Differential Equations Notes

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1 Introduction to Differential Equations

1.1 Background

In a variety of subject areas, mathematical models are developed to aid in understanding. These models often yield an equation that contains derivatives of an unknown function. Such an equation is called a differential equation.

One example is free fall of a body. An object is released from a certain height above the ground and falls under the force of gravity. Newton's second law states that an object's mass times its acceleration equals the total force acting on it.

$$m\frac{\mathrm{d}^2h}{dt^2} = -mg$$

We have $h(t)$ as position, $\frac{\mathrm{d}h}{\mathrm{d}t}$ as velocity and $\frac{\mathrm{d}^2h}{\mathrm{d}t}$ as acceleration. The independent variable is t and the dependent variable is h.

$m\frac{\mathrm{d}^2h}{dt^2} = -mg$ is a differential equation and $h(t)$ is the unknown function that we are trying to find.

From this we have $\frac{\mathrm{d}^2h}{\mathrm{d}t^2} = -g$ and the integral of this is $\frac{\mathrm{d}h}{\mathrm{d}t} = -gt + C_1$. To find h we integrate again and we get $h(t) = -\frac{gt^2}{2} + C_1t + C_2$.

Another example is the decay of a radioactive substance. The rate of decay is proportional to the amount of radioactive substance present.

$$\frac{\mathrm{d}A}{\mathrm{d}t} = -kA, \qquad k > 0$$

where A is the unknown amount of radioactive substance present at time t and k Is the proportionality constant.

We are looking for $A(t)$ that satisfies this equation. We can solve this from $\frac{1}{A}\mathrm{d}A = k\mathrm{d}t$ and integrating both sides we get that $\ln|A| + C_1 = -kt + C_2$. We can rewrite this as $\ln|A| = -kt + C$. So, $e^{-kt+C} = A$.

So $A(t) = e^{-kt} + e^C$, so $A(t) = Ce^{-kt}$. Remember A is the dependent variable and t is the independent variable.

Notice that the solution of a differential equation is a function, not merely a number.

When a mathematical model involves the rate of change of one variable with respect to another, a differential equation is apt to appear.

Terminology

If an equation involves the derivative of one variable with respect to another, then the former is called a dependent variable and the latter an independent variable.

In $\frac{\mathrm{d}h}{\mathrm{d}t}$, h is dependent and t is independent.

A differential equation involving only ordinary derivatives with respect to a single independent variable is called an ordinary differential equation. A differential equation involving partial derivatives with respect to more than one independent variable is a partial differential equation.

For example we have $z = f(x, y) = 4x^2 + 5xy$, so $\frac{\partial z}{\partial x} = 8x + 5y$ and that is partial differentiation.

The order of a differential equation is the order of the highest-order derivatives present in the equation.

For example, $\frac{\mathrm{d}^2h}{dt^2} = -g$ has a order of 2.

A linear differential equation is one in which the dependent variable y and its derivatives appear in additive combinations of their first powers. A differential equation is linear if it has the format.

$$a_n(x)\frac{\mathrm{d}^n y}{dx^n} + a_{n-1}(x)\frac{\mathrm{d}^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = F(x)$$

$2x + 3y = 7$ is linear, $2x^2 + 5xy + 7y + 8y = 1$ is second-degree. Nothing can have a second degree for this to be linear.

You are just looking at the dependent variable and the derivatives and adding their powers.

If an ordinary differential equation is not linear, we call it nonlinear.

Example

For each differential equation, classify as ODE or PDE, linear or nonlinear, and indicate the dependent/independent variables and order.

(a) $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + a\frac{\mathrm{d}x}{\mathrm{d}t} + kx = 0$

Dependent is x, independent is t and the order is 2. This is an ODE and linear.

(b) $\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = x - 2y$

The dependent variable is u and the independent variables are x, y, so this is a PDE. The order is 1.

(c) $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + y^3 = 0$

The dependent variable is y, the independent variable is x, the order is 2 and this is an ODE and this is nonlinear.

(d) $t^3\frac{\mathrm{d}x}{\mathrm{d}t} = t^3 + x$

Dependent is x, independent is t, order is 1, this is an ODE. We can rewrite this as $t^3\frac{\mathrm{d}x}{\mathrm{d}t} - 1x = t^3$, and this matches the form of the linear equation so this is linear.

(e) $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - y\frac{\mathrm{d}y}{\mathrm{d}x} = \cos x$

The dependent is y, the independent is x, the order is 2 and this is an ODE and this is nonlinear because of $y\frac{\mathrm{d}y}{\mathrm{d}x}$.

1.2 Solutions and Initial Value Problems

An nth-order ordinary differential equation is an equality relating the independent variable to the nth derivative (and usually lower-order derivatives as well) of the dependent variable.

Example

Identify the order, independent and dependent variable.

(a) $x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} + y = x^3$. Independent: x, dependent: y, order: 2

(b) $\sqrt{1 - \left(\frac{\mathrm{d}^2y}{\mathrm{d}t^2}\right)} - y = 0$. Independent: t, dependent: y, order: 2

(c) $\frac{\mathrm{d}^4x}{\mathrm{d}t^4} = xt$. Independent: t, dependent: x, order: 4. (This is also linear.)

A general form for an nth-order equation with x independent, y dependent can be expressed as

$$F(x, y, \frac{\mathrm{d}y}{\mathrm{d}x}, \ldots, \frac{\mathrm{d}^n y}{\mathrm{d}x^n}) = 0$$

where F is a function that depends on x, y, and the derivatives of y up to order n. We assume the equations holds for all x in an open interval I. In many cases, we can isolate the highest-order term and write the previous equation as

$$\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = f\left(x, y, \frac{\mathrm{d}y}{\mathrm{d}x}, \dots, \frac{\mathrm{d}^{n-1}y}{\mathrm{d}x^{n-1}}\right)$$

This is called the normal form.

A function $\phi(x)$ that when substituted for y in either the previous two equations satisfies the equation for all x in the interval I is called an explicit solution to the equation on I.

Example

Show that $\phi(x) = x^2 - x^{-1}$ is an explicit solution to the linear equation $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - frac2x^2y = 0$ but $\psi(x) = x^3$ is not.

So we have $y = x^2 - x^{-1}$. The first derivative of this is $2x + 1x^{-2}$. The second derivative is $y'' = 2 - 2x^{-3}$. If we plug in the values we end up getting from the derivatives, we get that $2 - 2x^{-3} - 2x + 2x^{-3} = 0$, so this is satisfied.

For the second part, the first derivative is $3x^2$ and the second derivative is $6x$. Plugging this in, we get $4x$ which is not 0, so $\psi(x)$ is not a solution.

Example

Show that for any choice of the constants c_1 and c_2, the function $\phi(x) = c_1e^{-x} + c_2e^{2x}$ is an explicit solution to the linear equation $y'' - y' - 2y = 0$.

We have that the first derivative is $-c_1e^{-x} + 2c_2e^{2x}$ and the second derivative is $c_1e^{-x} + 4c_2e^{2x}$. When we plug this in, we find that this does satisfy the solution for the differential equation.

Methods for solving differential equations do not always yield an explicit solution for the equation. A solution may be defined implicitly.

Example

Show that the relation $y^2 - x^3 + 8 = 0$ implicitly defines a solution to the nonlinear equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x^2}{2y}$ on the interval $(2, \infty)$.

We have from the given that $y = \pm\sqrt{x^3 - 8}$. The derivative (of the positive version) of this is $\frac{3x^2}{2\sqrt{x^3-8}}$. This is the same and defined on the interval.

A relation $G(x, y) = 0$ is said to be an implicit solution to the previous equation on the interval I if it defines one or more explicit solutions on I.

Example

Show that $x + y + e^{xy} = 0$ is an implicit solution to the nonlinear equation $(1 + xe^{xy})\frac{\mathrm{d}y}{\mathrm{d}x} + 1 + ye^{xy} = 0$.

Taking the derivative of both sides gets us that $1 + \frac{\mathrm{d}y}{\mathrm{d}x} + e^{xy}\frac{\mathrm{d}}{\mathrm{d}x}(xy) = 0$. This does simplify to what was given in the problem.

Example

Verify that for every constant C the relation $4x^2 - y^2 = C$ is an implicit solution to $y\frac{\mathrm{d}y}{\mathrm{d}x} - 4x = 0$. Graph the solution curves for $C = 0, \pm 1, \pm 4$.

The derivative of what is given is $8x - 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$. This simplifies to what is given, so it is clearly an implicit solution.

For $C = 0$, the solution curves for this is $2x = y$ and $-2x = y$.

For $C = \pm 4$, the solution curves is given by a hyperbola $\frac{x^2}{-}\frac{y^2}{4} = 1$.

The collection of all solutions in the previous example is called a one-parameter family of solutions.

In general, the methods for solving nth-order differential equations evoke n arbitrary constants. We often can evalute these constants if we are given n initial values $y(x_0), y'(x_0), \ldots, y^{(n-1)}(x_0)$.

Definition

By an initial value problem for an nth-order differential equation

$$F(x, y, \frac{\mathrm{d}y}{\mathrm{d}x}, \ldots, \frac{\mathrm{d}^n y}{\mathrm{d}x^2}) = 0$$

we mean: Find a solution so the differential equation on an interval I that satisfies at x_0 the n initial conditions

$$y(x_0) = y_0, \quad \frac{\mathrm{d}y}{\mathrm{d}x}(x_0) = y_1 \cdots \frac{\mathrm{d}^{n-1}y}{\mathrm{d}x^{n-1}}(x_0) = y_{n-1}$$

where $x_0 \in I$ and $y_0, y_1, \ldots, y_{n-1}$ are constants.

