$ for $a \leq t \leq b$, then

$$\begin{aligned} \int_C \mathbf{F} \cdot \mathbf{n} \mathrm{d}s &= \int_a^b (f(t)y'(t) - g(t)x'(t)) \mathrm{d}t \\ &= \int_C f \mathrm{d}y - g \mathrm{d}x \end{aligned}$$

1.3 Conservative Vector Fields

Definition: Simple and Closed Curves

Suppose a curve C (in $\mathbb{R}^2$ or $\mathbb{R}^3$) is described parametrically by $\mathbf{r}(t)$, where $a \leq t \leq b$.

- Then C is a simple curve if $\mathbf{r}(t_1) \neq \mathbf{r}(t_2)$ for all t_1 and t_2, with $a < t_1 < t_2 < b$; that is, C never intersects itself between its endpoints.
- The curve C is closed if $\mathbf{r}(a) = \mathbf{r}(b)$; that is, the initial and terminal points of C are the same.

Definition: Connected and Simply Connected Regions

- An open region R in $\mathbb{R}^2$ (or D in $\mathbb{R}^3$) is connected if it is possible to connect any two points in R by a continuous curve lying in R. (Think: R is in one piece)
- An open region R is simply connected if every closed simple curve in R can be deformed and contracted to a point in R. (Think: R has no holes.)

Recall that all points of an open set are interior points. An open set does not contain any of its boundary points.

Definition: Conservative Vector Fields

A vector field $\mathbf{F}$ is said to be conservative on a region (in $\mathbb{R}^2$ or $\mathbb{R}^3$) if there exists a scalar function φ such that $\mathbf{F} = \nabla\varphi$ on that region.

Recall that when $\mathbf{F} = \nabla\varphi$, the function φ is a potential function for $\mathbf{F}$.

Note: any function of the form $\varphi(x, y) = xy + C$ would be a potential function for $\mathbf{F}$.

Definition: Test for Conservative Vector Fields

Suppose $\mathbf{F} = \langle f, g\rangle$ has continuous first partial derivatives on a connected and simply connected region D in $\mathbb{R}^2$.

If $\mathbf{F}$ is conservative $\implies$ there is a function φ such that $\mathbf{F} = \nabla\varphi$.

$$\implies (1) f = \varphi_x \qquad \text{and} \qquad (2)\ g = \varphi_y.$$

Now by taking partial derivatives,

$$f_y = \varphi_{xy} \qquad \text{and} \qquad g_x = \varphi_{yx}.$$

By equality of mixed partial derivatives,

$$\varphi_{xy} = \varphi_y x.$$

Thus we can conclude, $f_y = g_x$.

The other direction is also true. That is, if $f_y = g_x$ then $\mathbf{F}$ is conservative.

This provides us with a test for a conservative vector field in two dimensions, which can be extended to the following test for vector fields in three dimensions.

Theorem 1.3: Test for Conservative Vector Fields

Let $\mathbf{F} = \langle f, g, h\rangle$ be a vector field defined on a connected and simply connected region D in $\mathbb{R}^3$, where f, g, and h have continuous first partial derivatives on D.

Then $\mathbf{F}$ is a conservative vector field on D if and only if

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}, \qquad \frac{\partial f}{\partial z} = \frac{\partial h}{\partial x} \qquad \text{and} \qquad \frac{\partial g}{\partial z} = \frac{\partial h}{\partial y}.$$

For vector fields in $\mathbb{R}^2$, we have the single condition $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$.

Theorem 1.4: Fundamental Theorem for Line Integrals

Let R be a region in $\mathbb{R}^2$ or $\mathbb{R}^3$ and let φ be a differentiable potential function defined on R. If $\mathbf{F} = \nabla\varphi$ (which means that $\mathbf{F}$ is conservative), then

$$\int_C \mathbf{F}\cdot\mathbf{T}\mathrm{d}s = \int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \varphi(B) - \varphi(A)$$

for all points A and B in R and all piecewise-smooth oriented curves C in R from A to B.

Why? Let $\mathbf{r}(t)$ be any parameterization of C for $a \leq t \leq b$ and one can use the chain rule to show that $\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \mathbf{F} \cdot \mathbf{r}'(t)$. This,

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_a^b \mathbf{F} \cdot \mathbf{r}'(t)\mathrm{d}t = \int_a^b \frac{\mathrm{d}\varphi}{\mathrm{d}t}\mathrm{d}t = \varphi(B) - \varphi(A)$$

Interpretation: If $\mathbf{F}$ is a conservative vector field, then the value of a line integral of $\mathbf{F}$ depends only on the endpoints of the path!

Definition: Path Independence

Let $\mathbf{F}$ be a continuous vector field with domain R. If

$$\int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{C_2} \mathbf{F}\mathrm{d}\mathbf{r}$$

for all piecewise-smooth curves C_1 and C_2 in R with the same initial and terminal points, then the line integral is independent of path.

