1 Laplace Transforms

1.1 Definition of the Laplace Transform

Definition

Let $f(t)$ be a function on $[0, \infty)$. The Laplace transform of f is the function F defined by the integral

$$F(s) = \int_0^\infty e^{-st} f(t) \mathrm{d}t$$

The domain of $F(s)$ is all the values of s for which the integral above exists. The Laplace transform of f is denoted by both F and $\mathcal{L}\{f\}$.

Example

Determine the Laplace transform of the constant function $f(t) = 1, t \geq 0$.

Let $F(s) = \int_0^\infty e^{-st} 1 \mathrm{d}t = \int_0^\infty e^{-st} \mathrm{d}t$. This is equal to $-\frac{1}{s} e^{-st}$ with bounds ∞ and 0.

Remember this is an improper integral where we have $\lim_{b \to \infty} -\frac{1}{s} e^{-st}$ from 0 to b.

This gives $-\frac{1}{s} e^{-sb} - \frac{1}{s} e^{0}$ on the inside of the limit, so we get $\lim_{b \to \infty} \left[-\frac{1}{s} e^{-sb} + \frac{1}{s}\right]$.

The above equals $\lim_{b \to \infty} \left[-\frac{1}{s} \cdot \frac{1}{e^{rb}} + \frac{1}{s}\right]$.

The restriction is $s > 0$ because $\frac{1}{e^{sb}}$ has to be greater than 0.

Our result ends up being $\frac{1}{s}$.

$\mathcal{L}\{1\} = \frac{1}{s}$.

Example

Determine the Laplace transform of $f(t) = t$.

We have $\mathcal{L}\{t\} = \int_0^\infty e^{-st} t \mathrm{d}t = \lim_{b \to \infty} \left[\int_0^b e^{-st} t \mathrm{d}t\right]$.

Integrating by parts gives the inside equal to $-\frac{1}{s} \cdot t \cdot \frac{1}{e^{st}} - \frac{1}{s^2} e^{-st}$ with bounds 0 to b.

Plugging this in gives $\lim_{b \to \infty} -\frac{1}{s} \cdot \frac{b}{e^{sb}} - \frac{1}{s} \cdot \frac{1}{e^{sb}} + \frac{1}{s^2}$.

We see that $\frac{b}{e^{sb}}$ is indeterminate, so using L'Hopital's Rule, the derivative is $\frac{1}{se^{sb}}$ and the limit as b approaches ∞ gives this as 0.

We are left with $\frac{1}{s^2}$.

$\mathcal{L}\{t\} = \frac{1}{s^2}$.

We will see that $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$.

Example

Determine the Laplace transform of $f(t) = e^{at}$, where a is a constant.

The integral is $\int_0^\infty e^{-st} \cdot e^{at} \mathrm{d}t = \int_0^\infty e^{-(s-a)t} \mathrm{d}t$.

Integrating this gives $-\frac{1}{s-a}e^{-(s-a)t}$ evaluated from 0 to ∞.

As t goes to infinity, we get 0 and then we get $0 - \frac{-1}{s-a}e^0 = \frac{1}{s-a}$.

So $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$.

If we were to find the Laplace of e^{5t}, from the above example it would be $\frac{1}{s-5}$.

Example

Find $\mathcal{L}\{\sin bt\}$, where b is a nonzero constant.

The integral this time is $\int_0^\infty e^{-st} \cdot \sin bt \mathrm{d}t$.

Integrating gives $-\frac{1}{s}\sin bte^{-st} + \frac{b}{s}\left[-\frac{1}{s}\cos bte^{-st} - \int -\frac{1}{s}e^{-st}(-b)\sin bt \mathrm{d}t\right]$.

(Do this example later)

Involves factoring Laplace stuff.

$\mathcal{L}\{\sin bt\} = \frac{b}{s^2+b^2}$.

Example

Determine the Laplace transform of

$$f(t) = \begin{cases} 2 & 0 < t < 5 \\ 0 & 5 < t < 10 \\ e^{4t} & t > 10 \end{cases}$$

To do this, you just do $\int_0^\infty e^{-st}f(t)\mathrm{d}t = \int_0^5 e^{-st} \cdot 2\mathrm{d}t + \int_5^{10} e^{-st} \cdot 0\mathrm{d}t + \int_1 0^\infty e^{-st} \cdot e^{4t}\mathrm{d}t$.

Evaluating this gives the laplace as $-\frac{2}{s}e^{-5s} + \frac{2}{s} + \frac{1}{s-4}e^{-(s-4)10}$

An important property of the Laplace transform is its linearity. That is, the Laplace transform $\mathcal{L}$ is a linear operator.

