1 Matrix Methods for Linear Systems

1.1 Introduction to Matrix Methods

The product of a matrix and a column vector is defined to be the collection of dot products of the rows of the matrix with the vector, arranged as a column vector:

$$\begin{bmatrix} \text{row \# 1} \\ \text{row \# 2} \\ \vdots \\ \text{row \# 3} \end{bmatrix} \begin{bmatrix} v \end{bmatrix} = \begin{bmatrix} \text{row \# 1} \cdot v \\ \text{row \# 2} \cdot v \\ \vdots \\ \text{row \# 3} \cdot v \end{bmatrix}$$

where the vector v has n components; the dot product of two n-dimensional vectors is computed in the obvious way:

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \cdot \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$$

Example

Express the system as a matrix equation.

$$x_1' = 2x_1 + t^2 x_2 + (4t + e^t)x_4$$
$$x_2' = (\sin t)x_2 + (\cos t)x_3$$
$$x_3' = x_1 + x_2 + x_3 + x_4$$
$$x_4' = 0$$

This is simply written as

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{bmatrix} = \begin{bmatrix} 2 & t^2 & 0 & (4t + e^t) \\ 0 & \sin t & \cos t & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

In general, if a system or differential equation is expresses as

$$x_1' = a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n$$
$$x_2' = a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n$$
$$\vdots$$
$$x_n' = a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n$$

it is said to be a linear homogeneous system in normal form. The matrix form of such a system is

$$\mathbf{X}' = \mathbf{A}\mathbf{X}$$

where $\mathbf{A}$ is the coefficient matrix

$$\mathbf{A} = \mathbf{A}(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \ldots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \ldots & a_{2n}(t) \\ \vdots & \vdots & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \ldots & a_{nn}(t) \end{bmatrix}$$

and $\mathbf{X}$ is the solution vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Example

Express the differential equation for the undamped, unforced mass-spring oscillator

$$my'' + ky = 0$$

as an equivalent system of first-order equations in normal form, expressed in matrix notation.

We have that $y' = v$ and $v' = -\frac{k}{m}y$.

So we can write this as $\begin{bmatrix} y \\ v \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -k/m & 0 \end{bmatrix} \begin{bmatrix} y \\ v \end{bmatrix}$.

We can write this as $a_n x'_n + a_{n-1} x_n + \cdots + a_1 x_2 + a_0 x_1 = 0$.

Which can be rewritten as $x'_n = -\frac{a_0}{a_n} x_1 - \frac{a_1}{a_n} x_2 - \cdots - \frac{a_{n-1}}{a_n} x_n$.

Using this can make it easy to get to matrix notation

Example

A coupled mass-spring oscillator is governed by the system

$$2\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + 6x - 2y = 0$$
$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 2y - 2x = 0$$

Let $x_1 = x$, $x_2 = x'$, $x_3 = y$, $x_4 = y'$.

This gives us $x'_1 = x_2$, $x'_2 = -3x_1 + x_3$, $x'_3 = x_4$, $x'_4 = x_1 - 2x_3$.

So the matrix form can be easily answered from that.

1.2   Review 1: Linear Algebraic Equations

A set of equations of the form

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

(where the a_{ij}'s and b_i's are given constants) is called a linear system of n algebraic equations in the n unknowns $x_1, x_2, \ldots x_n$.

The Gauss-Jordan elimination algorithm uses elimination to uncouple the system making the values of the unknowns apparent.

Example

Solve the system

$$2x_1 + 6x_2 + 8x_3 = 16$$
$$4x_1 + 15x_2 + 19x_3 = 38$$
$$2x_1 \qquad + 3x_3 = 6$$

Solving the coefficient matrix gives you $(0, 0, 2) = (x_1, x_2, x_3)$.

Exercise Solve the system

$$
\begin{aligned}
x_1 + 2x_2 + 4x_3 + x_4 &= 0 \\
-x_1 - 2x_2 - 2x_3 \quad\ \ &= 1 \\
-2x_1 - 4x_2 - 8x_3 + 2x_4 &= 4 \\
x_1 + 4x_2 + 2x_3 \quad\ \ &= -3
\end{aligned}
$$

Example

Solve the system

$$
\begin{aligned}
2x_1 + 4x_2 + x_3 &= 8 \\
2x_1 + 4x_2 \quad\ \ &= 6 \\
-4x_1 - 8x_2 + x_3 &= -10
\end{aligned}
$$

We will end up getting $x_1 + 2x_2 = 3$ and $x_3 = 2$, and x_2 has infinite solutions, and is called a free variable.

So $x_2 = t, x_1 = -2t + 3$, and $x_3 = 2$.