Example

Show that $\phi(x) = \sin x - \cos x$ is a solution to the initial value problem

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + y = 0; \qquad y(0) = -1 \qquad \frac{\mathrm{d}y}{\mathrm{d}x}(0) = 1$$

We have $y = \sin x - \cos x$, $y' = \cos x + \sin x$, and $y'' = -\sin x + \cos x$. These satisfy the conditions.

Theorem 1.1: Existence and Uniqueness of Solution

Consider the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y), \qquad y(x_0) = y_0$$

If f and $\partial f / \partial y$ are continuous functions in some rectangle

$$R = \{(x, y) : a < x < b, c < y < d\}$$

that contains the point (x_0, y_0), then the initial value problem has a unique solution $\phi(x)$ in some interval $x_0 - \delta < x < x_0 \delta$, where δ is a positive number.

Example

Does the theorem above imply the existence for this problem.

$3\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 - xy^3, \qquad y(1) = 6$

The derivative exists for all (x, y) and is continuous in all intervals containing $x = 1$ and all rectangular regions containing $(1, 6)$.

When we consider the partial derivatives, $\partial f / \partial y = -xy^2$, and this exists and is continuous for all rectangular regions in the xy plane.

1.3 Direction Fields

One technique useful in visualizing (graphing) the solutions to a first-order differential equation is to sketch the direction field for the equation. A first-order equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$$

specifies a slope at each point in the xy-plane where f is defined.

Definition

A plot of short line segments drawn at various points in the xy-plane showing the slope of the solution curve htere is called a direction field for the differential equation.

For example, consider the equation $\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 - y$. When we plug in the point $(1, 0)$, the slope is 1. When the point is $(0, 1)$ the slope is -1. Notice the further right we get, the steeper the graph goes. When we look at this function, $f(x, y) = x^2 - y$, so taking the partial of y results in -1 which is continuous so there is a solution curve at any point.

Note that we are basically just drawing slope fields from AP Calculus BC.

When we consider an equation $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y}{x}$, we have a unique solution when $x \neq 0$ because $f(x, y)$ is continuous if $x \neq 0$ and $\frac{\partial f}{\partial y} = -\frac{1}{x}$, so if $x \neq 0$, then there is a unique solution.

Example

Consider the direction field for $\frac{\mathrm{d}y}{\mathrm{d}x} = 3y^{2/3}$. Is there a unique solution passing through $(2, 0)$?

$\frac{\partial f}{\partial y} = \frac{2}{\sqrt[3]{y}}$ is not continuous when $y = 0$.

Example

The logistic equation of the population p (in thousands) at time t of a certain species is given by $\frac{\mathrm{d}p}{\mathrm{d}t} = p(2 - p)$. Use its direction field to answer the following questions.

(a) If the initial population is $3000[p(0) = 3]$, what can you say about the limiting population $\lim_{t\to\infty} p(t)$?

We start at the point $(0, 3)$ and as the field approaches $p = 2$, the rate of change becomes 0 so the limit is equal to 2.

(b) Can a population of 1000 ever decline to 500?

No

(c) Can a population of 1000 ever increase to 3000.

No

A differential equation $\frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y)$ is autonomous if the independent variable t does not appear explicitly: $\frac{\mathrm{d}y}{\mathrm{d}t} = f(y)$. An autonomous equation has the following properties

- The slopes in the direction field are all identical among horizontal lines
- New solutions can be generated from old ones by time shifting [i.e., replacing $y(t)$ with $y(t - t_0)$]

The constant, or equilibrium, solutions $y(t) = c$ for autonomous equations are of particular interest. The equilibrium $y = c$ is called a stable equilibrium, or sink, if neighboring solutions are attracted to it as $t \to \infty$. Equilibria that repel neighboring solutions, are known as sources; all other equilibria are called nodes. Sources and nodes are unstable equilibria. (Nodes are sometimes called semi stable).

A phase line indicates the zeros and signs of $f(y)$ to describe the nature of the equilibrium solutions for an autonomous equation.

Example

SKetch the phase line for $y' = -(y-1)(y-3)(y-5)^2$ and state the nature of its equilibria.

We have $y' = 0$ and $y = 1, 3, 5$. When $y = 0$, then $y' < 0$ that means that it is decreasing for values below 1. When we let $y = 2$ then $y' > 0$, so any value between 1 and 3 is increasing. When we put $y = 4$, then $y' < 0$ so any values between 3 and 5 are decreasing. When $y = 6$ then $y' < 0$ so it is decreasing.

At $y = 5$, above it is is attracting but repeling below, so it is semi-stable or a node. At $y = 3$, it is stable or an attractor because it is approaching on both sides. At $y = 1$, it is a repeller.

Hand sketching the direction field for a differential equation is often tedious. Fortunately, several software programs are available for this task. When hand sketching is necessary, the method of isoclines can be helpful reducing the work.

A isocline for the differential equation

$$y' = f(x, y)$$

is a set of points in the xy-plane where all the solutions have the same slope $\frac{\mathrm{d}y}{\mathrm{d}x}$; thus, it is a level curve for the function $f(x, y)$.

Example

Find isocline curves of $y' = f(x, y) = x + y$ for a few select values of c. Use the isoclines to draw hash marks with slope c along the isocline $f(x, y) = c$.

When $c = 0, y' = 0, x + y = 0$ and $y = -x$. When $c = 1, y' = 1, x + y = 1$ and $y = -x + 1$. When $c = 2, y' = 2, x + y = 2$ and $y = -x + 2$.

1.4 The Approximation Method of Euler

Euler's method (or the tangent-line method) is a procedure for constructing approximate solutions to an initial value problem for a first-order differential equation

$$y' = f(x, y), \qquad y(x_0) = y_0$$

Euler's method can be summarized by the recursive formulas $x_{n+1} = x_n + h$ and $y_{n+1} = y_n + hf(x_n, y_n)$, where $n = 0, 1, 2, \ldots$.

h is the step size, y' is the m of the tangent line. Remember that $y - y_0 = m(x - x_0)$ and that $y - y_0$ is just $f(x_0, y_0)(x - x_0)$ so $y = y_0 + f(x_0, y_0) \cdot h$.

Example

Use Euler's method with step size $h = 0.1$ to approximate the solution to the initial value problem

$$y' = x\sqrt{y}, \qquad y(1) = 4$$

at the points $x = 1.1, 1.2, 1.3, 1.4$ and 1.5.

We know the x points so we can find the y values from $y_n = y_{n-1} + f(x_{n-1}, y_{n-1})(0.1)$.

We have $(1, 4)$ then $(1.1, 4.2)$ and continuing the calculations, we get $(1.2, 4.43)$, $(1.3, 4.68)$, $(1.4, 4.96)$ and $(1.5, 5.27)$.

Example

Use Euler's method to find approximations to the solution of the initial value problem

$$y' = y, \qquad y(0) = 1$$

at $x = 1$, taking $1, 2, 4, 8$ and 16 steps.

Let $y = e^x$ and a point $(0, 1)$. The recursion formula is $y_n = y_{n-1} + f(x_{n-1} + y_{n-1})h$.

If we use a step size of 1, then $y(1) = 2$.

If we use a step size of 0.5 then $y(1) = 2.25$

If we use a step size of 0.25 then $y(1) = 2.44$

Using technology we can see with a step size of 0.125 that $y(1) = 2.57$ and with a step size of 0.0625, $y(1) = 2.64$.

2 First-Order Differential Equations

2.1 Separable Equations

Definition

If the right-hand side of the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$$

can be expressed as a function $g(x)$ that depends only on x times a function $p(y)$ that depends only on y, then the differential equation is called separable.

To solve the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = g(x)p(y)$$

multiply by $\mathrm{d}x$ and by $h(y) = 1/p(y)$ to obtain

$$h(y)\mathrm{d}y = g(x)\mathrm{d}x$$

Then integrate both sides and you end up getting $H(y) = G(x)+C$, where we have merged the two constants of integration into a single symbol C. The last equation gives an implicit solution to the differential equation.

Example

Solve the nonlinear equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x-5}{y^2}$$

This can be rewritten as $y^2\mathrm{d}y = (x-5)\mathrm{d}x$. Integrating both sides results in $\frac{y^3}{3} = \frac{x^2}{2} - 5x + C$. To get the explicit form just solve for y, which is trivial.

Example

Solve the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y-1}{x+3} \qquad y(-1) = 0$$

Doing Calc BC stuff gives us $y = 1 - \frac{1}{2}(x+3)$.

Be careful because you can be losing solutions. Ok bye!

2.2 Linear Equations

Remember a linear first-order equation is an equation that can be expressed in the form

$$a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0 y = b(x)$$

where $a_1(x)$, $a_0(x)$, and $b(x)$ depend only on the independent variable x, not on y.

Method for solving linear equation:

- Write the equation in the standard form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)$$

- Calculate the integrating factor $\mu(x)$ by the formula

$$\mu(x) = \exp\left[\int P(x)\mathrm{d}x\right]$$

- Multiply the equation in standard form by $\mu(x)$ and, recalling that the left-hand side is just $\frac{\mathrm{d}}{\mathrm{d}x}[\mu(x)y]$, obtain

$$\mu(x)\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)\mu(x)y = \mu(x)Q(x)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}[\mu(x)y] = \mu(x)Q(x)$$

- Integrate the last equation and solve for y by dividing by $\mu(x)$ to obtain.