Theorem 1.5

Let $\mathbf{F}$ be a continuous vector field on an open connected region R in $\mathbb{R}^2$. If

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

is independent of path, then $\mathbf{F}$ is conservative; that is, there exists a potential function φ such that $\mathbf{F} = \nabla\varphi$ on R.

Line Integrals on Closed Curves Notation: We will ues $\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ to denote a line integral over a closed curve C.

Theorem 1.6

Let R be an open connected region in $\mathbb{R}^2$ or $\mathbb{R}^3$. Then $\mathbf{F}$ is a conservative vector field on R if an only if $\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0$ on all simple closed piecewise-smooth oriented curves C in R.

Why?
$\mathbf{F}$ is a conservative $\implies \oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \varphi(B) - \varphi(A) = \varphi(A) - \varphi(A) = 0$
$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0 \implies 0 = \int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} + \int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$
$\implies \int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = -\int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{-C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$

1.4 Green's Theorem

Theorem 1.7: Green's Theorem - Circulation Form

Let C be a simple closed piecewise-smooth curve, oriented counterclockwise, that encloses a connected and simply connected region R in the plane. Assume $\mathbf{F} = \langle f, g\rangle$ where f and g have continuous first partial derivatives in R. Then

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \oint_C f\mathrm{d}x + g\mathrm{d}y = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) \mathrm{d}A$$

Green's Theorem relates the circulation on C to a double integral over the region R.

If needed: $\oint_{-C} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = -\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$

Definition: Two-Dimensional Curl

The two-dimensional curl of the vector field $\mathbf{F} = \langle f, g\rangle$ is

$$\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}$$

If the curl is zero throughout a regio, the vector field is irrotational throughout that region.

Recall: If $\mathbf{F} = \langle f, g\rangle$ is conservative then $f_y = g_x$

Thus, $g_x - f_y = 0$ and the curl of $\mathbf{F}$ is zero.

Under the conditions of Green's Theorem: $\oint_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) \mathrm{d}A = 0.$

Circulation integrals of conservative vector fields are always zero!

Theorem 1.8: Area of a Plane Region by Line Integrals

Let C be a simple closed piecewise-smooth curve, oriented counterclockwise, that encloses a connected and simply connected region R in the plane.

Then the area of R is given by:

$$\oint_C x\mathrm{d}y = -\oint_C y\mathrm{d}x = \frac{1}{2}\oint_C (x\mathrm{d}y - y\mathrm{d}x)$$

Theorem 1.9: Green's Theorem - Flux Form

Let C be a simple closed piecewise-smooth curve, oriented counterclockwise, that encloses a connected and simply connected region R in the plane. Assume $\mathbf{F} = \langle f, g\rangle$ where f and g have continuous first partial derivatives in R. Then

$$\oint_C \mathbf{F}\cdot\mathbf{n}\mathrm{d}s = \oint_C f\mathrm{d}y - g\mathrm{d}x = \iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right)\mathrm{d}A$$

Interpretation:

- Green's theorem says that the net divergence throughout the region R equals the flux across the boundary of R.

Definition: Two-Dimensional Divergence

The two-dimensional divergence of the vector field $\mathbf{F} = \langle f, g\rangle$ is

$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$$

If the divergence is zero throughout a region, the vector field is source free throughout that region.

The outward flux of a source free vector field is always zero!

1.5 Divergence and Curl

Definition: Divergence

The divergence of a vector field $\mathbf{F} = \langle f, g, h \rangle$ that is differentiable on a region of $\mathbb{R}^3$ is

$$\text{div}\mathbf{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

If div $\mathbf{F} = 0$, the vector field is source free.

Divergence measures the expansion or contraction of the vector field at each point.

Del operator: $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$

Alternation notation: $\nabla \cdot \mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle f, g, h \rangle = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} = \text{div}\mathbf{F}$

Theorem 1.10: Divergence of Radial Vector Fields

For a real number, p, the divergence of the radial vector field

$$\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|^p} = \frac{\langle x, y, z \rangle}{(x^2 + y^2 + z^2)^{p/2}} \qquad \text{is} \qquad \nabla \cdot \mathbf{F} = \frac{3 - p}{|\mathbf{r}|^p}$$

Definition: Curl

The curl of a vector field $\mathbf{F} = \langle f, g, h \rangle$ that is differentiable in a region of $\mathbb{R}^3$ is

$$\text{curl}\mathbf{F} = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\mathbf{i} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\mathbf{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\mathbf{k}$$

Curl is a measure of rotation within a vector field at each point.

We can express the curl as a cross product:

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}$$

If curl $\mathbf{F} = \mathbf{0}$, the vector field is irrotational.

The **k**-component of the curl (or the two-dimensional curl) gives the rotation of the vector field in the xy-plane at a point.

