

1 First-Order Differential Equations

1.1 Separable Equations

Definition

If the right-hand side of the equation

$$\frac{dy}{dx} = f(x, y)$$

can be expressed as a function $g(x)$ that depends only on x times a function $p(y)$ that depends only on y , then the differential equation is called separable.

To solve the equation

$$\frac{dy}{dx} = g(x)p(y)$$

multiply by dx and by $h(y) = 1/p(y)$ to obtain

$$h(y)dy = g(x)dx$$

Then integrate both sides and you end up getting $H(y) = G(x) + C$, where we have merged the two constants of integration into a single symbol C . The last equation gives an implicit solution to the differential equation.

Example

Solve the nonlinear equation

$$\frac{dy}{dx} = \frac{x-5}{y^2}$$

This can be rewritten as $y^2 dy = (x-5)dx$. Integrating both sides results in $\frac{y^3}{3} = \frac{x^2}{2} - 5x + C$.

To get the explicit form just solve for y , which is trivial.

Example

Solve the initial value problem

$$\frac{dy}{dx} = \frac{y-1}{x+3} \quad y(-1) = 0$$

Doing Calc BC stuff gives us $y = 1 - \frac{1}{2}(x+3)$.

Be careful because you can be losing solutions. Ok bye!

1.2 Linear Equations

Remember a linear first-order equation is an equation that can be expressed in the form

$$a_1(x) \frac{dy}{dx} + a_0 y = b(x)$$

where $a_1(x)$, $a_0(x)$, and $b(x)$ depend only on the independent variable x , not on y .

Method for solving linear equation:

- Write the equation in the standard form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

- Calculate the integrating factor $\mu(x)$ by the formula

$$\mu(x) = \exp \left[\int P(x) dx \right]$$

- Multiply the equation in standard form by $\mu(x)$ and, recalling that the left-hand side is just $\frac{d}{dx}[\mu(x)y]$, obtain

$$\begin{aligned} \mu(x) \frac{dy}{dx} + P(x)\mu(x)y &= \mu(x)Q(x) \\ \frac{d}{dx}[\mu(x)y] &= \mu(x)Q(x) \end{aligned}$$

- Integrate the last equation and solve for y by dividing by $\mu(x)$ to obtain.

Example

Find the general solution to

$$\frac{1}{x} dy - \frac{2y}{x^2} = x \cos x \quad x > 0$$

We have $\frac{dy}{dx} - \frac{2}{x}y = x^2 \cos x$.

The integrating factor $\mu(x) = e^{\int P(x) dx}$ which in this case is $e^{-2 \int \frac{1}{x} dx}$ and this is equivalent to $\frac{1}{x^2}$.

Using this we can multiply through in standard form then we have $\frac{1}{x^2} \frac{dy}{dx} - \frac{2}{x^3}y = \cos x$.

The left side is just $\frac{d}{dx} \left(\frac{1}{x^2} y \right) = \cos x$.

Integrating and solving for y we get that $y = x^2 \sin x + Cx^2$.

Example

For the initial value problem

$$y' + y = \sqrt{1 + \cos^2 x} \quad y(1) = 4$$

find the value of $y(2)$.

Our $P(x)$ is 1 here, so $\mu = e^x$.

So the equation after multiplying through by it gives us that $\mu y' + \mu y = \mu \sqrt{1 + \cos^2 x}$, or $e^x y' + e^x y = e^x \sqrt{1 + \cos^2 x}$.

This is equivalent to basically $\frac{d}{dx}(e^x y) = e^x \sqrt{1 + \cos^2 x}$.

This is $e^x y = \int e^x \sqrt{1 + \cos^2 x} dx$.

Using a calculator $y(2) = 2.127$.

Theorem 1.1: Existence and Uniqueness of Solution

Suppose $P(x)$ and $Q(x)$ are continuous on an interval (a, b) that contains the point x_0 . Then for any choice of initial value y_0 , there exists a unique solution $y(x)$ on (a, b) to the initial value problem

$$\frac{dy}{dx} + P(x)y = Q(x) \quad y(x_0) = y_0$$

In fact the solution is given for a suitable value of C .

1.3 Exact Equations

Definition: Exact Differential Form

The differential form $M(x, y)dx + N(x, y)dy$ is said to be exact in a rectangle R if there is a function $F(x, y)$ such that

$$\frac{\partial F}{\partial x}(x, y) = M(x, y) \quad \text{and} \quad \frac{\partial F}{\partial y}(x, y) = N(x, y)$$

for all (x, y) in R . That is, the total differential of $F(x, y)$ satisfies

$$dF(x, y) = M(x, y)dx + N(x, y)dy$$

If $M(x, y)dx + N(x, y)dy$ is an exact differential form, then the equation

$$M(x, y)dx + N(x, y)dy = 0$$

is called an exact equation.

Theorem 1.2: Test for Exactness

Suppose the first partial derivatives of $M(x, y)$ and $N(x, y)$ are continuous in a rectangle R . Then

$$M(x, y)dx + N(x, y)dy = 0$$

is an exact equation in R if and only if the compatibility condition

$$\frac{\partial M}{\partial y}(x, y) = \frac{\partial N}{\partial x}(x, y)$$

holds for all (x, y) in R .