Exercise Solve the system

$$
\begin{aligned}
x_1 - x_2 + 2x_3 + 2x_4 &= 0 \\
2x_1 - 2x_2 + 4x_3 + 3x_4 &= 1 \\
3x_1 - 3x_2 + 6x_3 + 9x_4 &= -3 \\
4x_1 - 4x_2 + 8x_3 + 8x_4 &= 0
\end{aligned}
$$

1.3   Review 2: Matrices and Vectors

A matrix is a rectangular array of numbers arranged in rows and columns. An $m \times n$ matrix, that is, a matrix with m rows and n columns is usually denoted by

$$
A := \begin{bmatrix}
a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\
a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\
\vdots & \vdots & \vdots & \dots & \vdots \\
a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn}
\end{bmatrix}
$$

Where the element in the ith row and jth column is a_{ij}. The notation $[a_{ij}]$ is used to designate A.

A square matrix has the same number of rows and columns. A diagonal matrix is a square matrix with only zero entries off the main diagonal. A column matrix, or vector, is an $n \times 1$ matrix. An $m \times n$ matrix whose entries are all zero is called a zero matrix. Matrices are denoted by boldfaces capital letters and vectors by boldfaced lower case letters.

The sum of two $m \times n$ matrices is given by

$$
\mathbf{A} + \mathbf{B} = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]
$$

To multiply a matrix by a scalar (number), multiply each element in the matrix by that number:

$$
r\mathbf{A} = r[a_{ij}] = [ra_{ij}]
$$

The notation $-\mathbf{A}$ stands for $(-1)\mathbf{A}$.

Properties of Matrix Addition and Scalar Multiplication

- $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$

- $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$
- $\mathbf{A} + \mathbf{0} = \mathbf{A}$
- $\mathbf{A} + (-\mathbf{A}) = \mathbf{0}$
- $r(\mathbf{A} + \mathbf{B}) = r\mathbf{A} + r\mathbf{B}$
- $(r + s)\mathbf{A} = r\mathbf{A} + s\mathbf{A}$
- $r(s\mathbf{A}) = (rs)\mathbf{A} = s(r\mathbf{A})$

Exercise Perform the indicated operation: $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Exercise Perform the indicated operation: $3 \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

The product of a matrix A and a column vector x is the column vector composed of dot products of the rows of A with x. AB is only defined when the number of columns of A matches the number of rows of B.

Exercise Perform the indicated operation: $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

Exercise Perform the indicated operation: $\begin{bmatrix} 1 & 0 & 1 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & x \\ -1 & -1 & y \\ 4 & 1 & z \end{bmatrix}$

Properties of Matrix Multiplication

- $(\mathbf{A}\mathbf{B})\mathbf{C} = \mathbf{A}(\mathbf{B}\mathbf{C})$
- $(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{A}\mathbf{C} + \mathbf{B}\mathbf{C}$
- $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{A}\mathbf{B} + \mathbf{A}\mathbf{C}$
- $(r\mathbf{A})\mathbf{B} = r(\mathbf{A}[B]) = \mathbf{A}(r\mathbf{B})$

Let A be an $m \times n$ matrix and let x and y be $n \times 1$ vectors. Then Ax is an $m \times 1$ vector so we can think of multiplication by A as defining an operator that maps $n \times 1$ vectors into $m \times 1$ vectors. Multiplication by A defines a linear operator since $\mathbf{A}(\mathbf{x} + \mathbf{y}) = \mathbf{A}\mathbf{x} + \mathbf{A}\mathbf{y}$ and $\mathbf{A}(\mathbf{r}\mathbf{x}) = \mathbf{r}\mathbf{A}\mathbf{x}$.

Examples of linear operations are:

1. Stretching or contracting the components of a vector by constant factors
2. rotating a vector through some angle about a fixed axis
3. reflecting a vector in a plane mirror

We express the linear system

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

In matrix notation as $\mathbf{A}\mathbf{x} = \mathbf{b}$ where $\mathbf{A}$ is the coefficient matrix, $\mathbf{x}$ is the vector of unknowns, and $\mathbf{B}$ is the vector of constants occurring on the right-hand side:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

If $b = 0$, the system $Ax = b$ is said to be homogeneous.

The matrix obtained from A by interchaing its rows and columns is called the transpose of A and is denoted by A^T.

Exercise Find A^T if $\mathbf{A} = \begin{bmatrix} 1 & 2 & 6 \\ -1 & 2 & -1 \end{bmatrix}$.

There is a multiplicative identity in matrix algebra, namely, a square diagonal matrix $\mathbf{I}$ with ones down the main diagonal. Multiplying $\mathbf{I}$ on the right or left by any other matrix (with compatible dimensions) reproduces the latter matrix.