Example

Find the general solution to

$$\frac{1}{x}\mathrm{d}y\mathrm{d}x - \frac{2y}{x^2} = x\cos x \qquad x > 0$$

We have $\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{2}{x}y = x^2\cos x$.

The integrating factor $\mu(x) = e^{\int P(x)\mathrm{d}x}$ which in this case is $e^{-2\int \frac{1}{x}\mathrm{d}x}$ and this is equivalent to $\frac{1}{x^2}$.

Using this we can multiply through in standard form then we have $\frac{1}{x^2}\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{2}{x^3}y = \cos x$.

The left side is just $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x^2}\cdot\right) = \cos x$.

Integrating and solving for y we get that $y = x^2\sin x + Cx^2$.

Example

For the initial value problem

$$y' + y = \sqrt{1+\cos^2 x} \qquad y(1) = 4$$

find the value of $y(2)$.

Our $P(x)$ is 1 here, so $\mu = e^x$.

So the equation after multiplying through by it gives us that $\mu y' + \mu y = \mu\sqrt{1+\cos^2 x}$, or $e^x y' + e^x y = e^x\sqrt{1+\cos^2 x}$.

This is equivalent to basically $\frac{\mathrm{d}}{\mathrm{d}x}(e^x y) = e^x\sqrt{1+\cos^2 x}$.

This is $e^x y = \int e^x\sqrt{1+\cos^2 x}\mathrm{d}x$.

Using a calculator $y(2) = 2.127$.

Theorem 2.1: Existence and Uniqueness of Solution

Suppose $P(x)$ and $Q(x)$ are continuous on an interval (a,b) that contains the point x_0. Then for any choice of initial value y_0, there exists a unique solution $y(x)$ on (a,b) to the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Qx \qquad y(x_0) = y_0$$

In fact the solution is given for a suitable value of C.

2.3 Exact Equations

Definition: Exact Differential Form

The differential form $M(x, y)\mathrm{d}x + N(x, y)\mathrm{d}y$ is said to be exact in a rectangle R is there is a function $F(x, y)$ such that

$$\frac{\partial F}{\partial x}(x, y) = M(x, y) \qquad \text{and} \qquad \frac{\partial F}{\partial y}(x, y) = N(x, y)$$

for all (x, y) in R. That is, the total differential of $F(x, y)$ satisfies

$$\mathrm{d}F(x, y) = M(x, y)\mathrm{d}x + N(x, y)\mathrm{d}y$$

If $M(x, y)\mathrm{d}x + N(x, y)\mathrm{d}y$ is an exact differential form, then the equation

$$M(x, y)\mathrm{d}x + N(x, y)\mathrm{d}y = 0$$

is called an exact equation.

Theorem 2.2: Test for Exactness

Suppose the first partial derivatives of $M(x, y)$ and $N(x, y)$ are continuous in a rectangle R. Then

$$M(x, y)\mathrm{d}x + N(x, y)\mathrm{d}y = 0$$

is an exact equation in R if and only if the compatibility condition

$$\frac{\partial M}{\partial y}(x, y) = \frac{\partial N}{\partial x}(x, y)$$

holds for all (x, y) in R.

Example

Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2xy^2 + 1}{2x^2 y}$$

Ok so this is not separable or linear, so we use exactness.

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{2xy^2 + 1}{2x^2 y} &= 0 \\
\mathrm{d}y + \frac{2xy^2 + 1}{2x^2 y}\mathrm{d}x &= 0 \\
\frac{2xy^2 + 1}{2x^2 y}\mathrm{d}x + 1\mathrm{d}y &= 0
\end{aligned}$$

This is the same form we want.

Another form we can get is $(2xy^2 + 1)\mathrm{d}x + 2x^2 y\mathrm{d}y = 0$.

Another form we can get is $1\mathrm{d}x + \frac{2x^2 y}{2xy^2+1}\mathrm{d}y = 0$.

We are now looking for a $F(x, y) = c$ and we know this is true when $\frac{\partial m}{\partial y} = \frac{\partial n}{\partial x}$.

So the second one of these is probably the best, so we now have $m = 2xy^2 + 1$ and $n = 2x^2 y$.

Doing the partial of m with respect to y we get $4xy$ and the partial of n with respect to x is $4xy$ and these are the same.

Let $F(x, y) = x^2y^2 + x = C$. The partial of this function with respect to x is $2xy^2 + 1$ and the partial of this function with respect to y is $2x^2y$ and this is the same as previous.

Method for Solving Exact Equations:

- If $M\mathrm{d}x + N\mathrm{d}y = 0$ is exact, then $\partial F/\partial x = M$. Integrate this last equation with respect to x to get

$$F(x, y) = \int M(x, y)\mathrm{d}x + g(y)$$

- To determine $g(y)$, take the partial derivative with respect to y of both sides of the aboev equation and substitute N for $\partial F/\partial y$. We can now solve for $g'(y)$.
- Integrate $g'(y)$ to obtain $g(y)$ up to a numerical constant. Substituting $g(y)$ into the equation from step 1 gives $F(x, y)$
- The solution to $M\mathrm{d}x + N\mathrm{d}y = 0$ is given implicitly by

$$F(x, y) = C$$

(Alternatively, starting with $\partial F/\partial y = N$, the implicit solution can be foudn by first integrating with respect to y.)

Example

Solve

$$(2xy - \sec^2 x)\mathrm{d}x + (x^2 + 2y)\mathrm{d}y = 0$$

Let m be the first term and n be the second term, and the partial derivatives of these are the same, so they are exact.

Let $F(x, y) = \int 2xy - \sec^2 x\mathrm{d}x$. When we integrate this, we get $yx^2 - \tan x$. In this case, the constant is anything with y, so the integral is equivalent to $yx^2 - \tan x + g(y)$.

Now we take the $\frac{\partial F}{\partial y} = x^2 - 0 + g'(y)$. These two are n so $x^2 + 2y = x^2 + g'(y)$, so solving for $g(y)$ we get that this is equal to $y^2 + C$.

So $F(x, y) = xy^2 - \tan x + y^2 = C$.

Exercise Solve $(1 + e^x y + xe^x y)\mathrm{d}x + (xe^x + 2)\mathrm{d}y = 0$.

Solution: $x + xye^x + 2y = C$.

Example

Solve

$$(x + 3x^3 \sin y)\mathrm{d}x + (x^4 \cos y)\mathrm{d}y = 0$$

Doing the partials originally makes them not equal to each other.

We can get this to exact form by multiplying through by x^{-1}. When we do this we get $(1+3x^2 \sin y)\mathrm{d}x + x^3 \cos y\mathrm{d}y = 0$ and the partials of these are the same.

x^{-1} is called an integrating factor.

Integrating m with respect to x, we get that $F(x, y) = \int 1 + 3x^2 \sin y\mathrm{d}x = x + \sin y \cdot x^3 + g(y)$.

Doing the partial of F with respect to y, we get $\frac{\partial F}{\partial y} = x^3 \cos y = 0 + x^3 \cos y + g'(y)$, and this gets that $g(y) = C$.

So the answer is $x + x^3 \sin y = C$.

2.4 Special Integrating Factors

Definition

If the equation

$$M(x, y)\mathrm{d}x + N(x, y)\mathrm{d}y = 0$$

is not exact, but the equation

$$\mu(x, y)M(x, y)\mathrm{d}x + \mu(x, y)N(x, y)\mathrm{d}y = 0$$

which results from multiplying the first equation by the function $\mu(x, y)$, is exact, then $\mu(x, y)$ is called an integrating factor of the first equation.

Theorem 2.3: Special Integrating Factors

If $(\partial M/\partial y - \partial N/\partial x)/N$ is continuous and depends only on x, then

$$\mu(x) = \exp\left[\int \left(\frac{\partial M/\partial y - \partial N/\partial x}{N}\right) \mathrm{d}x\right]$$

is an integrating factor for an equation. If $(\partial N/\partial x - \partial M/\partial y)/M$ is continuous and depends only on y, then

$$\mu(y) = \exp\left[\int \left(\frac{\partial M/\partial x - \partial N/\partial y}{M}\right) \mathrm{d}y\right]$$

is an integrating factor for the same equation.

Method for Finding Special Integrating Factors:

If $M\mathrm{d}x + N\mathrm{d}y = 0$ is neither separable nor linear, compute $\partial M/\partial y$ and $\partial N/\partial x$. If $\partial M/\partial y = \partial N/\partial x$, then the equation is exact. If it is not exact, consider

$$\frac{\partial M/\partial y - \partial N/\partial x}{N}$$

If this is a function of just x, then an integrating factor is given by the formula above of $\mu(x)$. If not consider

$$\frac{\partial N/\partial x - \partial M/\partial y}{M}$$

If this is a function of just y¡ then an integrating factor is given by above of $\mu(y)$.

Example

Solve $(2x^2 + y)\mathrm{d}x + (x^2y - x)\mathrm{d}y = 0$

When we do the partials, we get that $1 \neq 2xy - 1$.

So lets look at $\frac{\partial m/\partial y - \partial n/\partial x}{N}$, which is $\frac{1-(2xy-1)}{x^2y-x} = \frac{-2}{x}$ which is just a function of x. So we have that $\mu = e^{\int -\frac{2}{x}\mathrm{d}x}$, so we don't have to look at the one in terms of y.