Theorem 1.1

Let f, f_1, and f_2 be functions whose Laplace transforms exist for $s > \alpha$ and let c be a constant. Then, for $s > \alpha$,

$$\mathcal{L}\{f_1 + f_2\} = \mathcal{L}\{f_1\} + \mathcal{L}\{f_2\}$$
$$\mathcal{L}\{cf\} = c\mathcal{L}\{f\}$$

Exercise Determine $\mathcal{L}\{11 + 5e^{4t} - 6\sin 2t\}$.

A function $f(t)$ on $[a, b]$ is said to have a jump discontinuity at $t_0 \in (a, b)$ if $f(t)$ is discontinuous at t_0, but the one-sided limits

$$\lim_{t \to t_0^-} f(t) \quad \text{and} \quad \lim_{t \to t_0^+} f(t)$$

exist as finite numbers.

Definition

A function $f(t)$ is said to be piecewise continuous on a finite interval $[a, b]$ if $f(t)$ is continuous at every point in $[a, b]$, except possibly for a finite number of points at which $f(t)$ has a jump discontinuity.

A function $f(t)$ is said to be piecewise continuous on $[0, \infty)$ if $f(t)$ is piecewise continuous on $[0, N]$ for all $N > 0$.

In contrast, the function $f(t) = 1/t$ is not piecewise continuous on any interval containing the origin, since it has an "infinite jump" at the origin.

A function that is piecewise continuous on a finite interval is not necessarily integrable over that interval. However, piecewise continuity on $[0, \infty)$ is not enough to guarantee the existence (as a finite number) of the improper integral over $[0, \infty)$; we also need to consider the growth of the integrand for large t. The Laplace transform of a piecewise continuous function exists, provided the function does not grow "faster than an exponential".

Definition

A function $f(t)$ is said to be of exponential order α if there exist positive constants T and M such that

$$|f(T)| \leq Me^{\alpha t}$$

for all $t \geq T$.

Theorem 1.2

If $f(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order α, then $\mathcal{L}\{f\}(s)$ exists for $s > a$.

Here are common Laplace transforms:

- $\mathcal{L}\{1\} = \frac{1}{s}$
- $\mathcal{L}\{t\} = \frac{1}{s^2}$
- $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$
- $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$
- $\mathcal{L}\{\sin bt\} = \frac{b}{s^2+b^2}$
- $\mathcal{L}\{\cos bt\} = \frac{s}{s^2+b^2}$

1.2 Properties of the Laplace Transform

Theorem 1.3

If the Laplace transform $\mathcal{L}\{f\}(s) = F(s)$ exists for $s > \alpha$, then

$$\mathcal{L}\{e^{\alpha t} f(t)\}(s) = F(s - a)$$

for $s > \alpha + a$

Example

Determine the Laplace transform of $e^{\alpha t} \sin bt$

We know the Laplace of $\sin bt$ is equal to $\frac{b}{s^2+b^2}$.

Multiplying by $e^{\alpha t}$ just shifts it $F(s - \alpha) = \frac{b}{(s-\alpha)^2+b^2}$

Theorem 1.4

Let $f(t)$ be continuous on $[0, \infty)$ and $f'(t)$ be piecewise continuous on $[0, \infty)$, with both of exponential order α. Then for $s > \alpha$,

$$\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0)$$

Theorem 1.5

Let $f(t), f'(t), \ldots, f^{(n-1)}(t)$ be continuous on $[0, \infty)$ and let $f^{(n)}(t)$ be piecewise continuous on $[0, \infty)$, with all these functions of exponential order α. Then, for $s > \alpha$,

$$\mathcal{L}\{f^{(n)}\}(s) = s^n \mathcal{L}\{f\}(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$$

Example

Using the above theorems and the fact that $\mathcal{L}\{\sin bt\}(s) = \frac{b}{s^2+b^2}$, determine $\mathcal{L}\{\cos bt\}$

We know that $f'(t) = b\cos bt$ from this. So $\mathcal{L}\{b\cos bt\} = s\mathcal{L}\{\sin bt\} - f(0)$.

We know that $b\mathcal{L}\{\cos bt\} = s\mathcal{L}\{\sin bt\}$, since $f(0) = 0$.

So simplifying gives the Laplace transform as $\frac{s}{s^2+b^2}$

Example

Prove the following identity for continous functions $f(t)$ (assuming the transforms exist):

$$\mathcal{L}\left\{\int_0^t f(\tau)\mathrm{d}\tau\right\}(s) = \frac{1}{s}\mathcal{L}\{f(t)\}(s)$$

We know $g(t) = \int_0^t f(\tau)\mathrm{d}\tau$. From this we know $g'(t) = f(t)$.

We get that $\mathcal{L}\{g'(t)\} = s\mathcal{L}\{g(t)\} - g(0)$. and that $\mathcal{L}\{f(t)\} = s\mathcal{L}\{\int_0^t f(\tau)\mathrm{d}\tau\}$.

We also know $g(0) = 0$.