Exercise Demonstrate the identity property for $\mathbf{A} = \begin{bmatrix} 1 & 2 & 6 \\ -1 & 2 & -1 \end{bmatrix}$

Some square matrices $\mathbf{A}$ can be paired with other square matrices $\mathbf{B}$ having the property that $\mathbf{BA} = \mathbf{I}$. When this happens,

1. $\mathbf{B}$ is the unique matrix satisfying $\mathbf{BA} = \mathbf{I}$ and

2. $\mathbf{B}$ also satisfies $\mathbf{AB} = \mathbf{I}$.

In such a case, $\mathbf{B}$ is the inverse of $\mathbf{A}$ and write $\mathbf{B} = \mathbf{A}^{-1}$. A matrix that has no inverse is said to be singular.

When an inverse for the coefficient matrix A in a system of linear equations $A\mathbf{x} = \mathbf{b}$, the solution can be calculated directly by $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$.

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. The matrix A is invertible if and only if $ad - bc \neq 0$. If $ad - bc = 0$, then A does not have a multiplicative inverse.

Exercise If $A = \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}$, solve $Ax = b$ where $b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Finding the Inverse of a Matrix. Row operations include

- Interchanging two rows of the matrix

- Multiplying a row of the matrix by a nonzero scalar

- Adding a scalar multiple of one row of the matrix to another row

If the $n \times n$ matrix $\mathbf{A}$ has an inverse, $\mathbf{A}^{-1}$ can be found by performing row operations on the $n \times 2n$ matrix $[\mathbf{A}|\mathbf{I}]$ obtained by writing $\mathbf{A}$ and $\mathbf{I}$ side by side. If the procedure produces a new matrix in the form $[\mathbf{I}|\mathbf{B}]$, then $\mathbf{A}^{-1} = \mathbf{B}$.

Exercise Find the inverse of $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix}$

For a 2×2 matrix $\mathbf{A}$, the determinant of $\mathbf{A}$, denoted by $\det \mathbf{A}$ or $|\mathbf{A}|$, is defined by

$$\det \mathbf{A} := \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_21$$

The determinant of a 3×3 matrix A can be defined in terms of its cofactor expansion about the first row

$$\det \mathbf{A} := \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Exercise Find the determinant $\begin{vmatrix} 2 & 4 \\ 1 & 1 \end{vmatrix}$

Exercise Find the determinant $\begin{vmatrix} 1 & 2 & 1 \\ 0 & 3 & 5 \\ 2 & 1 & -1 \end{vmatrix}$

Theorem 1.1

Let $\mathbf{A}$ be an $n \times n$ matrix. The following statements are equivalent:

- **A** is singular (does not have an inverse).

- The determinant of **A** is zero.

- **Ax = 0** has nontrivial solutions ($\mathbf{x} \neq 0$)

- The columns (rows) of **A** form a linearly dependent set.

The columns of A are linearly dependent means there exist scalars $c_1, \ldots, c_n$ not all zero, such that

$$c_1 a_1 + c_2 a_2 + \cdots + c_n a_n = 0$$

where a_j is the vector forming the jth column of **A**.

If **A** is a singular square matrix $(\det \mathbf{A} = 0)$ then **Ax = 0** has infinitely many solutions.

If **A** is singular, **Ax = b** either has no solutions or it has infinitely many of the form

$$x = x_p + x_h$$

where x_p is a particular solution to **Ax = b** and x_h is any of the infinite solutions to **Ax = 0**, the homogeneous system.

Exercise In a previous section, we saw the system

$$\begin{bmatrix} 2 & 4 & 1 \\ 2 & 4 & 0 \\ -4 & -8 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \\ -10 \end{bmatrix}$$

has solutions $x_1 = 3 - 2s, x_2 = s, x_3 = 2$ where $-\infty < s < \infty$.

1. Write the solution in matrix notation and identify x_p and x_h.

2. Verify $\det \mathbf{A} = 0$

3. Give the identity that exhibits the linear dependence of the columns of **A**.

If **A** is a nonsingular square matrix (i.e., **A** has an inverse and $\det \mathbf{A} \neq 0$), then **Ax = 0** has **x = 0** as its only solution and the unique solution to **Ax = b** is $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$.

If the entries $a_{ij}(t)$ in a matrix $\mathbf{A}(t)$ are functions of the variable t, then $\mathbf{A}(t)$ is a matrix function of t. Similarly, if the entries $x_i(t)$ of a vector $\mathbf{x}(t)$ are functions of t, then $\mathbf{x}(t)$ is a vector function of t.