Doing the integral of all this gives us that $e^{-2\ln x} = x^{-2}$. So when we multiply through by x^{-2}, we get that $(2 + x^{-2}y)\mathrm{d}x + (y - x^{-1})\mathrm{d}y = 0$.

The partials are equal to each other, so this equation is now exact.

Now we find $F(x, y)$ by integrating m, so $\int(2 + x^{-2}y)\mathrm{d}x = 2x + -x^{-1}y + g(y) = F(x)$

Now we differentiation with respect to y so $\frac{\partial F}{\partial y} = y - x^{-1} = -x^{-1} + g'(y)$, so $g(y) = \frac{y^2}{2}$

The solution is therefore $2x - x^{-1}y + \frac{y^2}{2} = C$.

2.5 Substitutions and Transformations

Substitution Procedure:

- Identify the type of equation and determine the appropriate substitution or transformation
- Rewrite the original equation in terms of new variables
- Solve the transformed equation
- Express the solution in terms of the original variables

Definition: Homogeneous Equation

If the right-hand side of the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$$

can be expressed as a function of the ratio y/x alone, then we say the equation is homogeneous.

To solve a homogeneous equation, use the substitution $v = \frac{y}{x}$; $\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$ to transform the equation into a separable equation.

Example

Solve $(xy + y^2 + x^2)\mathrm{d}x - x^2\mathrm{d}y = 0$.

Solving for $\frac{\mathrm{d}y}{\mathrm{d}x}$ we get that this is equal to $\frac{-x^2-y^2-xy}{-x^2}$ and this simplifies to $1 + \left(\frac{y}{x}\right)^2 + \frac{y}{x}$.

This is equivalent to $v + x\frac{\mathrm{d}v}{\mathrm{d}x} = 1 + v^2 + v$. We end up getting that $\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{v^2+1}{x}$ and this can be done by separation. The solution is $y = x\tan(\ln|x| + C)$ after solving.

To solve an equation of the form $\frac{\mathrm{d}y}{\mathrm{d}x} = G(ax+by)$, use the substitution $z = ax+by$ to transform the equation into a separable equation.

Example

Solve $\frac{\mathrm{d}y}{\mathrm{d}x} = y - x - 1 + (x - y + 2)^{-1}$

First we have $\frac{\mathrm{d}y}{\mathrm{d}x} = -(x - y) - 1 + (x - y + 2)^{-1}$

So substituting with $z = x - y$, we have that $\frac{\mathrm{d}z}{=}1 - \frac{\mathrm{d}z}{\mathrm{d}x}$. Knowing this, the equation is equal to $1 - \frac{\mathrm{d}z}{\mathrm{d}x} = -z - 1 + (z + 2)^{-1}$. From this this simplifies to $\frac{\mathrm{d}z}{\mathrm{d}x} = z + 2 - (z + 2)^{-1}$.

So now we write this into a separable equation with $\frac{(z+2)\mathrm{d}z}{(z+2)^2-1} = \mathrm{d}x$.

Separating by parts and substituting gives $(x - y + 2)^2 = ce^{2x} + 1$

Definition: Bernoulli Equation

A first-order equation that can be written in the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)y^n$$

where $P(x)$ and $Q(x)$ are continuous on the interval (a, b) and n is a real number, is called a Bernoulli equation.

To solve a Bernoulli equation use the substitution $v = y^{1-n}$ to transform the equation into a linear equation.

Example

Solve $\frac{\mathrm{d}y}{\mathrm{d}x} - 5y = -\frac{5}{2}xy^3$.

From above, we have $v = y^{-2}$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = -2y^{-3}\frac{\mathrm{d}u}{\mathrm{d}x}$ and $-\frac{1}{2}\frac{\mathrm{d}v}{\mathrm{d}x} = y^{-3}\frac{\mathrm{d}y}{\mathrm{d}x}$.

So multiplying through by y^{-3} and substituting, we get that $-\frac{1}{2}\frac{\mathrm{d}v}{\mathrm{d}x} - 5v = -\frac{5}{2}x$.

This is equal to $\frac{\mathrm{d}v}{\mathrm{d}x} + 10v = 5x$. The integrating factor here is $\mu = e^{\int P(x)\mathrm{d}x}$, which is e^{10x} in this case.

Multiplying through by μ, we get that $e^{10x}\frac{\mathrm{d}v}{\mathrm{d}x} + 10ve^{10x} = 5xe^{10}x$ and the LHS should be equal to $\frac{\mathrm{d}}{\mathrm{d}x}(e^{10x}v) = 5xe^{10x}$.

Using elementary integration techniques the answer is $y^{-2} = \frac{x}{2} - \frac{1}{20} + Ce^{-10x}$.

3 Mathematical Models and Numerical Methods Involving First-Order Equations

3.1 Compartmental Analysis

We assume that the growth rate is proportional to the population present. A mathematical model called the Malthusian, or exponential law of population growth, model is given by

$$\frac{\mathrm{d}p}{\mathrm{d}t} = kp, \qquad p(0) = p_0$$

where $k > 0$ is the proportionality constant of the growth rate and p_0 is the population at time $t = 0$.

Example

In 1790, the population of the United States was 3.93 million, and in 1890 it was 62.98 million. Assuming the Mathusian model, estimate the U.S. population as a function of time.

Let $t = 0$ be year 1790. Using separation of variables and integration from $\frac{\mathrm{d}p}{\mathrm{d}t} = kp$, we get that $\ln P = kt + C$. Solving for P we get that $e^{\ln P} = P = e^{kt+C}$, so this is equal to $P = Ce^{kt}$.

At $P(0) = 3.93$, and using this we can find k. $3.93 = Ce^0$, so $C = 3.93$. So we have $P(t) = P_0 e^{kt}$. We can now find k by plugging in $P(100) = 62.98$ from this, and solving for k, we get that $k \approx 0.027742$.

So $P(t) = 3.93e^{.027742t}$

The Malthusian model considered only death by natural casues. Other factors such as premature deaths due to malnutrition, inadequate medical supplies, communicable diseases, violent crimes, etc involve a competition within the population. The logistic model is given by

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -Ap(p - p_1), \qquad p(0) = p_0$$

Note that this equilibrium has two equilibrium solutions $p(t) = p_1$ and $p(t) = 0$.

Example

Taking the population of 3.93 million as the initial population and given the 1840 and 1890 populations of 17.07 and 62.98 million respectively, use the logistic model to estimate the population at time t.

We have $P(50) = 17.07$ and $P(100) = 62.98$. Using a previously derived formula, $p(t) = \frac{p_0 p_1}{p_0 + (p_1 - p_0)e^{-Ap_1 t}}$, we get $17.07 = \frac{3.93p_1}{3.93 + (p_1 - 3.93)e^{-50Ap_1}}$ and $62.98 = \frac{3.93p_1}{3.93 + (p_1 - 3.93)e^{-100Ap_1}}$.

The answers are that $p_1 \approx 251.78$ and $A \approx 0.0001210$, so the logistic equation ended up being $p(t) = \frac{989.5}{3.93 + 247.85e^{-0.030463t}}$.

Example

Suppose a student carrying a flu virus returns to an isolated college campus of 1000 students. If it is assumed that the rate at which the virus spreads is proportional not only to the number x of infected students but also to the number of students not infected, determine the number of infected students after 6 days if it is further observed that after 4 days, $x(4) = 50$.

We are solving the initial value problem $\frac{\mathrm{d}x}{\mathrm{d}t} = kx(1000 - x), x(0) = 1$. Doing some stuff we get that $\ln \frac{x}{1000-x} = 1000kt + C$.

After doing some more things we get that $x = \frac{1000Ce^{1000kt}}{1+Ce^{1000kt}}$ and from this we can get that $C = \frac{1}{999}$.

So the function is now $x(t) = \frac{\frac{1000}{999}e^{1000kt}}{1+\frac{1}{999}e^{1000kt}}$, so simplifying we get that $x(t) = \frac{1000}{999e^{-1000kt}+1}$.

Doing some plugging in stuff we get that $k \approx -.9906$, so after 6 days approximately 276 students are infected.

3.2 Numerical Methods: A Closer Look At Euler's Algorithm

The numerical method defined by the formula

$$y_{n+1} = y_n + h\frac{f(x_n, y_n) + f(x_{n+1}, y^*_{n+1})}{2}$$

where

$$y^*_{n+1} = y_n + hf(x_n, y_n)$$

is known as the improved Euler's method.

The improved Euler's method is an example of a predictor-corrector method.

It is recommended you use technology to find the answer.

3.3 Higher-Order Numerical Methods: Taylor and Runge-Kutta

We wish to obtain a numerical approximation of the solution $\phi(x)$ to the initial value problem

$$y' = f(x, y), \qquad y(x_0) = y_0$$

To derive the Tayler methods, let $\phi_n(x)$ be the exact solution to the initial value problem

$$\phi_n'(x) = f(x, \phi_n), \qquad \phi_n(\text{problem}x_n) = y_n$$

The Taylor series for $\phi_n(x)$ about the point x_n is

$$\phi_n(x) = \phi_n(x_n) + h\phi_n'(x_n) + \frac{h}{2!}\phi_n''(x_n) + \dots$$

where $h = x - x_n$. Since ϕ_n satisfies the initial value, we can write this series in the form

$$\phi_n(x) = y_n + hf(x_n, y_n) + \frac{h^2}{2!}\phi_n''(x_n) + \dots$$

Example

Determine the recursive formula for the Taylor method of order 2 for the initial value

$$y' = \sin(xy), y(0) = \pi$$

We know that $\phi_n''(x_n) = \frac{\partial f}{\partial x}(x_n, y_n) + \frac{\partial f}{\partial y}(x_n, y_n) \cdot \frac{\mathrm{d}y}{\mathrm{d}x}$.