So the Laplace of the function is equal to $\frac{1}{s}\mathcal{L}\{f(t)\}$.

Theorem 1.6

Let $F(s) = \mathcal{L}\{f\}(s)$ and assume $f(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order α. Then, for $s > \alpha$,

$$\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \frac{\mathrm{d}^n F}{\mathrm{d}s^n}(s)$$

Example

Determine $\mathcal{L}\{t\sin bt\}$.

We know $f(t) = \sin bt$ and that $n = 1$.

This is equal to $(-1)^1 \frac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}\{\sin bt\}$.

We end up getting $-\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{b}{s^2+b^2}\right)$.

We end up getting $\frac{2bs}{(s^2+b^2)^2}$.

Here are some basic properties of Laplace Transforms

- $\mathcal{L}\{f+g\} = \mathcal{L}\{f\} + \mathcal{L}\{g\}$.
- $\mathcal{L}\{cf\} = c\mathcal{L}\{f\}$ for any constant c.
- $\mathcal{L}\{e^{at}f(t)\}(s) = \mathcal{L}\{f\}(s-a)$
- $\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0)$

- $\mathcal{L}\{f''(s)\} = s^2\mathcal{L}\{f\}(s) - sf(0) - f'(0)$
- $\mathcal{L}\{f^{(n)}\}(s) = s^n\mathcal{L}\{f\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$
- $\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}s^n}(\mathcal{L}\{f\}(s))$

1.3 Inverse Laplace Transform

Example

Solve the initial value problem

$$y'' - y = -t \qquad y(0) = 0, \qquad y'(0) = 1$$

We can say that $\mathcal{L}\{y'' - y\} = \mathcal{L}\{-t\}$.

Using properties we know that $\mathcal{L}\{y''\} - \mathcal{L}\{y\} = -\mathcal{L}\{t\}$

This is equal to $s^2\mathcal{L}\{y\} - sy(0) - y'(0) = \mathcal{L}\{y\} = -\frac{1}{s^2}$.

Now plugging in $\mathcal{L}\{y(t)\} = Y(s)$, we get $s^2Y(s)1 - Y(s) = -\frac{1}{s^2}$

Simplifying gives $Y(s)(s^2 - 1) = \frac{s^2-1}{s^2}$.

We see that $Y(s) = \frac{1}{s^2}$. This is the Laplace of t, so $y(t) = t$.

Definition

Given a function $F(s)$, if there is a function $f(t)$ that is cintinuous on $[0, \infty)$ and satisfies

$$\mathcal{L}\{f\} = F$$

then we say that $f(t)$ is the inverse Laplace transform of $F(s)$ and employ the notation $f = \mathcal{L}^{-1}\{F\}$.

Example

Determine $\mathcal{L}\{F\}$ for $F(s) = \frac{2}{s^2}$.

The Inverse Laplace transform of this is t^2.

Determine it for $F(s) = \frac{3}{s^2+9}$.

This is $\sin 3t$ from the definition.

Determine it for $\frac{s-1}{s^2-2s+5}$.

This simplifies to $\frac{s-1}{(s-1)^2+4} = F(s-1)$. This is the same as $\cos 2t$ but shifted by 1. The Inverse Laplace transform ends up being $e^t \cos 2t$.

Theorem 1.7

Assume that $\mathcal{L}^{-1}\{F\}, \mathcal{L}^{-1}\{F_1\}$, and $\mathcal{L}^{-1}\{F_2\}$ exist and are continuous on $[0, \infty)$ and let c be any constant. Then

$$\mathcal{L}^{-1}\{F_1 + F_2\} = \mathcal{L}^{-1}\{F_1\} + \mathcal{L}^{-1}\{F_2\}$$
$$\mathcal{L}^{-1}\{cF\} = c\mathcal{L}^{-1}\{F\}$$

Example

Determine $\mathcal{L}^{-1}\left\{\frac{5}{s-6}-\frac{6s}{s^2+9}+\frac{3}{2s^2+8s+10}\right\}$.

The first two terms of this gives $5e^6t - 6\cos 3t$.

For the last term, We see that $\frac{1}{2(s^2+4s+5)}$ lets us put $\frac{3}{2}$ in the front and we can complete the square for this for the denominator to give $\frac{1}{(s+2)^2+1}$.

The last term ends up being $\frac{3}{2}e^{-2t}\sin t$.

Exercise Determine $\mathcal{L}^{-1}\{\frac{5}{s+2}^4\}$

Exercise Determine $\mathcal{L}^{-1}\{\frac{3s+2}{s^2+2s+10}\}$.