The other components of the curl give similar information about the rotation of the vector field.

Theorem 1.11: Curl of a Conservative Vector Field

Suppose $\mathbf{F}$ is a conservative vector field on an open region D of $\mathbb{R}^3$. Let $\mathbf{F} = \nabla\varphi$, where φ is a potential function with continuous second partial derivatives on D.

Then curl $\mathbf{F}=\mathbf{0}$ and F is irrotational.

Theorem 1.12: Divergence of the Curl

Suppose $\mathbf{F} = \langle f, g, h \rangle$, where f, g, and h have continuous second partial derivatives, then

$$\text{div curl } \mathbf{F} = 0.$$

That is, the divergence of the curl is zero.

Note: If $\mathbf{F}$ is a vector field in $\mathbb{R}^3$ then div $\mathbf{F}$ is a scalar valued function and not a vector field. Hence, curl div $\mathbf{F}$ is not defined.

General Rotation Field:

$\mathbf{F} = \mathbf{a} \times \mathbf{r} = \langle a_1, a_2, a_3 \rangle \times \langle x, y, z \rangle$.

- The vector **a** is the axis of rotation for the vector field **F**.
- The length of the curl of **F**, $|\nabla \times \mathbf{F}| = 2|\mathbf{a}|$.
- The divergence of the vector field **F** is zero, or **F** is source free.

1.6 Surface Integrals

Recall a curve in $\mathbb{R}^2$ is defined parametrically by $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, for $a \leq t \leq b$.

For a surface in $\mathbb{R}^3$ we'll need two parameters and three dependent variables:

$$\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$$

A cylinder with radius $a > 0$ and height $h > 0$ can be described parametrically as $\mathbf{r}(u, v) = \langle a\cos u, a\sin u, v \rangle$ where $0 \leq u \leq 2\pi$ and $0 \leq v \leq h$.

A cone with radius $a > 0$ and height $h > 0$ can be described parametrically as $\mathbf{r}(u, v) \left\langle \frac{av}{h}\cos u, \frac{av}{h}\sin u, v \right\rangle$ where $0 \leq u \leq 2\pi$ and $0 \leq v \leq h$.

A sphere with radius $a > 0$ can be described parametrically as $\mathbf{r}(u, v) = \langle a\sin u\cos v, a\sin u\sin v, a\cos u \rangle$, where $0 \leq u \leq \pi$ and $0 \leq v \leq 2\pi$.

For an explicitly defined surface:

$$z = g(x, y) \text{ on } R = (x, y) : a \leq x \leq b, c \leq y \leq d$$

can be parametrically described:

$$\mathbf{u,v} = \langle u, v, g(u, v) \rangle$$

where $a \leq u \leq b$ and $c \leq v \leq d$.

Now we will develop the surface integral of a scalar-valued function f defined on a smooth parametrized surface S.

$$\iint_S f(x, y, z) \mathrm{d}S$$

Applications:

- Compute the surface area of S.
- Compute the mass of a thin sheet described by the surface S with mass density function f.
- Compute the average value of f over the surface S.

Definition: Surface Integrals of Scalar-Valued Functions

Let f be a continuous scalar-valued function on a smooth surface S given parametrically by $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$, where u and v vary over the rectangle $R = (u, v) : a \leq u \leq b, c \leq v \leq d$. Assume also that the tangent vectors

$$\mathbf{t}_u = \frac{\partial \mathbf{r}}{\partial u} = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle \quad \text{and} \quad \mathbf{t}_v = \frac{\partial \mathbf{r}}{\partial v} = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle$$

are continuous on R and the normal vector $\mathbf{t}_u \times \mathbf{t}_v$ is nonzero on R.

The surface integral of f over S is

$$\iint_S f(x, y, z) \mathrm{d}S = \iint_R f(x(u, v), y(u, v), z(u, v)) |\mathbf{t}_u \times \mathbf{t}_v| \mathrm{d}A$$

If $f(x, y, z) = 1$, this integral equals the surface area of S.

Theorem 1.13: Evaluation of Surface Integrals of Scalar-Valued Functions on Explicitly Defined Surfaces

Let f be a continuous scalar-valued function on a smooth surface S given parametrically by $z = g(x, y)$, for (x, y) in a region R. The surface integral of f over S is

$$\iint_S f(x,y,z)\mathrm{d}S = \iint_R f(x,y,g(x,y))\sqrt{z_x^2 + z_y^2 + 1}\mathrm{d}A$$

If $f(x, y, z) = 1$, the surface integral equals the area of the surface.

Orientable Surfaces: To be orientable, a surface must have a choice of normal vectors that varies continuously over the surface. (The surface is two-sided.)

If a surface encloses a region then we will choose normal vectors to point in the outward direction. For other surfaces, we must specify the direction of the normal vector.