Example

Solve the differential equation

$$\frac{dy}{dx} = -\frac{2xy^2 + 1}{2x^2y}$$

Ok so this is not separable or linear, so we use exactness.

$$\begin{aligned} \frac{dy}{dx} + \frac{2xy^2 + 1}{2x^2y} &= 0 \\ dy + \frac{2xy^2 + 1}{2x^2y} dx &= 0 \\ \frac{2xy^2 + 1}{2x^2y} dx + 1 dy &= 0 \end{aligned}$$

This is the same form we want.

Another form we can get is $(2xy^2 + 1)dx + 2x^2ydy = 0$.

Another form we can get is $1dx + \frac{2x^2y}{2xy^2+1}dy = 0$.

We are now looking for a $F(x, y) = c$ and we know this is true when $\frac{\partial m}{\partial y} = \frac{\partial n}{\partial x}$.

So the second one of these is probably the best, so we now have $m = 2xy^2 + 1$ and $n = 2x^2y$.

Doing the partial of m with respect to y we get $4xy$ and the partial of n with respect to x is $4xy$ and these are the same.

Let $F(x, y) = x^2y^2 + x = C$. The partial of this function with respect to x is $2xy^2 + 1$ and the partial of this function with respect to y is $2x^2y$ and this is the same as previous.

Method for Solving Exact Equations:

- If $Mdx + Ndy = 0$ is exact, then $\partial F/\partial x = M$. Integrate this last equation with respect to x to get

$$F(x, y) = \int M(x, y)dx + g(y)$$

- To determine $g(y)$, take the partial derivative with respect to y of both sides of the above equation and substitute N for $\partial F/\partial y$. We can now solve for $g'(y)$.
- Integrate $g'(y)$ to obtain $g(y)$ up to a numerical constant. Substituting $g(y)$ into the equation from step 1 gives $F(x, y)$
- The solution to $Mdx + Ndy = 0$ is given implicitly by

$$F(x, y) = C$$

(Alternatively, starting with $\partial F/\partial y = N$, the implicit solution can be found by first integrating with respect to y .)

Example

Solve

$$(2xy - \sec^2 x)dx + (x^2 + 2y)dy = 0$$

Let m be the first term and n be the second term, and the partial derivatives of these are the same, so they are exact.

Let $F(x, y) = \int 2xy - \sec^2 x dx$. When we integrate this, we get $yx^2 - \tan x$. In this case, the constant is anything with y , so the integral is equivalent to $yx^2 - \tan x + g(y)$.

Now we take the $\frac{\partial F}{\partial y} = x^2 - 0 + g'(y)$. These two are n so $x^2 + 2y = x^2 + g'(y)$, so solving for $g(y)$ we get that this is equal to $y^2 + C$.

So $F(x, y) = xy^2 - \tan x + y^2 = C$.

Exercise Solve $(1 + e^xy + xe^xy)dx + (xe^x + 2)dy = 0$.

Solution: $x + xye^x + 2y = C$.

Example

Solve

$$(x + 3x^3 \sin y)dx + (x^4 \cos y)dy = 0$$

Doing the partials originally makes them not equal to each other.

We can get this to exact form by multiplying through by x^{-1} . When we do this we get $(1 + 3x^2 \sin y)dx + x^3 \cos y dy = 0$ and the partials of these are the same.

x^{-1} is called an integrating factor.

Integrating m with respect to x , we get that $F(x, y) = \int 1 + 3x^2 \sin y dx = x + \sin y \cdot x^3 + g(y)$.

Doing the partial of F with respect to y , we get $\frac{\partial F}{\partial y} = x^3 \cos y = 0 + x^3 \cos y + g'(y)$, and this gets that $g(y) = C$.

So the answer is $x + x^3 \sin y = C$.

1.4 Special Integrating Factors

Definition

If the equation

$$M(x, y)dx + N(x, y)dy = 0$$

is not exact, but the equation

$$\mu(x, y)M(x, y)dx + \mu(x, y)N(x, y)dy = 0$$

which results from multiplying the first equation by the function $\mu(x, y)$, is exact, then $\mu(x, y)$ is called an integrating factor of the first equation.

Theorem 1.3: Special Integrating Factors

If $(\partial M/\partial y - \partial N/\partial x)/N$ is continuous and depends only on x , then

$$\mu(x) = \exp \left[\int \left(\frac{\partial M/\partial y - \partial N/\partial x}{N} \right) dx \right]$$

is an integrating factor for an equation. If $(\partial N/\partial x - \partial M/\partial y)/M$ is continuous and depends only on y , then

$$\mu(y) = \exp \left[\int \left(\frac{\partial N/\partial x - \partial M/\partial y}{M} \right) dy \right]$$

is an integrating factor for the same equation.