Method of Partial Fractions - A rational function of the form $\frac{P(s)}{Q(s)}$, where $P(s)$ and $Q(s)$ are polynomials with the degree of P less than the degree of Q has a partial fraction expansion whose form is based on the linear and quadratic factors of $Q(s)$. We consider the three cases:

1. Nonrepeated linear factors
2. Repeated linear factors
3. Quadratic factors

Nonrepeated Linear Factors - If $Q(s)$ can be factored into a product of distinct linear factors, $Q(s) = (s-r_1)(s-r_2)\dots(s-r_n)$, where the r_i's are all distinct real numbers, then the partial fraction expansion has the form

$$\frac{P(s)}{Q(s)} = \frac{A_1}{s-r_1} + \frac{A_2}{s-r_2} + \cdots + \frac{A_n}{s-r_n}$$

where the A_i's are real numbers.

Example

Determine $\mathcal{L}^{-1}\{F\}$, where $F(s) = \frac{7s-1}{(s+1)(s+2)(s-3)}$.

The decomposition is equal to $\frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s-3}$.

Solving for A, B, C gives $2, -3, 1$ respectively.

We end up getting $\frac{2}{s+1} + \frac{-3}{s+2} + \frac{1}{s-3}$. This gives us $2e^{-t} - 3e^{-2t} + e^{3t}$.

Repeated Linear Factors - Let $s-r$ be a factor of $Q(s)$ and suppose $(s-r)^m$ is the highest power of $s-r$ that divides $Q(s)$. Then the portion of the partial fraction expansion of $P(s)/Q(s)$ that corresponds to the term $(s-r)^m$ is

$$\frac{A_1}{s-r} + \frac{A_2}{(s-r^2)} + \cdots + \frac{A_m}{(s-r)^m}$$

where the A_i's are real numbers.

Example

Determine $\mathcal{L}\left\{\frac{s^2+9s+2}{(s-1)^2(s+3)}\right\}$.

We end up getting $\frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+3}$.

Solving for A, B, C gives $2, 3, -1$ respectively.

This gives $2e^t + 3t^t t - e^{-3t}$.

Quadratic Factors - Let $(s-\alpha)^2 + \beta^2$ be a quadratic factor of $Q(s)$ that cannot be reduced to linear factors with real coefficients. Suppose m is the highest power of $(s-\alpha)^2+\beta^2$ that divides $Q(s)$. THen the portion of the partial fraction expansion that corresponds to $(s-\alpha)^2+\beta^2$ is

$$\frac{C_1 s + D_1}{(s-\alpha)^2+\beta^2} + \frac{C_2 s + D_2}{[(s-\alpha)^2+\beta^2]^2} + \cdots + \frac{C_m s + D_m}{[(s-\alpha)^2+\beta^2]^m}$$

When looking up Laplace transforms, the following equivalent form is more convenient

$$\frac{A_1(s-\alpha)+\beta B_1}{(s-\alpha)^2+\beta^2}+\frac{A_2(s-\alpha)\beta B_2}{[(s-\alpha)^2+\beta^2]^2}+\cdots+\frac{A_m(s-\alpha)+\beta B_m}{[(s-\alpha)^2+\beta^2]^m}$$

Example

Determine $\mathcal{L}^{-1}\left\{\frac{2s^2+10s}{(s^2-2s+5)(s+1)}\right\}$.

The partial fraction is $\frac{As+B}{(s^2-2s+5)}+\frac{C}{s+1}$.

Solving the system gives $A, B, C = 3, 5, -1$.

So we are now finding the Laplace transform of $\frac{3s+5}{(s-1)^2+4}0\frac{1}{s+1}$.

The first term of this can be rewritten as $\frac{3(s-1)+8}{(s-1)^2+4}$.

The transform ends up being $3e^t\cos 2t+4e^t\sin 2t-e^{-t}$.

1.4 Solving Initial Value Problems

Method of Laplace Transforms

To solve initial value problems:

- Take the Laplace transforms of both sides of the equation
- Use the properties of the Laplace transform and the initial conditions to obtain an equation for the Laplace transform of the solution and then solve this equation for the transform
- Determine the inverse Laplace transform of the solution by looking it up in a table or by using a suitable method (such as partial fractions) in combination with the table.

Example

Solve the initial value problem

$$y''-2y'+5y=-8e^{-t} \qquad y(0)=2, \qquad y'(0)=12$$

This is equal to $\mathcal{L}\{y''\}-2\mathcal{L}\{y'\}+5\mathcal{L}\{y\}=-8\mathcal{L}\{e^{-t}\}$.

This ends up being $s^2\mathcal{L}\{y\}-sy(0)-y'(0)-2[s\mathcal{L}\{y\}-y(0)]+5\mathcal{L}\{y\}=-8\frac{1}{s+1}$.

We know that $\mathcal{L}\{y\}=Y(s)$.

So $Y(s)[s^2-2s+5]-2s-12+4=\frac{-8}{s+1}$.

This is $Y(s)(s^2-2s+5)=2s+8-\frac{8}{s+1}$.