Definition

Flux Integrals: Consider a continuous vector field $\mathbf{F} = \langle f, g, h\rangle$. Let S be a smooth oriented surface with unit normal vector $\mathbf{n}$.

The flux integral

$$\iint_S \mathbf{F}\cdot\mathbf{n}\mathrm{d}S$$

computes the net flux of the vector field across the surface.

$\mathbf{F}\cdot\mathbf{n} = |\mathbf{F}||\mathbf{n}|\cos\theta = |\mathbf{F}|\cos\theta$.

The flux integral adds up the components of the vector field $\mathbf{F}$ normal to the surface.

Definition: Surface Integral of a Vector Field

Suppose $\mathbf{F} = \langle f, g, h\rangle$ is a continuous vector field on a region of $\mathbb{R}^3$ containing a smooth oriented surface S. If S is defined parametrically as $\mathbf{u},\mathbf{v} = \langle x(u, v), y(u, v), z(u, v)\rangle$, for (u, v) in a region R, then

$$\iint_S \mathbf{F}\cdot\mathbf{n}\mathrm{d}S = \iint_R \mathbf{F}\cdot(\mathbf{t}_u\times\mathbf{t}_v)\mathrm{d}A$$

where $\mathbf{t}_u = \frac{\partial\mathbf{r}}{\partial u} = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right\rangle$ and $\mathbf{t}_v = \frac{\partial\mathbf{r}}{\partial v} = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right\rangle$ are continuous on R, and the normal vector $\mathbf{t}_u\times\mathbf{t}_v$ is nonzero on R, and the direction of the normal vector is consistent with the orientation of S.

Definition: Surface Integral of a Vector Field

If S is defined in the form $z = w(x, y)$ for (x, y) in a region R, then

$$\iint_S \mathbf{F}\cdot\mathbf{n}\mathrm{d}S = \iint_R \mathbf{F}\cdot(\mathbf{t}_u\times\mathbf{t}_v)\mathrm{d}A = \iint_R (-fz_x - gz_y + h)\mathrm{d}A$$

1.7 Stokes' Theorem

Theorem 1.14: Stokes' Theorem

Let S be an oriented surface in $\mathbb{R}^3$ with a piecewise-smooth closed boundary C whose orientation is consistent with that of S. Assume $\mathbf{F} = \langle f, g, h\rangle$ is a vector field whose components have continuous first partial derivatives on S. Then

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \mathrm{d}S$$

where $\mathbf{n}$ is the unit normal vector to S determined by the orientation of S.

Stokes' Theorem says that the line integral around the boundary C of the tangential component of $\mathbf{F}$ is equal to the surface integral over S of the normal component of the curl of $\mathbf{F}$.

The right hand rule relates the orientations of S and C and determines the choice of normal vectors.

Stokes' Theorem is the three-dimensional version of the circulation form of Green's Theorem.

Note: If a closed curve C is the boundary of two different smooth oriented surfaces S_1 and S_2 which both have orientation consistent with that of C, then the integrals of $(\nabla \times \mathbf{F}) \cdot \mathbf{n}$ on the two surfaces are equal.

1.8 Divergence Theorem

Theorem 1.15: Divergence Theorem

Let $\mathbf{F}$ be a vector field whose components have continuous first partial derivatives in a connected and simply connected region D in $\mathbb{R}^3$ enclosed by an oriented surface S. Then

$$\iint_S \mathbf{F} \cdot \mathbf{n} \mathrm{d}S = \iiint_D \nabla \cdot \mathbf{F} \mathrm{d}V$$

where $\mathbf{n}$ is the outward unit normal vector to S.

The Divergence Theorem says that the flux of $\mathbf{F}$ across the boundary surface of D is equal to the triple integral over the divergence of $\mathbf{F}$ over D.

Theorem 1.16: Divergence Theorem for Hollow Regions

Suppose the vector field $\mathbf{F}$ satisfies the conditions of the Divergence Theorem on a region D in $\mathbb{R}^3$ bounded by two oriented surfaces S_1 and S_2 where S_1 lies within S_2. Let S be the entire boundary of $D (S = S_1 \cup S_2)$ and let $\mathbf{n}_1$ and $\mathbf{n}_2$ be the outward unit normal vectors for S_1 and S_2 respectively. Then

$$\iiint_D \nabla \cdot \mathbf{F} \mathrm{d}V = \iint_S \mathbf{F} \cdot \mathbf{n} \mathrm{d}S = \iint_{S_2} \mathbf{F} \cdot \mathbf{n}_2 \mathrm{d}S - \iint_{S_1} \mathbf{F} \cdot \mathbf{n}_1 \mathrm{d}S$$

This form ot the Divergence Theorem is applicable to vector fields that are not differentiable at the origin, as is the case with some important radial vector fields of the form:

$$\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|^p}$$