Method for Finding Special Integrating Factors:

If $Mdx + Ndy = 0$ is neither separable nor linear, compute $\partial M/\partial y$ and $\partial N/\partial x$. If $\partial M/\partial y = \partial N/\partial x$, then the equation is exact. If it is not exact, consider

$$\frac{\partial M/\partial y - \partial N/\partial x}{N}$$

If this is a function of just x , then an integrating factor is given by the formula above of $\mu(x)$. If not consider

$$\frac{\partial N/\partial x - \partial M/\partial y}{M}$$

If this is a function of just y , then an integrating factor is given by above of $\mu(y)$.

Example

Solve $(2x^2 + y)dx + (x^2y - x)dy = 0$

When we do the partials, we get that $1 \neq 2xy - 1$.

So let's look at $\frac{\partial m/\partial y - \partial n/\partial x}{N}$, which is $\frac{1 - (2xy - 1)}{x^2y - x} = \frac{-2}{x}$ which is just a function of x . So we have that $\mu = e^{\int -\frac{2}{x} dx}$, so we don't have to look at the one in terms of y .

Doing the integral of all this gives us that $e^{-2 \ln x} = x^{-2}$. So when we multiply through by x^{-2} , we get that $(2 + x^{-2}y)dx + (y - x^{-1})dy = 0$.

The partials are equal to each other, so this equation is now exact.

Now we find $F(x, y)$ by integrating m , so $\int (2 + x^{-2}y)dx = 2x + -x^{-1}y + g(y) = F(x)$

Now we differentiate with respect to y so $\frac{\partial F}{\partial y} = y - x^{-1} = -x^{-1} + g'(y)$, so $g(y) = \frac{y^2}{2}$

The solution is therefore $2x - x^{-1}y + \frac{y^2}{2} = C$.

1.5 Substitutions and Transformations

Substitution Procedure:

- Identify the type of equation and determine the appropriate substitution or transformation
- Rewrite the original equation in terms of new variables
- Solve the transformed equation
- Express the solution in terms of the original variables

Definition: Homogeneous Equation

If the right-hand side of the equation

$$\frac{dy}{dx} = f(x, y)$$

can be expressed as a function of the ratio y/x alone, then we say the equation is homogeneous.

To solve a homogeneous equation, use the substitution $v = \frac{y}{x}$; $\frac{dy}{dx} = v + x \frac{dv}{dx}$ to transform the equation into a separable equation.

Example

Solve $(xy + y^2 + x^2)dx - x^2dy = 0$.

Solving for $\frac{dy}{dx}$ we get that this is equal to $\frac{-x^2 - y^2 - xy}{-x^2}$ and this simplifies to $1 + \left(\frac{y}{x}\right)^2 + \frac{y}{x}$.

This is equivalent to $v + x \frac{dv}{dx} = 1 + v^2 + v$. We end up getting that $\frac{dv}{dx} = \frac{v^2 + 1}{x}$ and this can be done by separation. The solution is $y = x \tan(\ln|x| + C)$ after solving.

To solve an equation of the form $\frac{dy}{dx} = G(ax + by)$, use the substitution $z = ax + by$ to transform the equation into a separable equation.

Example

Solve $\frac{dy}{dx} = y - x - 1 + (x - y + 2)^{-1}$

First we have $\frac{dy}{dx} = -(x - y) - 1 + (x - y + 2)^{-1}$

So substituting with $z = x - y$, we have that $\frac{dz}{dx} = 1 - \frac{dz}{dx}$. Knowing this, the equation is equal to $1 - \frac{dz}{dx} = -z - 1 + (z + 2)^{-1}$. From this this simplifies to $\frac{dz}{dx} = z + 2 - (z + 2)^{-1}$.

So now we write this into a separable equation with $\frac{(z+2)dz}{(z+2)^2 - 1} = dx$.

Separating by parts and substituting gives $(x - y + 2)^2 = ce^{2x} + 1$

Definition: Bernoulli Equation

A first-order equation that can be written in the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

where $P(x)$ and $Q(x)$ are continuous on the interval (a, b) and n is a real number, is called a Bernoulli equation.

To solve a Bernoulli equation use the substitution $v = y^{1-n}$ to transform the equation into a linear equation.

Example

Solve $\frac{dy}{dx} - 5y = -\frac{5}{2}xy^3$.

From above, we have $v = y^{-2}$ and $\frac{dv}{dx} = -2y^{-3}\frac{dy}{dx}$ and $-\frac{1}{2}\frac{dv}{dx} = y^{-3}\frac{dy}{dx}$.

So multiplying through by y^{-3} and substituting, we get that $-\frac{1}{2}\frac{dv}{dx} - 5v = -\frac{5}{2}x$.

This is equal to $\frac{dv}{dx} + 10v = 5x$. The integrating factor here is $\mu = e^{\int P(x)dx}$, which is e^{10x} in this case.

Multiplying through by μ , we get that $e^{10x}\frac{dv}{dx} + 10ve^{10x} = 5xe^{10x}$ and the LHS should be equal to $\frac{d}{dx}(e^{10x}v) = 5xe^{10x}$.

Using elementary integration techniques the answer is $y^{-2} = \frac{x}{2} - \frac{1}{20} + Ce^{-10x}$.