This ends up being $Y(s)=\frac{2s}{s^2-2s+5}+\frac{8}{s^2-2s+5}-\frac{8}{(s+1)(s^2-2s+5)}$.

Simplifying ends up getting $\frac{2s^2+10s}{(s+1)(s^2-2s+5)}$.

Doing partial fraction decomposition gives $\frac{3s+5}{s^2-2s+5}+\frac{-1}{s+1}=\frac{3(s-1)+8}{(s-1)^2+4}+\frac{-1}{s+1}$.

The Inverse Laplace of this is $3e^t\cos 2t+4e^t\sin 2t-e^{-t}$.

Exercise Solve the initial value problem

$$y''+4y'-5y=te^t \qquad y(0)=1 \qquad y'(0)=0$$

Example

Solve the initial value proiblem

$$w''(t) - 2w'(t) + 5w(t) = -8e^{\pi - t} \qquad w(\pi) = 2 \qquad w'(\pi) = 12$$

Let's introduce a new function $y(t) = w(t + \pi)$.

Replace t with $t + \pi$ in this equation and we get $w''(t + \pi) - 2w'(t + \pi) + 5w(t + \pi) = -8e^{\pi - (t+\pi)}$.

Substituting the derivatives gives $y''(t) - 2y'(t) + 5y(t) = -8e^{-t}$.

This basically comes out to $y = 3e^t \cos 2t + 4e^t \sin 2t - e^{-t}$.

Replacing everything with $t - \pi$ gives $3e^{t-\pi} \cos 2(t - \pi) + 4e^{t-\pi} \sin 2(t - \pi) - e^{-(t-\pi)} = y(t - \pi)$.

This gives $w(t) = 3e^{t-\pi} \cos 2t + 4e^{t-\pi} \sin 2t - e^{-(t-\pi)}$.

1.5 Transforms of Discontinuous Functions

Definition

The unit step function $u(t)$ is defined to by

$$u(t) := \begin{cases} 0, & t < 0, \\ 1, & 0 < t \end{cases}$$

Example

Graph $u(t), u(t - a)$, and $Mu(t - a)$.

The graph of $u(t)$ is just as given above.

The graph of $u(t - a)$ is just a horizontal shift.

The graph of $Mu(t - a)$ will just have the one with 1 multiplied by M

Definition

The rectangular window function $\prod_{a,b}(t)$ is defined by

$$\prod_{a,b}(t) := u(t - a) - u(t - b) = \begin{cases} 0, & t < a \\ 1, & a < t < b \\ 0, & b < t \end{cases}$$

Example

Write the function

$$f(t) = \begin{cases} 3 & t < 2 \\ 1 & 2 < t < 5 \\ t & 5 < t < 8 \\ t^2/10 & 8 < t \end{cases}$$

In terms of window and step functions.

This is $3\prod_{0,2}(t) + 1\prod_{2,5}(t) + t\prod_{5,8}(t) + \frac{t^2}{10}u(t - 8)$.

Also this can be written as $3u(t) - 2u(t-2) + (t-1)u(t-5) + (\frac{t^2}{10} - t)u(t-8)$.

$$\mathcal{L}\{u(t-a)\}(s) = \frac{e^{-as}}{s}$$

Theorem 1.8

Let $F(s) = \mathcal{L}\{f\}(s)$ exist for $s > \alpha \geq 0$. If a is a positive constant, then

$$\mathcal{L}\{f(t-a)u(t-a)\}(s) = e^{-\alpha s}F(s)$$

and, conversely, an inverse Laplace transform of $e^{-as}F(s)$ is given by

$$\mathcal{L}^{-1}\{e^{-asF(s)}\}(t) = f(t-a)u(t-a)$$

$$\mathcal{L}\{g(t)u(t-a)\}(s) = e^{-as}\mathcal{L}\{g(t+a)\}(s)$$

Example

Determine the Laplace transform of $t^2u(t-1)$.

$a =$ from here, and $g(t) = t^2$.

We take $\mathcal{L}\{g(t)u(t-1)\} = e^{-s} \cdot \mathcal{L}\{g(t+1)\}$.

Replacing $g(t)$ gives that $t^2 + 2t + 1$ for the inside, so the Answer ends up being $e^{-s} \cdot [\frac{2!}{s^3} + \frac{2}{s^2} + \frac{1}{s}]$.

Example

Determine $\mathcal{L}\{(\cos t)u(t-\pi)\}$.

This has $a = \pi$. So we can see that We are doing $e^{-\pi s}\mathcal{L}\{g(t+\pi)\}$.

$g(t) = \cos t$, so $g(t+\pi) = \cos(t+\pi) = \cos t \cos \pi - \sin t \sin \pi = -\cos t$.

So the Laplace is $e^{-\pi s} \cdot -1 \cdot \frac{s}{s^2+1}$.

Exercise Determine $\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2}\right\}$ and sketch its graph.