A matrix $\mathbf{A}(t)$ is said to be continuous at t_0 if each entry $a_{ij}(t)$ is continuous at t_0. $\mathbf{A}(t)$ is differentiable at t_0 if each entry $a_{ij}(t)$ is differentiable at t_0.

$$\frac{d\mathbf{A}}{dt}(t_0) = \mathbf{A}'(t_0) := [a'_{ij}(t_0)]$$

$$\int_a^b \mathbf{A}(t)dt := \left[\int_a^b a_{ij}(t)dt \right]$$

Exercise Let $\mathbf{A}(t) = \begin{bmatrix} t^2 + 1 & \cos t \\ e^t & 1 \end{bmatrix}$

1. Find: $\mathbf{A}'(t)$

2. Find: $\int_0^1 \mathbf{A}(t)dt$

Differentiation Formulas for Matrix Functions:

- $\frac{d}{dt}(\mathbf{CA}) = \mathbf{C}\frac{d\mathbf{A}}{dt}$ (**C** a constant matrix)

- $\frac{d}{dt}(\mathbf{A} + \mathbf{B}) = \frac{d\mathbf{A}}{dt} + \frac{d\mathbf{B}}{dt}$

- $\frac{d}{dt}(\mathbf{AB}) = \mathbf{A}\frac{d\mathbf{B}}{dt} + \frac{d\mathbf{A}}{dt}\mathbf{B}$

Exercise Show that $\mathbf{x}(t) = \begin{bmatrix} \cos \omega t \\ \sin \omega t \end{bmatrix}$ is a solution of the matrix differential equation $\mathbf{x}' = \mathbf{Ax}$ where $\mathbf{A} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}$

1.4 Linear Systems in Normal Form

A system of n linear differential equations is in normal form if it is expressed as

$$\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t)$$

where $x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$, $f(t) = \begin{bmatrix} f_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$ and $A(t) = \begin{bmatrix} a_{ij}(t) \end{bmatrix}$ is an $n \times n$ matrix.

A system is called homogeneous when $\mathbf{f}(t) = 0$; otherwise it is called nonhomogeneous. When the elements of A are all constants, the system is said to have constant coefficients.

The initial value problem for the normal system is the problem of finding a differentiable vector function $\mathbf{x}(t)$ that satisfies the system on an interval I and also satisfies the initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$ where t_0 is a given point of I and $\mathbf{x}_0$ is a given vector.

Theorem 1.2

Suppose $\mathbf{A}(t)$ and $\mathbf{f}(t)$ are continuous on an open interval I that contains the point t_0. Then, for any choice of the initial vector $\mathbf{x}_0$, there exists a unique solution $\mathbf{x}(t)$ on the whole interval I to the initial value problem

$$\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t), \qquad \mathbf{x}(t_0) = \mathbf{x}_0$$

Definition

The m vector functions $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are said to be linearly dependent on an interval I if there exist constants $c_1, \ldots, c_n$, not all zero, such that

$$c_1\mathbf{x}_1(t) + \cdots + c_n\mathbf{x}_n(t) = \mathbf{0}$$

for all t in I. If the vectors are not linearly dependent, they are said to be linearly independent on I.

Example

Show that the vector functions $x_1(t) = \begin{bmatrix} e^t \\ 0 \\ e^t \end{bmatrix}$, $x_2(t) = \begin{bmatrix} 3e^t \\ 0 \\ 3e^t \end{bmatrix}$, and $x_3 = \begin{bmatrix} t \\ 1 \\ 0 \end{bmatrix}$ are linearly dependent on $(-\infty, \infty)$.

They are dependent because let $c_1 = -3, c_2 = 1, c_3 = 0$ to get $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Example

Show that the vector functions $x_1(t) = \begin{bmatrix} e^{2t} \\ 0 \\ e^{2t} \end{bmatrix}$, $x_2(t) = \begin{bmatrix} e^{2t} \\ e^{2t} \\ -e^{2t} \end{bmatrix}$, and $x_3(t) = \begin{bmatrix} e^t \\ 2e^t \\ e^t \end{bmatrix}$ are linearly independent on $(-\infty, \infty)$.

The only way we can get $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is if the three constants are 0.

A set of vector functions $x_1(t), x_2(t), \ldots, x_n(t)$ each having n components is linearly independent on an interval I if we can find one point t_0 in I where the determinant $\det \begin{bmatrix} x_1(t_0) & \ldots & x_n(t_0) \end{bmatrix}$ is not zero. We call this determrinant the Wronksian. (This was previously defined)

> **Theorem 1.3**
>
> Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ be n linearly independent solutions to the homogeneous system
>
> $$\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t)$$
>
> on the interval I, where $\mathbf{A}(t)$ is an $n \times n$ matrix function continuous on I. Then every solution to the above on I can be expressed in the form
>
> $$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + \cdots + c_n\mathbf{x}_n(t)$$

A set of solutions $\{x_1, \ldots x_n\}$ that are linearly independent on I is called a fundamental solution set for the homogeneous system on I. The linear combination written with arbitrary constants, is referred to as the general solution to the homogeneous system.