So $\frac{\partial f}{\partial x} = \cos(xy) \cdot y$ and $\frac{\partial f}{\partial y} = \cos(xy) \cdot x$.

Putting this all together we get that $\phi_n(x) = y_{n+1} = y_n + h\sin(x_n y_n) + \frac{h^2}{2}[y\cos(xy) + x\cos(xy) \cdot \sin(xy)]$.

Doing some magic, $x_{n+1} = x_n + h$ and $y_{n+1} = y_n + h\sin(x_n y_n) + \frac{h^2}{2}[y_n\cos(x_n y_n) + \frac{x_n}{2}\sin(2x_n y_n)]$.

4 Linear Second-Order Equations

4.1 Introduction: The Mass-Spring Oscillator

A damped mass-spring oscillator consists of a mass m attached to a spring fixed at one end. Devise a differential equation that governs the motion of this oscillator, taking into account the forces acting on it due to the spring elasticity, damping friction, and possible external influences.

Newton's second law - force equals mass times acceleration ($F = ma$) - is the most commonly encountered differential equation. It is an ordinary differential equation of the second order since acceleration is the second derivative of position (y) with respect to time ($a = \mathrm{d}^2 y/\mathrm{d}t^2$).

If the spring is unstretched and the inertial mass m is still, the system is at equilibrium. We stretch the coordinate y of the mass by its displacement from the equilibrium position.

When the mass m is displaced from equilibrium, the spring is stretched or compressed and it exerts a force that resists the displacement. For most springs this force is directly proportional to the displacement y and is given by Hooke's law.

$$F_{\text{spring}} = -ky$$

where the positive constant k is known as the stiffness (spring constant) and the negative sign reflects the opposing nature of the force. Hooke's law is only valid for sufficiently small displacements.

Usually all mechanical systems also experience friction. For vibrational motion this force is usually modeled accurately by a term proportional to velocity:

$$F_{\text{friction}} = -b\frac{\mathrm{d}y}{\mathrm{d}t} = -by'$$

where $b \geq 0$ is the damping coefficient and the negative sign reflects the opposing nature of the force.

The other forces on the oscillator are usually regarded as external to the system. Although they may be gravitational, electrical, or magnetic, commonly the most important external forces are transmitted to the mass by shaking the supports holding the system. For now we refer to the combined external forces by a single known function $F_{\text{ext}}(t)$. Newton's law provides the differential equation for the mass-spring oscillator:

$$my'' = -ky - by' + F_{\text{ext}}(t)$$

or

$$my'' + by' + ky = F_{\text{ext}}(t)$$

Example

Verify that if $b = 0$ and $F_{\text{ext}} = 0$, that the above equation has a solution of the form $y(t) = \cos(\omega t)$ and the angular frequency ω increases with k and decreases with m.

The differential equation is $my'' + by' + ky = F_{\text{ext}}$ and that $my'' + ky = 0$.

Since we are given what $y(t)$ is, taking the derivative of the differential equation gives that $-m\omega^2\cos(\omega t) + k\cos(\omega t) = 0$, and solving for ω, we get that $\omega = \sqrt{\frac{k}{m}}$.

If k increases, ω increases, and if m increases, ω decreases.

Example

Verify that the exponentially damped sinusoid given by $y(t) = e^{-3t}\cos 4t$ is a solution to the above differential equation if $F_{\text{ext}} = 0, m = 1, k = 25$, and $b = 6$.

From the differential equation $my'' + by' + ky = F_{\text{ext}}$, we can plug in stuff and we get that $y'' + by' + 25y =$

0.

Since we are given $y(t)$ we can find the derivatives and substitute. What we are given is quite long, but essentially it cancels out to $0 = 0$, which means it is a solution to the system.

Exercise Verify that the simple exponential function $y(t) = e^{-5t}$ is a solution to the above differential equation if $F_{\text{ext}} = 0$, $m = 1$, $k = 25$, and $b = 10$.

Sometimes the external force will make the system look somewhat erratic. There are many real world examples where the external force must defintely be taken into account.

4.2 Homogeneous Linear Equations: The General Solution

A second-order constant-coefficient differential equation has the form

$$ay'' + by' + cy = f(t) \qquad (a \neq 0)$$

A homogeneous second-order constant-coefficient differential equation is the special case with $f(t) = 0$.

$$ay'' + by' + cy = 0 \qquad (a \neq 0)$$

A solution of this equation has the form $y = e^{rt}$. The resulting equation $ar^2 + br + c = 0$ is called the auxiliary equation (or characteristic equation) associated with the homogeneous equation.

Example

Find a pair of solutions to

$$y'' + 5y' - 6y = 0$$

Plugging in $y = e^{rt}$, we get that $e^{rt}(r^2+5r-6)$ after substituting. Solving the quadratic $r^2+5r-6 = 0$, we get that $r = -6$ and $r = 1$, which is $y = e^{-6t}$ and $y = e^t$.

Note that the zero function, $y(t) = 0$ is always a solution to an equation above (figure this out later). In addition when we have a pair of solutions $y_1(t)$ and $y_2(t)$, we can construct an infinite number of other solutions by forming linear combinations:

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

for any choice of the constants c_1 and c_2. This is a two-paramter solution form since there are two unknown constants. To find a specific solution, two initial conditions are needed.

Example

Solve the initial value problem

$$y'' + 2y' - y = 0 \qquad y(0) = 0, \qquad y'(0) = -1$$

Doing the default substitution, our auxiliary equation is $r^2 + 2r - 1 = 0$.

The solutions from this are $r = -1 + \sqrt{2}$ and $r = -1 - \sqrt{2}$. Remember $y = e^{rt}$. Using the initial conditions, we get that $c_1 = -\frac{\sqrt{2}}{4}$ and $c_2 = \frac{\sqrt{2}}{4}$.

Theorem 4.1

For any real numbers $a(\neq 0), b, c, t_0, Y_0$, and Y_1, there exists a unique soultion to the initial value problem.

$$ay'' + by' + cy = 0 \qquad y(t_0) = Y_0 \qquad y'(t_0) = y_1$$

The solution is valid for all t in $(-\infty, \infty)$

Definition

A pair of functions $y_1(t)$ and $y_2(t)$ is said to be linearly independent on the interval t if and only if neither of them is a constant multiple of the other on all of t. We say that y_1 and y_2 are linearly dependent on t if one of them is a constant multiple of the other on all of t.

Theorem 4.2

If $y_1(t)$ and $y_2(t)$ are any two solutions to the differential equation that are linearly independent on $(-\infty, \infty)$, then unique constants c_1 and c_2 can always be found so that $c_1 y_1(t) + c_2 y_2(t)$ satisfies the initial value problem on $(-\infty, \infty)$.

Definition: Wronksian

Suppose each of the functions $f_1(x), f_2(x), \ldots f_n(x)$ possess at least $n-1$ derivatives.

The determinant

$$W(f_1, f_2, \ldots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

is called the Wronksian of the functions.

Theorem 4.3

Let $y_1, y_2, \ldots, y_n$ be n solutions of the homogeneous linear nth-order differential equation on an interval I. Then the set of solutions is linearly independent on I if and only if $W(y_1, y_2, \ldots, y_n) \neq 0$ for every x in the interval.

Distinct real roots: If the auxiliary equation has distinct real roots r_1 and r_2, then both $y_1(t) = e^{rt}$ and $y_2(t) = e^{rt}$ are solutions to the above differential equation and $y(t) = e_1 e^{rt} + e_2 e^{rt}$ is a general solution.

Repeated root: if the auxiliary equation has a repeated root r, then both $y_1(t) = e^{rt}$ and $y_2(t) = te^{rt}$ are solutions to the differential equation and $y(t) = e_1 e^{rt} + e_2 te^{rt}$ is a general solution.

A homogeneous linear nth-order equation has a general solution of the form

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t)$$

where the individual solutions $y_i(t)$ are linearly independent, i.e. no $y_i(t)$ is expressible as a linear combination of the others.

4.3 Auxiliary Equations with Complex Roots

The simple harmonic equation $y'' + y = 0$ so called because of its relation to the fundamental vibration of a musical tone, has as solutions $y_1(t) = \cos t$ and $y_2(t) = \sin t$.

When $b^2 - 4ac < 0$, the roots of the auxiliary equation $ar^2 + br + c = 0$ associated with the homogeneous equation $ay'' + by' + cy = 0$ are the complex conjugate numbers $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$ where $\alpha = -\frac{b}{2a}$ and $\beta = \frac{\sqrt{4ac - b^2}}{2a}$

Combing the solutions $e^{r_1 t}$ and $e^{r_2 t}$ with Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$, yields complex function solutions

$$e^{(\alpha+i\beta)t} = e^{at}(\cos\beta t + i\sin\beta t) \text{ and } e^{(\alpha-i\beta)t} = e^{at}(\cos\beta t - i\sin\beta t)$$

Example

Solve the initial value problem $y'' + 2y' + 2y = 0$ given $y(0) = 0$ and $y'(0) = 2$.

Using the auxiliary form of the equation we have $r^2+2r+2 = 0$. From the quadratic formula, $r = -1 \pm i$, the two roots are $r_1 = -1 + i$ and $r_2 = -1 - i$.