Example

The current I in an LC series circuit is governed by the initial value problem

$$I'' + 4I(t) = g(t) \qquad I(0) = 0 \qquad I'(0) = 0$$

where

$$g(t) = \begin{cases} 1 & 0 < t < 1 \\ -1 & \quad 1 < t < 2 \\ 0 & 2 < t \end{cases}$$

Determine the current as a function of time t.

$g(t) = 1\prod_{0,1} + -1\prod_{1,2} = 1[u(t-0) - u(t-1)] - 1[u(t-1) - u(t-2)]$. This is equal to $g(t) = 1u(t-0) - 2u(t-1) + u(t-2)$.

This simplifies to $1 - 2u(t-1) + u(t-2)$

The Laplace of the initial value problem is $s^2\mathcal{L}\{I\} - sI(0) - I'(0) + 4\mathcal{L}\{I\} = \mathcal{L}\{1 - 2u(t-1) + u(t-2)\}$

We end up getting $(s^2 + 4)\mathcal{L}\{I\} = \frac{1}{s} - \frac{2e^{-s}}{s} + \frac{e^{-2s}}{s}$.

We get that $\mathcal{L}\{I\} = \frac{1}{s(s^2+4)} - 2e^{-s}\left[\frac{1}{s(s^2+4)}\right] + e^{-2s}\left[\frac{1}{s(s^2+4)}\right]$.

Using partial fraction decomposition of $\frac{1}{s(s^2+4)}$ gives $\frac{1}{4} \cdot \frac{1}{s} + -\frac{1}{4} \cdot \frac{s}{s^2+4}$.

If we call what we got above to be $F(s)$, we get $F(s) - 2e^{-s}F(s) + e^{-2s}F(s)$.

The inverse of what we have is $I = \mathcal{L}^{-1}\{F(s)\} - 2\mathcal{L}^{-1}\{e^{-s}F(s)\} + \mathcal{L}^{-1}\{e^{-2s}F(s)\}$.

Doing Laplace stuff gives $I = \frac{1}{4} - \frac{1}{4}\cos 2t - 2\left[\frac{1}{4} - \frac{1}{4}\cos 2(t-1)\right]u(t-1) + \left[\frac{1}{4} - \frac{1}{4}\cos 2(t-2)\right]u(t-2)$.

1.6 Transforms of Periodic and Power Functions

Definition

A function $f(t)$ is said to be periodic of period T $(\neq 0)$ if

$$f(t+T) = f(t)$$

for all t in the domain of f.

To specify a periodic function, it is sufficient to give its values over one period.

The square wave function can be epxressed as

$$f(t) = \begin{cases} 1, & 0 < t < 1 \\ -1, & 1 < t < 2 \end{cases}$$

and $f(t)$ has period 2.

For convenience, we introduce a notation for a "windowed" version of a periodic function (using a rectangular window whose width is the period T)

$$f_T(t) := f(T)\prod_{0,T}(t) = f(t)[u(t) - u(t-T)] = \begin{cases} f(t), & 0 < t < T \\ 0, & \text{otherwise} \end{cases}$$

Theorem 1.9

If f has period T and is piecewise continuous on $[0, T]$, then the Laplace transform $F(s) = \int_0^\infty e^{-st} f(t)\mathrm{d}t$ and $F_T(s) = \int_0^T e^{-st} f(t)\mathrm{d}t$ are related by

$$F_T(s) = F(s)[1 - e^{-sT}]$$

or

$$F(s) = \frac{F_T(s)}{1 - e^{-st}}$$

Example

Determine $\mathcal{L}\{f\}$, where f is the square wave function.

The function of the step function gives

$$f_T(t) = 1 \prod_{0,1} + -1 \prod_{1,2} = u(t) - 2u(t-1) + u(t-2)$$

The Laplace of this gives $\frac{e^0}{s} - \frac{2e^{-s}}{s} + \frac{e^{-2s}}{s} = \frac{1-2e^{-s}+e^{-2s}}{s}$

$F(s)$ is just $\frac{F_T(s)}{1-e^{-2s}} = \frac{1-e^{-s}}{s(1+e^{-s})}$.

1.7 Convolution

Definition

Let $f(t)$ and $g(t)$ be piecewise continuous on $[0, \infty)$. The convolution of $f(t)$ and $g(t)$, denoted $f * g$, is defined by

$$(f * g)(t) := \int_0^t f(t-v)g(v)\mathrm{d}v$$

Example

Find the convolution of t and t^2.

Let $f(t) = t$ and $g(t) = t^2$

$t * t^2 = \int_0^t (t - v) \cdot v^2 \mathrm{d}v$

So let's integrate. We get $\frac{tv^3}{3} - \frac{v^4}{4}$. Putting in the bounds gives $\frac{t^4}{12}$.