If we take the vectors in a fundamental solution set and let them form the columns of a matrix $\mathbf{X}(t)$.

$$\mathbf{X}(t) = \begin{bmatrix} \mathbf{x}_1(t) & \mathbf{x}_2(t) & \ldots & \mathbf{x}_n(t) \end{bmatrix} = \begin{bmatrix} x_{1,1}(t) & x_{1,2}(t) & \ldots & x_{1,n}(t) \\ x_{2,1}(t) & x_{2,2}(t) & \ldots & x_{2,n}(t) \\ \vdots & \vdots & & \vdots \\ x_{n,1}(t) & x_{n,2}(t) & \ldots & x_{n,n}(t) \end{bmatrix}$$

Then the matrix $\mathbf{X}(t)$ is called a fundamental matrix for the homogeneous system.

> **Example**
>
> Verify that the set $S = \left\{ \begin{bmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix}, \begin{bmatrix} -e^{-t} \\ 0 \\ e^{-t} \end{bmatrix}, \begin{bmatrix} 0 \\ e^{-t} \\ -e^{-t} \end{bmatrix} \right\}$ is a fundamental solution set for the system
>
> $$\mathbf{x}'(t) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$
>
> $textbf{x}(t)$ on the interval $(-\infty, \infty)$ and find a fundamental matrix for the system. Determine a general solution for the system.
>
> Testing the three matrices in the system gives the correct resulting vector, and finding the Wronksian shows us that the columns are linearly independent, so the general solution is
>
> $$x = c_1 \begin{bmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix} + c_2 \begin{bmatrix} -e^{-t} \\ 0 \\ e^{-t} \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ e^{-t} \\ -e^{-t} \end{bmatrix}$$

> **Theorem 1.4**
>
> Let $\mathbf{x}_p$ be a particular solution to the nonhomogeneous system
>
> $$\mathbf{x}'(t) = \mathbf{A}(t) + \mathbf{f}(t)$$
>
> on the interval I, and let $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ be a fundamental solution set on I for the corresponding homogeneous system $\mathbf{x}(t) = \mathbf{A}(t)\mathbf{x}(t)$. Then every solution to the nonhomogeneous system on I can be expressed in the form
>
> $$\mathbf{x}(t) = \mathbf{x}_p(t) + c_1\mathbf{x}_1(t) + \cdots + c_n\mathbf{x}_n(t)$$
>
> where $c_1, \ldots, c_n$ are constants.

Approach to Solving Normal Systems:

1. To determine a general solution to the $n \times n$ homogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x}$:

   - Find a fundamental solution set $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ that consists of n linearly independent solutions to the homogeneous system.

- Form the linear combination

$$\mathbf{x} = \mathbf{X}\mathbf{c} = c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n$$

where $\mathbf{c} = \text{col}(c_1, \ldots, c_n)$ is any constant vector and $\mathbf{X} = \begin{bmatrix} \mathbf{x}_1 & \ldots & \mathbf{x}_n \end{bmatrix}$ is the fundamental matrix, to obtain a general solution.

2. To determine a general solution to the nonhomogeneous system $\mathbf{x}'(t) = \mathbf{A}\mathbf{x} + \mathbf{f}$:

- Find a particular solution $\mathbf{x}_p$ to the nonhomogeneous system.

- Form the sum of the particular solution and the general solution $\mathbf{X}\mathbf{c} = c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n$ to the corresponding homogeneous system in part 1,

$$\mathbf{x} = \mathbf{x}_p + \mathbf{X}\mathbf{c} = \mathbf{x}_p + c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n$$

to obtain a general solution to the given system.

1.5 Homogeneous Linear Systems with Constant Coefficients

We now define a procedure for obtaining a general solution for the homogeneous system

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$$

Definition

Let $\mathbf{A} = \begin{bmatrix} a_{ij} \end{bmatrix}$ be an $n \times n$ constant matrix. The eigenvalues of $\mathbf{A}$ are those (real or complex) numbers r for which $(\mathbf{A} - r\mathbf{I}) = \mathbf{0}$ has at least one nontrivla solution $\mathbf{u}$. The corresponding nontrivial solutions $\mathbf{u}$ are called the eigenvectors of $\mathbf{A}$ associated with r.

Finding eigenvalues of a matrix A is equivalent to finding the zeroes of the polynomial $p(r) = \det(A - rI)$. The equation $\det(A - rI) = 0$ is called the characteristic equation of A and $p(r)$ is the characteristic polynomial of A.