The solution is therefore $y_1 = e^{(-1+i)t}$ and $y_2 = e^{(-1-i)t}$

From the form given from euler's formula earlier, $y_1 = e^{-t}(\cos t + i \sin t)$ and $y_2 = e^{-t}(\cos t - i \sin t)$, so our general solution is $y = c_1 e^{-t}(\cos t + i \sin t) + c_2 e^{-t}(\cos t - i \sin t)$.

Plugging the initial conditions, we get that $0 = c_1 + c_2$.

The derivative of the general solution is $y' = c_1 e^{-t}(-\sin t + i \cos t) + (\cos t + i \sin t) \cdot c_1(-1)e^{-t} + c_2 e^{-t}(-\sin t - i \cos t) + (\cos t - i \sin t) \cdot c_2(-1)e^{-t}$. Plugging in the initial conditions gives $2 = c_1 i - c_1 - c_2 i - c_2$.

Factoring we get $2 = c_1(i - 1) + c_2(-i - 1)$. Using some substitution $c_2 = i$ and $c_1 = -i$. Plug this in the general solution to solve.

If the auxiliary equation has complex conjugate roots $a \pm i\beta$, then two linearly independent solutions to the equation are

$$e^{\alpha t} \cos \beta t \text{ and } e^{\alpha t} \sin \beta t$$

and a general solution is

$$y(t) = c_1 e^{\alpha t} \cos \beta t + c_2 e^{\alpha t} \sin \beta t$$

where c_1 and c_2 are arbitrary constants.

Example

Find a general solution to $y'' + 2y' + 4y = 0$.

Using the auxiliary equation the roots are $-1 \pm \sqrt{3}i$. Using what was given above, the general solution is $y = c_1 e^{-t} \cos(\sqrt{3}t) + c_2 e^{-t} \sin(\sqrt{3}t)$.

Example

Newton's second law implies the position $y(t)$ of the mass m is governed by the second-order differential equation $my''(t) + by'(t) + ky(t) = 0$ where the terms are physically identified as my being interial, by is damping and ky is stiffness. Determine the equation of motion for a spring system when $m = 36$ kg, $b = 12$ kg/sec (which is equivalent to 12 N - sec/m), $k = 37$ kg/sec^2, $y(0) = 0.7$ m and $y'(0) = 0.1$ m/sec. Also find $y(10)$, the displacement after 10 sec.

The differential equation is $36y'' + 12y' + 37$, so the roots are $-\frac{1}{6} \pm i$.

Doing the methods explained above, the solution is $y = .7e^{-t/6} \cos t + \frac{13}{60} e^{-t/6} \sin t$, and $y(10) \approx -.13$ m.

Exercise Interpret the equation $y'' + 5y' - 6y = 0$ in terms of the mass-spring system.

4.4 Nonhomogeneous Equations: the Method of Undetermined Coefficients

The method of Undetermined Coefficients is the technique used to guess a solution's form based on the form of the nonhomogeneous function $f(t)$ in a linear equation with constant coefficients such as $ay''+by'+cy = f(t)$.

For example the particular solution to $ay'' + by' + cy = Ct^m$ is of the form $y_p(t) = A_m t^m + \cdots + A_1 t + A_0$.

Example

Find a particular solution to $y'' + 3y' + 2y = 10e^{3t}$.

Our guess based on the form of $f(t) = 10e^{3t}$ is that $y = Ae^{3t}$ is the guess form of the particular solution, so we know that $y' = 3A3^{3t}$ and $y'' = 9Ae^{3t}$.

Substituting this in gives us $9Ae^{3t} + 3(3Ae^{3t}) + 2(Ae^{3t}) = 10e^{3t}$. Simplifying this gives us $20Ae^{3t} = 10e^{3t}$ so $A = 1/2$.

So our particular solution is $\frac{1}{2}e^{3t}$.

Example

Find a particular solution to $y'' + 3y' + 2y = \sin t$.

Let $y = A\sin t + B\cos t$ as the form of the particualr solution.

Substituting and solving should result in $A = 1/10$ and $B = -3/10$.

This example suggests an equation of the form $ay'' + by' + cy = C\sin\beta t$ (or $C\cos\beta t$) will have a particular solution of the form $y_p(t) = A\cos\beta t + B\sin\beta t$.

Example

Find a particular solution to $y'' + 4y = 5t^2e^t$.

Let $y = At^2e^t + Bte^t + Ce^t = e^t(At^2 + Bt + C)$. The result should be $A = 1, B = -4/5, C = -2/25$.

To find a particular soultion to the differential equation

$$ay'' + by' + cy = Ct^m e^{rt}$$

where m is a nonnegative integer, use the form

$$y_p(t) = t^s(A_m t^m + \cdots + A_1 t + A_0)e^{rt}$$

with

1. $s = 0$ if r is not a root of the associated auxiliary equation;
2. $s = 1$ if r is a simple root of the associated auxiliary equation;
3. $s = 2$ if r is a double root of the associated auxiliary equation.

To find a solution to the differential equation $ay'' + by' + cy = Ct^m e^{\alpha t}\cos\beta t$ or equal to $Ct^m e^{\alpha t}\sin\beta t$ for $\beta \neq 0$, use the form $y_p(t) = t^s(A_m t^m + \ldots A_1 t + A_0)e^{\alpha t}\cos\beta t + t^s(B_m t^m + \cdots + B_1 t + B_0)e^{\alpha t}\sin\beta t$, with

1. $s = 0$ if $\alpha + i\beta$ is not a root of the associated auxiliary equation; and
2. $s = 1$ if $\alpha + i\beta$ is a root of the associated auxiliary equation

4.5 The Superposition Principle and Undetermined Coefficients Revisited

Theorem 4.4: Superposition Principle

Let y_1 be a solution to the differential equation

$$ay'' + by' + cy = f_1(t)$$

and y_2 is a solution to

$$ay'' + by' + cy = f_2(t)$$

then for any constants k_1 and k_2, the function $k_1y_1 + k_2y_2$ is a solution to the differential equation

$$ay'' + by' + cy = k_1f_1(t) + k_2f_2(t)$$

Example

Find a particular solution to

$$y'' + 3y' + 2y = 3t + 10e^{3t}$$

The solution for equal to $3t$ is $y = \frac{3t}{2} - \frac{9}{4}$ and for $10e^{3t}$ is $y = \frac{e^{3t}}{2}$ so the solution is $y = \frac{3t}{2} - \frac{9}{4} + \frac{e^{3t}}{2}$.

Exercise Find a particualr solution to $y'' + 3y' + 2y = -9t + 20e^{3t}$.

General solution for Nonhomogeneous Differential Equations: Let y_p be a particular solution to

$$ay'' + by' + cy = f(t)$$

and $c_1y_1 + c_2y_2$ be the general solution to the homogeneous equation

$$ay'' + by' + cy = 0$$

Then the general solution to the nonhomogeneous equation is given by

$$y(t) = y_p(t) + c_1y_1(t) + c_2y_2(t)$$

Theorem 4.5

For any real numbers $a(\neq 0), b, c, t_0, Y_0$, and Y_1, suppose $y_p(t)$ is a particular solution to above in an interval I containing t_0 and that $y_1(t)$ and $y_2(t)$ are linearly independent solutions to the associated homogeneous equation in I. Then there exists a unique solution in I to the initial value problem.

$$ay'' + by' + cy = f(t) \qquad y(t_0) = Y_0 \qquad y'(t_0) = Y_1$$

Example

Given that $y_p(t) = t^2$ is a particular solution to

$$y'' - y = 2 - t^2$$

Find a general solution and a solution satisfying $y(0) = 1, y'(0) = 0$.

Our general solution using the auxiliary equation is $y = c_1e^t + c_2e^{-t}$.

Our particular solution will be in $At^2 + By + C$.

So $y = t^2 + c_1e^t + c_2e^{-t}$. Solving for c_1 and c_2 by finding the derivative of this and using the initial conditions, the specific solution is $y = t^2 + \frac{1}{2}e^t + \frac{1}{2}e^{-t}$.

Example

A mass-spring system is driven by a sinusodial external force $5\sin t + 5\cos t$. The mass equals 1, the spring constant equals 2, and the damping coefficient equals 2 (in appropriate units), so the motion is governed by the differential equation

$$y'' + 2y' + y = 5\sin t + 5\cos t$$

If the mass is initially located at $y(0) = 1$, with a velocity $y'(0) = 2$, find its equation of motion.

Finding the general solution to this we get that $y = c_1e^{-t}\cos t + c_2e^{-t}\sin t$.

From $y_p = A\sin t + B\cos t$, we should solve that

$$y = 3\sin t - \cos t + 2e^{-t}\cos t + e^{-t}\sin t$$

Example

Find a particular solution to

$$y'' - y = 8te^t + 2e^t$$

The general solution for this is $y = c_1e^t + c_2e^{-t}$. y_p is equal to $(At + B)e^t$. Doing some calculations should result in $y_p = (2t^2 - t)e^t$.

Method of Undetermined Coefficients (Revisited)

To find a particular solution to the differential equation

$$ay'' + by' + cy = P_m(t)e^{rt}$$

where $P_m(t)$ is a polynomial of degree m, use the form

$$y_p(t) = t^s(A_mt^m + \cdots + A_1t + A_0)e^{rt}$$

if r is not a root of the associated auxiliary equation, take $s = 0$; ir r is a simple root of the associated auxiliary equation, take $s = 1$; and if r is a double root of the associated auxiliary equation, take $s = 2$.