Theorem 1.10

Let $f(t), g(t)$, and $h(t)$ be piecewise continuous on $[0, \infty)$. Then

- $f * g = g * f$
- $f * (g + h) = (f * g) + (f * h)$
- $(f * g) * h = f * (g * h)$
- $f * 0 = 0$

Theorem 1.11

Let $f(t)$ and $g(t)$ be piecewise continuous on $[0, \infty)$ and of exponential order α and set $F(s) = \{f\}(s)$ and $G(s) = \mathcal{L}\{g\}(s)$. Then

$$\mathcal{L}\{f * g\}(s) = F(s)G(s)$$

or, equivalently,

$$\mathcal{L}^{-1}\{F(s)G(s)\}(t) = (f * g)(t)$$

Example

Use the convolution theorem to solve the initial value problem

$$y'' + y = g(t) \qquad y(0) = 0 \qquad y'(0) = 0$$

where $g(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order.

We can get that $\mathcal{L}\{y''\} + \mathcal{L}\{y\} = G(s)$ from the problem.

Doing the Laplace transform gives $s^2Y(s) - sy(0) - y'(0) + Y(s) = G(s)$.

This simplifies to $(s^2 + 1)Y(s) = G(s)$.

So $Y(s) = \frac{1}{s^2+1} \cdot G(s)$.

Taking the Laplace transform of both sides gives us $y(t) = \mathcal{L}\{\frac{1}{s^2+1}G(s)\}$.

The right side is just $\sin t * g(t)$.

We know that $y(t) = \int_0^t \sin(t - v)g(v)\mathrm{d}v$ from this.

Example

Use the convolution theorem to find $\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\}$.

From the Convolution Theorem, we find that $\mathcal{L}\{F(s)G(s)\} = f(t) * g(t)$.

From that definition, the laplace is $\sin t * \sin t$.

This is $\int_0^t \sin(t - v) \cdot \sin v\mathrm{d}v$.

Note that $\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$.

So applying this, we get that $\frac{1}{2}\int_0^t \cos[t - v - v] - \cos[t - v + v]\mathrm{d}v$.

This is equal to $\frac{1}{2}\int \cos[-(2v - t)] - \cos t\mathrm{d}v$.

Remember that $\cos(-A) = \cos A$.

So we end up getting $\frac{1}{2}\int \cos(2v - t) - \cos t\mathrm{d}v$.

Integrating gives $\frac{1}{2}[\frac{1}{2}\sin(2v - t) - v\cos t]$ from 0 to t.

Simplifying this gives you $\frac{\sin t - t\cos t}{2}$

Example

Solve the integro-differential equation

$$y'(t) = 1 - \int_0^t y(t - v)e^{-2v}\mathrm{d}v \qquad y(0) = 1$$

The integral in the expression is just a convolution.

The integral is $y * e^{-2t}$.

The Laplace transform of both sides results in $\mathcal{L}\{y'(t)\} = \mathcal{L}\{1\} - \mathcal{L}\{y(t) * e^{-2t}\}$.

So this is $sY(s) - y(0) = \frac{1}{s} - \mathcal{L}\{y(t)\} \cdot \mathcal{L}\{e^{-2t}\}$.

This is $sY(s) - 1 = \frac{1}{s} - Y(s) \cdot \frac{1}{s+2}$.

$(s + \frac{1}{s+2})Y(s) = 1 + \frac{1}{s}$.

We end up getting $\frac{s^2+2s+1}{s+2}Y(s) = 1 + \frac{1}{s}$.

Factoring and solving for $Y(s)$ gives $\frac{s+2}{(s+1)^2} \cdot \frac{s+1}{s}$.

This gives us $\frac{s+2}{s(s+1)}$.

Doing the partial fraction decomposition gives us $2 = A$ and $1 = -B$.

So we end up getting $\frac{2}{s} - \frac{1}{s+1}$. Taking the inverse laplace transform of both sides gives us $2 - e^{-t}$.

1.8 Impulses and the Dirac Delta Function

Definition

The Dirac delta function $\delta(t)$ is characterized by the following two properties:

$$\delta(t) = \begin{cases} 0, & t \neq 0, \text{“infinite”} \quad t = 0 \end{cases}$$

and

$$\int_{-\infty}^{\infty} f(t)\delta(t)\mathrm{d}t = f(0)$$

for any function $f(t)$ that is continuous on an open interval containing $t = 0$.

By shifting the argument of $\delta(t)$, we have $\delta(t-a) = 0.t \neq a$, and

$$\int_{-\infty}^{\infty} f(T)\delta(t-a)\mathrm{d}t = f(a)$$

for any function $f(t)$ that is continuous on an interval containing $t = a$.