Example

Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$

First find the characteristic equation, this is the determinant of $\begin{bmatrix} 2-r & -3 \\ 1 & -2-r \end{bmatrix}$. So the characteristic equation is $r^2 - 1$ and the eigenvalues are $r = -1$ and $r = 1$.

Now doing the procedure above, we find that for $r = -1$, the eigenvector is $u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and for $r = 1$, the eigenvector is $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

The collection of all eigenvectors associated with an eigenvalue forms a subspace when the zero vector is adjoined. These subspaces are called eigenspaces.

Exercise Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}$$

Theorem 1.5

Suppose the $n \times n$ constant matrix $\mathbf{A}$ has n linearly independent eigenvectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$. Let r_i be

the eigenvalue corresponding to $\mathbf{u}_i$. Then

$$\{e^{r_1 t}\mathbf{u}_1, e^{r_2 t}\mathbf{u}_2, \ldots, e^{r_n t}\mathbf{u}_n\}$$

is a fundamental solution set (and $\mathbf{X}(t) = \begin{bmatrix} e^{r_1 t}\mathbf{u}_1 & e^{r_2 t}\mathbf{u}_2 & \ldots & e^{r_n t}\mathbf{u}_n \end{bmatrix}$ is a fundamental matrix) on $(-\infty, \infty)$ for the homogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x}$. Consequently, a general solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is

$$\mathbf{x}(t) = c_1 e^{r_1 t}\mathbf{u}_1 + c_2 e^{r_2 t}\mathbf{u}_2 + \cdots + c_n e^{r_n t}\mathbf{u}_n$$

where $c_1, \ldots, c_n$ are arbitrary constants.

Example

Find a general solution of $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$, where $\mathbf{A} = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$

We get the eigenvalues as ± 1 from this. Therefore, when we find the the general solution, we can see that this is

$$x(t) = c_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}$$

, and this can be written in different ways, but this is a way to write the general solution.

Theorem 1.6

If $r_1, \ldots, r_m$ are distinct eigenvalues for the matrix $\mathbf{A}$ and $\mathbf{u}_i$ is an eigenvector associated with r_i, then $\mathbf{u}_1, \ldots, \mathbf{u}_m$ are linearly independent.

Corollary 1.7

If the $n \times n$ constant matrix $\mathbf{A}$ has n distinct eigenvalues $r_1, \ldots, r_n$ and $\mathbf{u}_i$ is an eigenvector associated with r_i, then

$$\{e^{r_1 t}\mathbf{u}_1, \ldots, e^{r_n t}\mathbf{u}_n\}$$

is a fundamental solution set for the homogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

Exercise Solve the initial value problem $\mathbf{x}'(T) = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix} \mathbf{x}(t)$ where $\mathbf{x}(0) = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$.

Definition

A real symmetric matrix $\mathbf{A}$ is a matrix with real entries that satisfies $\mathbf{A}^T = \mathbf{A}$.

If A is an $n \times n$ real symmetric matrix, there always exist n linearly independent eigenvectors.

Example

Find a general solution of $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$, where $\mathbf{A} = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$.

From this, we can find the eigenvalues are $r = 3$ and $r = -3$.

The eigenvectos for $r = 3$ are $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and for c_3 it is $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$, and the general solution can be found from this.

Second Solution:

Suppose r_1 is an eigenvalue of multiplicity two and that there is only one eigenvector associated with this value. A second solution can be found of the form

$$x_2 = Kte^{r_1 t} + Pe^{r_1 t}$$

where $\mathbf{K} = \begin{bmatrix} k_1 \\ \vdots \\ k_n \end{bmatrix}$ and $\mathbf{P} = \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix}$ satisfy $(A - r_1 I)K = 0$ and $(A - r_1 I)P = K$.

> **Example**
>
> Find the general solution of $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$, where $\mathbf{A} = \begin{bmatrix} 3 & -18 \\ 2 & -9 \end{bmatrix}$.
>
> The eigenvalue for this is $r = -3$ with multiplicity 2.
>
> If we find the first eigenvector we get $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$, and we can set this equal to k.
>
> Then we can use this to find the second eigenvector by doing $\begin{bmatrix} 6 & -18 \\ 2 & -6 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, and finding this eigenvector as $\begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$.
>
> This shows that $x_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} te^{-3t} + \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} e^{-3t}$.
>
> Therefore the general solution is $x = c_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{-3t}$ added on to c_2 times what x_2 is.