To find a particular solution to the differential equation

$$ay'' + by' + cy = P_m(t)e^{\alpha t}\cos\beta t + Q_n(t)e^{\alpha t}\sin\beta t, \beta \neq 0$$

where $P_m(t)$ is a polynomial of degree m and $Q_n(t)$ is a polynomial of degree n, use the form $y_p(t) = t^s(A_kt^k + \cdots + A_1t + A_0)e^{\alpha t}\cos\beta t + t^s(B_kt^k + \cdots + B_1t + B_0)e^{\alpha t}\sin\beta t$, where k is the larger of m and n. If $\alpha + i\beta$ is not a root of the associated auxiliary equation, take $s = 0$; if $\alpha + i\beta$ is a root of the associated auxiliary equation, take $s = 1$.

Exercise Write down the form of a particular solution to the equation $y'' + 2y' + 2y = 5e^{-t}\sin t + 5t^3e^{-t}\cos t$.

Exercise Write down the form of a particular solution to the equation $y''' + 2y'' + y' = 5e^{-t}\sin t + 3 + 7te^{-t}$.

4.6 Variation of Parameters

The Method of Undetermined Coefficients is a procedure for determining a particular solution when the equation has constant coefficients and the nonhomogeneous term is of a special type.

Variation of Paramters is a more general method for finding a particular solution.

Consider a linear second-order equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$$

in the standard form

$$y'' + P(x)y' + Q(x)y = f(x)$$

Obtain the solution to the associated homogeneous equation

$$y = c_1y_1(x) + c_2y_2(x)$$

And replace the constants with functions

$$y = u_1y_1(x) + u_2y_2(x)$$

Substituting into the DE yields the system:

$$\begin{aligned} y_1u_1' + y_2u_2' &= 0 \\ y_1'u_1' + y_2'u_2' &= f(x) \end{aligned}$$

By Cramer's Rule, the solution can be expressed in terms of determinants-

$$u_1' = \frac{W_1}{W} = \frac{-y_2 f(x)}{W} \qquad u_2' = \frac{W_2}{W} = \frac{y_1 f(x)}{W}$$

The functions u_1 and u_2 are found by integrating.

A particular solution is $y_p = u_1 y_1 + u_2 y_2$.

Example

Find a general solution on $(-\frac{\pi}{2}, \frac{\pi}{2})$ to $\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + y = \tan t$.

The standard form is $y'' + y = \tan t$.

The solutions to the homogeneous equation is $r = \pm i$, so $\alpha = 0$ and $\beta = 1$, so $y_1 = \cos t$ and $y_2 = \sin t$.

Using what was previously introduced, we end up with $y = c_1 \cos t + c_2 \sin t - \cos t \ln |\sec t + \tan t|$

Exercise Find a general solution on $(-\frac{\pi}{2}, \frac{\pi}{2})$ to $\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + y = \tan t + 3t - 1$.

4.7 Variable-Coefficient Equations

We now consider equations with variable coefficients of the form

$$a_2(t)y'' + a_1(t)y' + a_0(t)y = f(t)$$

Typically, the equation is divided by the nonzero coefficient $a_2(t)$ and is expressed in standard form

$$y''(t) + p(t)y' + q(t)y(t) = g(t)$$

Theorem 4.6

Suppose $p(t), q(t)$, and $g(t)$ are continuous on an interval (a, b) that contains the point t_0. The, for any choice of the initial values Y_0 and Y_1, there exists a unique solution $y(t)$ on the same interval (a, b) to the initial value problem

$$y''(t) + p(t)y'(t) + q(t)y(t) = g(t), \qquad y(t_0) = Y_0, \qquad y'(t_0) = Y_1$$

Example

Determine the largest interval for which the above theorem ensures the existence and uniqueness of a solution to the initial value problem

$$(t-3)y'' + y' + \sqrt{t}y = \ln t \qquad y(1) = 3, \qquad y'(1) = -5$$

First, let's put this into standard form. $y'' + \frac{1}{t-3}y' + \frac{\sqrt{t}}{t-3}y = \frac{\ln t}{t-3}$.

From this we can see that this is only continuous from $(0, 3)$.

Definition

A linear second-order equation that can be expressed in the form

$$at^2 y''(t) + bty'(t) + cy = f(t)$$

where a, b, and c are constants, is called a Cauchy-Euler, or equidimensional, equation.

To solve a Cauchy-Euler equation, for $t > 0$ look for solutions of the form

$$y = t^r$$

and substitute into the homogeneous form

$$at^2y''(t) + bty'(t) + cy = 0$$

the resulting equation $ar^2 + (b - a)r + c = 0$ is called the associated characteristic equation.

Example

Find two linearly independent solutions to the equation

$$3t^2y'' + 11ty' - 3y = 0 \qquad t > 0$$

The solutions are $y = t^r$ and plugging this into the equation results in $3r^2 + 8r - 3$, which factors to $r = \frac{1}{3}$ and $r = -3$.

The solutions are $y = t^{1/3}$ and $y = t^{-3}$.

If the roots of the associated characteristic equation r are equal, then independent solutions of the Cauchy-Euler equation on $(0, \infty)$ are given by

$$y_1 = t^r \qquad \text{and} \qquad y^2 = t^r \ln t$$

IF the roots are complex, $r = \alpha \pm \beta i$, then the independent solutions are given by

$$y_1 = t^a \cos(\beta \ln t) \qquad \text{and} \qquad y_2 = t^\alpha \sin(\beta \ln t)$$

Example

Find a pair of linearly independent solutions to the Cauchy-Euler equations for $t > 0$.

$t^2y'' + 5ty' + 5y = 0$

Answer: $y_1 = t^{-2}\cos(\ln t)$, $y_2 = t^{-2}\sin(\ln t)$

Exercise Do the same thing for $t^2y'' + ty' = 0$.

Lemma 4.7

If $y_1(t)$ and $y_2(t)$ are any two solutions to the homogeneous differential equation

$$y''(t) + p(t)y'(t) + q(t)y(t) = 0$$

on an interval I where the functions $p(t)$ and $q(t)$ are continuous and if the Wronksian is zero at any point t of I, then y_1 and y_2 are linearly dependent on I.

Theorem 4.8

If $y_1(t)$ and $y_2(t)$ are any two solutions to the homogeneous differential equation that are linearly independent on an interval I containing t_0, then unique constants c_1 and c_2 can always be found so that $c_1y_1(t) + c_2y_2(t)$ satisfies the initial conditions $y(t_0) = Y_0$, $y'(t_0) = Y_1$ for any Y_0 and Y_1.

$y_h = c_1y_1 + c_2y_2$ is called a general solution to the homogeneous differential equation $y''(t) + p(t)y'(t) + q(t)y(t) = 0$ if y_1 and y_2 are linearly independent solutions on I.

For the nonhomogeneous equation $y''(t) + p(t)y'(t) + q(t)y(t) = g(t)$ a general solution on I is given by $y = y_p + y_h$ where $y_h = c_1y_1 + c_2y_2$ is a general solution to the corresponding homogeneous equation on I and y_p is a particular solution on I.

If linearly independent solutions to the homogeneous equation are known, y_p can be determined by the variation of parameters method.

Theorem 4.9

If y_1 and y_2 are two linearly independent solutions to the homogeneous equation on an interval I where $p(t)$, $q(t)$, and $g(t)$ are continuous, then a particular solution is given by $y_p = u_1 y_1 + u_2 y_2$, where u_1 and u_2 are determined up to a constant by the pair of equations, $y_1 v_1' + y_2 v_2' = 0$, $y_1' v_1' + y_2' v_2' = g$, which have the solutions

$$v_1(t) = \int \frac{-g(t) y_2(t)}{W(y_1, y_2)(t)} \mathrm{d}t$$

and

$$v_2(t) = \int \frac{g(t) y_1(t)}{W(y_1, y_2)(t)} \mathrm{d}t$$

Note that the formulation above presumes that the differential equation has been put into standard form (that is divided by $a_2(t)$).

Theorem 4.10

Let $y_1(t)$ be a solution, not identically zero, to the homogeneous equation in an interval I. Then

$$y_2(t) = y_1(t) \int \frac{e^{-\int p(t) \mathrm{d}t}}{y_1(t)^2} \mathrm{d}t$$

is a second, linearly independent solution.

Example

Given that $y_1(t) = t$ is a solution to

$$y'' - \frac{1}{t} y' + \frac{1}{t^2} y = 0$$

use the Reduction of Order formula to determine a second linearly independent solution for $t > 0$.

y_2 is equal to $t \int \frac{e^{\ln t}}{t^2} \mathrm{d}t = t \ln t$ from the formula, so the general solution is $y = c_1 t + c_2 t \ln t$.

Exercise The following equation arises in the mathematical modeling of reverse osmosis.

$$(\sin t) y'' - 2(\cos t) y' - (\sin t) y = 0, \qquad 0 < t < \pi$$

Find a general solution.

5 Introduction to Systems

Apologies for no examples since I have really bad allergies as of writing this.

5.1 Differential Operators and the Elimination Method for Systems

$y'(t) = \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}y$ is used to emphasize that the derivative of a function y is the result of operation on the function y with the differentiation operator $\frac{\mathrm{d}}{\mathrm{d}t}$. Similarly $y''(t) = \frac{\mathrm{d}^2y}{\mathrm{d}t^2} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}t}y$. Commonly the symbol D is used instead of $\frac{d}{\mathrm{d}t}$.