When $t_0 = 0$, we derive from the limiting properties of the $\mathcal{F}_n$'s of a “function” δ that satisfies the first equation of this topic and the integral condition

$$\int_{-\infty}^{\infty} \delta(t)\mathrm{d}t = 1$$

The Laplace transform of the Dirac Delta function can be equickly derived from the property given above from shifting the argeumtn. Since $\delta(t-a) = 0$ for $t \neq a$, then setting $f(t) = e^{-st}$ in that function, we find for $a \geq 0$

$$\int_{0}^{\infty} \delta(t-a)\mathrm{d}t = \int_{-\infty}^{\infty} e^{-st}\delta(t-a)\mathrm{d}t = e^{-as}$$

Thus, for $a \geq 0$,

$$\mathcal{L}\{\delta(t-a)\}(s) = e^{-as}$$

Example

Use the Laplace transform to solve the initial value-value problem

$$y' + y = \delta(t-1), \qquad y(0) = 2$$

Taking the Laplace of both sides gives $sY(s) - y(0) + Y(s) = e^{-s}$.

Now we see that $Y(s) = \frac{1}{s+1}e^{-s} + \frac{2}{s+1}$.

This becomes $e^{-(t-1)}u(t-1) + 2e^{-t}$.

To write this as a piecewise function we can write this as $y(t) = \begin{cases} 2e^{-t} & 0 < t < 1 \\ e^{-t-1} + 2e^{-t} & t > 1 \end{cases}$.

Example

A mass attached to a spring is released from rest 1 m below the equilibrium position for the mass-spring system and begins to vibrate. After π seconds, the mass is struck by a hammer exerting an impulse on the mass. The system is governed by the symbolic initial value problem

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 9x = 3\delta(t-\pi); \qquad x(0) = 1, \qquad \frac{\mathrm{d}x}{\mathrm{d}t}(0) = 0$$

where $x(t)$ denotes the displacement from equilibrium at time t. Determine $x(t)$.

Doing the Laplace of the problem gives $s^2x(s) - s + 9x(s) = 3e^{-\pi s}$.

So we have $x(s) = \frac{s}{s^2+9} + \frac{3}{s^2+9}e^{-\pi s}$.

From this the inverse Laplace is $\cos(3t) + -\sin(3t)u(t-\pi)$.

1.9 Solving Linear Systems with Laplace Transforms

Example

Solve the initial value problem

$$\begin{aligned} x'(t) - 2y(t) &= 4t \qquad & x(0) = 4 \\ y'(t) + 2y(t) - 4x(t) &= -4t - 2 \qquad & y(0) = -5 \end{aligned}$$

Doing the Laplace of everything gives $sX(s) - x(0) - 2Y(s) = 4 \cdot \frac{1}{s^2}$ for the top equation and $sY(s) - y(0) + 2Y(s) - 4X(s) = -4 \cdot \frac{1}{s^2} - 2 \cdot \frac{1}{s}$ for the second equation.

After substituting we get

$$\begin{aligned} sX(s) - 2Y(S) &= \frac{4}{s^2} + 4 \\ -4X(s)(s+2)Y(s) &= -\frac{4}{s^2} - \frac{2}{s} - 5 \end{aligned}$$

By eliminating y, we get $X(s) = \frac{4s-2}{(s^2+2s-8)} = \frac{4s-2}{(s+4)(s-2)}$.

This is equivalent to $\frac{3}{s+4} + \frac{1}{s-2}$.

This gives us $x(t) = 3e^{-4t} + e^{2t}$.

We know from the problem that $y(t) = \frac{x'(t)-4t}{2}$.

So substituting values gives us $y(t) = \frac{1}{2}[-12e^{-4t} + 2e^{2t}] - 2t = -6e^{-4t} + e^{2t} - 2t$.

Example

Solve the initial value problem

$$x_1'' + 10x_1 - 4x_2 = 0$$
$$-4x_1 + x_2'' + 4x_2 = 0$$

subject to $x_1(0) = 0, \qquad x_1'(0) = 1, \qquad x_2(0) = 0, \qquad x_2'(0) = -1.$

The top equation's laplace transformation is $s_2x_1(s) - sx_1(0) - x_1'(0) + 10x_1(s) - 4x_2(s) = 0$.

The bottom equation becomes $-4x_1(s) + s^2x_2(s) - sx_2(0) - x_2'(0) + 4x_2(s) = 0$.

Solving the system of equations for $x_2(s)$ gives us $\frac{-s^2-6}{(s^2+12)(s^2+2)} = \frac{-2/5}{s^2+2} + \frac{-3/5}{s^2+12}$.

The Laplace gives $x_2(t) = -\frac{\sqrt{2}}{5}\sin(\sqrt{2}t) - \frac{\sqrt{3}}{10}\sin(2\sqrt{3}t)$.

Doing the derivatives gives us $x_1 = -\frac{\sqrt{2}}{10}\sin(\sqrt{2}t) + \frac{\sqrt{3}}{5}\sin(2\sqrt{3}t)$.