When the coefficient matrix A has only one eigenvalue associated with an eigenvalue r_1 of multiplicity three, we can find a second solution of the form

$$x_2 = Kte^{r_1 t} + Pe^{r_1 t}$$

and a third solution of the form

$$x_3 = K\frac{t^2}{2}e^{r_1 t} + Pte^{r_1 t} + Qe^{r_1 t}$$

where $\mathbf{K} = \begin{bmatrix} k_1 \\ \vdots \\ k_n \end{bmatrix}$, $\mathbf{P} = \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix}$, and $Q = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$ satfisfy

$$(A - r_1 I)K = 0$$
$$(A - r_1 I)P = K$$
$$(A - r_1 I)Q = P$$

Exercise Find the general solution $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$, where $\mathbf{A} = \begin{bmatrix} 2 & 1 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 2 \end{bmatrix}$.

1.6   Complex Eigenvalues

If the real matrix $\mathbf{A}$ has complex conjugate eigenvalues $\alpha \pm i\beta$ with corresponding eigenvectors $\mathbf{a} \pm i\mathbf{b}$, then two linearly independent real vector solutions to $\mathbf{x}'(t) = A\mathbf{x}(t)$ are

$$e^{\alpha t} \cos \beta t \mathbf{a} - e^{\alpha t} \sin \beta t \mathbf{b}$$

$$e^{\alpha t} \sin \beta t \mathbf{a} + e^{\alpha t} \cos \beta t \mathbf{b}$$

Example

Find a general solution of $\mathbf{x}'(t) = \begin{bmatrix} -1 & 2 \\ -1 & -3 \end{bmatrix} \mathbf{x}(t)$.

The eigenvalue for this is $-2 \pm i$.

The first vector gives you $e^{-2t}[\begin{bmatrix} 1 \\ -1 \end{bmatrix} \cos t - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \sin t]$ and the other gives $e^{-2t}[\begin{bmatrix} 1 \\ -1 \end{bmatrix} \sin t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cos t]$.

1.7   Nonhomogeneous Linear Systems

The Method of Undetermined Coefficients can be used to find a particular solution to the nonhomogeneous linear system

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{f}(t)$$

where A is an $n \times n$ constant matrix and the entries of $\mathbf{f}(t)$ are polynomials, exponential functions, sines and cosines, or finite sums and products of these functions.

Example

Find a general solution of $x'(t) = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} x(t) + t \begin{bmatrix} -9 \\ 0 \\ -18 \end{bmatrix}$.

The particular solution is $x_p = \begin{bmatrix} At + B \\ Ct + D \\ Et + F \end{bmatrix}$.

Solve the homogeneous equation now and it is $x_h = c_1 e^{3t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{-3t} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

We also can see now that $\begin{bmatrix} A \\ C \\ E \end{bmatrix} = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} At + B \\ Ct + D \\ Et + F \end{bmatrix} + \begin{bmatrix} -9t \\ 0 \\ -18t \end{bmatrix}$.

Multiplying the matrices gives $\begin{bmatrix} A \\ C \\ E \end{bmatrix} = \begin{bmatrix} At + B - 2Ct - 2D + 2Et + 2F - 9t \\ -2At - 2B + Ct + D + 2Et + 2F \\ 2At + 2B + 2Ct + 2D + Et + F - 18t \end{bmatrix}$.

We see that $A = t[A - 2C + 2E - 9] + [B - 2D + 2F]$

We have also $C = t[-2A + C + 2E] + [-2B + D + 2F]$.

We can also see $E = t[2A + 2C + e - 18] + [2B + 2D + F]$.

From these, we can write 6 equations to get the following matrix.

$$\begin{bmatrix} 1 & 0 & -2 & 0 & 2 & 0 & 9 \\ -1 & 1 & 0 & -2 & 0 & 2 & 0 \\ -2 & 0 & 1 & 0 & 2 & 0 & 0 \\ 0 & -2 & -1 & 1 & 0 & 2 & 0 \\ 2 & 0 & 2 & 0 & 1 & 0 & 18 \\ 0 & 2 & 0 & 2 & -1 & 1 & 0 \end{bmatrix}$$

Putting this in our calculator gives us the particular solution $x_p = \begin{bmatrix} 5t + 1 \\ 2t \\ 4t + 2 \end{bmatrix}$.

If $x_1, x_2, \ldots, x_n$ is a fundamental set of solutions of the homogeneous system $\mathbf{x}'(t) = A\mathbf{x}(t)$ on an interval

I, then it sgeneral solution on the interval is the linear combination $x = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$ or

$$\mathbf{X} = c_1 \begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{pmatrix} + c_2 \begin{pmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{n2} \end{pmatrix} + \cdots + c_n \begin{pmatrix} x_{1n} \\ x_{2n} \\ \vdots \\ x_{nn} \end{pmatrix} = \begin{pmatrix} c_1 x_{11} + c_2 x_{12} + \cdots + c_n x_{1n} \\ c_1 x_{21} + c_2 x_{22} + \cdots + c_n x_{2n} \\ \vdots \\ c_1 x_{n1} + c_2 x_{n2} + \cdots + c_n x_{nn} \end{pmatrix}$$

The solution can be written as $\mathbf{x}(t) = \mathbf{X}(t)\mathbf{C}$ where $\mathbf{C}$ is an $n \times 1$ column vector of arbitrary constants $c_1, c_2, \ldots, c_n$ and

$$\mathbf{X}(t) = \begin{bmatrix} \mathbf{x}_1(t) & \mathbf{x}_2(t) & \ldots & \mathbf{x}_n(t) \end{bmatrix} = \begin{bmatrix} x_{1,1}(t) & x_{1,2}(t) & \ldots & x_{1,n}(t) \\ x_{2,1}(t) & x_{2,2}(t) & \ldots & x_{2,n}(t) \\ \vdots & \vdots & & \vdots \\ x_{n,1}(t) & x_{n,2}(t) & \ldots & x_{n,n}(t) \end{bmatrix}$$

is the fundamental matrix of the system on ther interval.

Because a general solution to $\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t)$ is given by $\mathbf{x}(t) = \mathbf{X}(t)\mathbf{C}$ we seek a particular solution to the nonhomogeneous system $\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t)$ of the form $\mathbf{x}_p(t) = \mathbf{X}(t)\mathbf{v}(t)$ where $v(t) = \begin{bmatrix} v_1(t) \\ \vdots \\ v_n(t) \end{bmatrix}$ can be found by

$$\mathbf{v}(t) = \int \mathbf{X}^{-1}(t)\mathbf{f}(t)\mathrm{d}t$$

and

$$\mathbf{x}_p(t) = \mathbf{X}(t)\mathbf{v}(t) = \mathbf{X}(t)\int \mathbf{X}^{-1}(t)\mathbf{f}(t)\mathrm{d}t$$

Combining with the solution to the nonhomogeneous system gives the general solution

$$\mathbf{x}(t) = \mathbf{X}(t)C + \mathbf{X}(t)\int \mathbf{X}^{-1}(t)\mathbf{f}(t)\mathrm{d}t$$

Example

Find the solution to the initial value problem

$$\mathbf{x}'(t) = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} e^{2t} \\ 1 \end{bmatrix}, \qquad \mathbf{x}(0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Previously we found the homogeneous solution is $c_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-3t}$.

Now to find $v(t)$, remember we need to find $X(t)v(t)$ and $v(t) = \int X^{-1}(t)f(t)\mathrm{d}t$.

The inverse of $x(t)$ is $X^{-1} = \begin{bmatrix} \frac{1}{2}e^{-t} & -\frac{1}{2}e^{-t} \\ -\frac{1}{2}e^t & \frac{3}{2}e^t \end{bmatrix}$.

So $v(t) = \int \begin{bmatrix} \frac{1}{2}e^{-t} & -\frac{1}{2}e^{-t} \\ -\frac{1}{2}e^t & \frac{3}{2}e^t \end{bmatrix} \begin{bmatrix} e^{2t} \\ 1 \end{bmatrix} \mathrm{d}t$.

Integrating this out gives $v(t) = \begin{bmatrix} \frac{1}{2}e^t + \frac{1}{2}e^{-t} \\ -\frac{1}{2}e^{3t} + \frac{3}{2}e^t \end{bmatrix}$.

Now multiply this with $X^{-1}(t)$ to get $\begin{bmatrix} \frac{4}{3}e^{2t} + 3 \\ \frac{1}{3}e^{2t} + 2 \end{bmatrix}$ for x_p.

The general solution is $x = c_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + \begin{bmatrix} \frac{4}{3} \\ \frac{1}{3} \end{bmatrix} e^{2t} + \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

Now plugging in the initial conditions gives $c_1 = -\frac{3}{2}$ and $c_2 = -\frac{5}{6}$.

Plugging in everything gives $x = \begin{bmatrix} -\frac{9}{2}e^t - \frac{5}{6}e^{-t} + \frac{4}{3}e^{2t} + 3 \\ -\frac{3}{2}e^t - \frac{5}{6}e^{-t} + \frac{1}{3}e^{2t} + 2 \end{bmatrix}$.

$x = \begin{bmatrix} -\frac{9}{2}e^t - \frac{5}{6}e^{-t} + \frac{4}{3}e^{2t} + 3 \\ -\frac{3}{2}e^t - \frac{5}{6}e^{-t} + \frac{1}{3}e^{2t} + 2 \end{bmatrix}$