$y'' + 4y' + 3y = 0$ is represented by $(D^2 + 4D + 3)[y] = 0$.

We use the convention that the operator "product" is a composition when it operates on functions. *Exercise* Show that the operator $(D + 1)(D + 3)$ is the same as $D^2 + 4D + 3$.

Exercise Show that $(D + 3t)D$ is not the same as $D(D + 3t)$.

Because the "algebra" of differential operators follows the same rules as the algebra of polynomials, we can adapt the elimination method used to solve algebraic operations to solve any system of linear differential equations with constant coefficients.

Example

Solve the system

$$x'(t) = 3x(t) - 4y(t) + 1$$
$$y'(t) = 4x(t) - 7y(t) + 10t$$

First we can solve this by writing this in differential operator form:

$$(D - 3)x + 4y = 1$$
$$-4x + (D + 7)y = 10t$$

By eliminating this using algebra, we get that $(D^2+4D-5)y = 14-30t$ (this is essentially $y''+4y-5y = 14 - 30t$).

The first step to solving this nonhomogeneous equation is to solve the homogeneous equation.

The homogeneous equation results in giving us that $y_h = c_1e^{-5t} + c_2e^t$.

We can guess the particular solution as $y_p = At + B$, so $y'_p = A$ and $y''_p = 0$. So we have that $0 + 4A - 5(At + B) = 14 - 30t$. From this, we have $4A - 5B = 14$ and $-5A = -30$, so $A = 6$ and $B = 2$.

So the general solution for y is $y = C_1e^{-5t} + C_2e^t + 6t + 2$.

Now we need to find the function x.

We can find x from elimination once again.

Using the same methods above, we have $x = k_1e^{-5t} + k_2e^t + 8t + 5$.

We should only end up with two constants, so we need to find the relationship between the constants of y and x.

So using the derivative of x we get $-5k_1e^{-5t} + k_2e^t + 8 = 3k_1e^{-5t} + 3k_2e^t + 24t + 15 - 4C_1e^{-5t} - 4C_2e^t - 24t - 8 + 1$.

Simplifying we get $-5k_1e^{-5t} - k_2e^t = (3k_1 - 4C_1)e^{-5t} + (3k_2 - 4C_2)e^t$. So we have that $-5k_1 = 3k_1 - 4C_1$ and $k_2 = 3k_2 - 4C_2$, so $C_1 = 2k_1$ and $C_2 = \frac{1}{2}k_2$.

To find a general solution for the system

$$\begin{aligned} L_1[x] + L_2[y] &= f_1 \\ L_3[x] + L_4[y] &= f_2 \end{aligned}$$

where L_1, L_2, L_3, and L_4 are polynomials in $D = \mathrm{d}/\mathrm{d}t$.

- Make sure that the system is written in operator form.
- Eliminate one of the variables, say, y, and solve the resulting equation for $x(t)$. If the system is degenerate, stop! A separate analysis is required to determine whether or not there are solutions.
- (Shortcut) If possible, use the system to derive an equation that involves $y(t)$ but not its derivatives (Otherwise go to the next step). Substitute the found expression for $x(t)$ into this equation to get a formula for $y(t)$. The expressions for $x(t)$, $y(t)$ give the desired general solution.
- Eliminate x from the system and solve for $y(t)$. Solving for $y(t)$ gives more constants - in fact, twice as many as needed.
- Remove the extra constants by substituting the expressions for $x(t)$ and $y(t)$ into one or both of the equations in the system. Write the expressions for $x(t)$ and $y(t)$ in terms of the remaining constants.

Example

Find a general solution to

$$\begin{aligned} x''(t) + y'(t) - x(t) + y(t) &= -1 \\ x'(t) + y'(t) - x(t) &= t^2 \end{aligned}$$

Subtracting the two, we get that $x'' - x' + y = -1 - t^2$.

We will now solve for x. In differential operator form we have $(D^2 - 1)x + (D + 1)y = -1$ and $(D - 1)x + Dy = t^2$.

Eliminating for x gives the $x_h = C_1e^t + C_2te^t + C_3e^{-t}$.

The particular solution will be in the form $At^2 + Bt + C$. The first derivative of this is $2At + B$ and the second derivative is $2A$.

Plugging this in gives us that $(D^2-2D+1)(D+1)x = -2t-t^2$. This is equal to $(D^3-D^2-D+1)x = -2t - t^2$.

This is $2-2A-2At-B+At^2+Bt+C = -2t-t^2$. From this we see that $A = -1, B = -4, C = -6$.

The particular solution $x_p = -t^2 - 4t - 6$.

Now since we have $x = c_1e^t + C_2te^t + C_3e^{-t} - t^2 - 4t - 6$, we can solve this for the y.

Plugging the general solution in gives us $y = -c_1e^t - c_2e^t - c_2e^t - c_2te^t - c_1e^{-t} + 2 + c_1e^t + c_2e^t + c_2te^t - c_3e^{-t} - 2t - 4 - 1 - t^2$.

Simplifying this gives $y = -c_2e^t - 2c_3e^{-t} - t^2 - 2t - 3$.

5.2 Solving Systems and Higher-Order Equations Numerically

If equations have variable coefficients, the solution process is limited. The solutions can be expressed as infinite series which can be very laborious (with the exception of the Cauchy-Euler equation). Fortunately all cases, constant and variable coefficient, nonlinear and higher order equations and systems can be addressed numerically.

We will express differential equations as a system in normal form and used the basic Euler method for computer solution that can be "vectorized" to apply to such systems.

A system of m differential equations in the m unknown functions $x_1(t), x_2(t), \ldots, x_m(t)$ expressed as

$$\begin{aligned} x_1'(t) &= f_1(t, x_1, x_2, \ldots, x_m) \\ x_2'(t) &= f_2(t, x_1, x_2, \ldots, x_m) \\ &\vdots \\ x_m'(t) &= f_m(t, x_1, x_2, \ldots, x_m) \end{aligned}$$

is said to be in normal form. It can be expressed in vector form as $x' = t(t, x)$.

A single higher-order equation can always be converted to an equivalent system of first-order equations. To convert an mth-order differential equation

$$y^{(m)}(t) = f(t, y, y', \ldots, y^{(m-1)})$$

into a first-order system, introduce additional unknowns, the sequence of derivatives of y:

$$x_1(t) = y(t), x_2(t) = y'(t), \ldots, x_m(t) = y^{(m-1)}(t)$$

We obtain this system

$$\begin{aligned} x_1'(t) = y'(t) &= x_2(t) \\ x_2'(t) = y''(t) &= x_3(t) \\ &\vdots \\ x_{m-1}'(t) = y^{(m-1)}(t) &= x_m(t) \\ x_m'(t) = y^{(m)}(t) &= f(t, x_1, x_2, \ldots, x_m) \end{aligned}$$

Example

Convert the initial value problem

$$y''(t) + 3ty'(t) + y(t)^2 = \sin t \qquad y(0) = 1, y'(0) = 5$$

into an initial value problem for a system in normal form.

We have $x_1 = y, x_1' = y', x_2 = y'$ and $x_2' = y''$.

In normal form, we would have $y'' = \sin t - 3ty' + y^2$ which is what x_2' and y'' are equal to. Note that this ia function of $f(t, y, y')$.

We also have $x_1' = y' = x_2$ and we also know that $\sin t - 3tx_2 + x_1^2 = x_2'$.

We now have a system where $x_1' = x_2, x_2' = \sin t - 3tx_2 + x_1^2$ where we know $x_1(0) = 1$ and $x_2(0) = 5$.

6 Theory of Higher-Order Linear Differential Equations

6.1 Basic Theory of Linear Differential Equations

A linear differential equation of order n is an equation that can be written in the form

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_0(x)y(x) = b(x)$$

where $a_0(x), a_1(x), \ldots, a_n(x)$ and $b(x)$ depend on on x, not y. When $a_0, a_1, \ldots, a_n$ are all constants, we say this equation has constant coefficients; otherwise it has variable coefficients. If $b(x) = 0$, this equation is called homogeneous; otherwwise it is nonhomogeneous.

We assume $a_0(x), a_1(x), \ldots, a_n(x)$ and $b(x)$ are all continuous on an interval I and $a_n(x) \neq 0$ on I.

We can rewrite the equation in standard form

$$y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \cdots + p_n(x)y(x) = g(x)$$

where the functions $p_1(x), \ldots, p_n(x)$, and $g(x)$ are continuous on I.

Theorem 6.1

Suppose $p_1(x) \ldots p_n(x)$ and $g(x)$ are continuous on an interval (a, b) that contains the point x_0. Then, for any choice of the initial values, $\gamma_0, \gamma_1, \ldots, \gamma_{n-1}$, there exists a unique solution $y(x)$ on the whole interval (a, b) to the initial value problem

$$y^{(n)} + p_1(x)y^{(n-1)}(x) + \cdots + p_n(x)y(x) = g(x)$$

$$y(x_0) = \gamma_0, y'(x_0) = \gamma_1 \ldots y^{(n-1)}(x_0) = \gamma_{n-1}$$

Example

For the initial value problem

$$\begin{gathered} x(x-1)y''' - 3xy'' + 6x^2y' - (\cos x)y = \sqrt{x+5} \\ y(x_0) = 1, \qquad y'(x_0) = 0, \qquad y''(x_0) = 7 \end{gathered}$$

determine the values of x_0 and the intervals (a, b) containing x_0 for which the above theorem guarantees the existence of a unique solution on (a, b).

We know that $x \neq 0, 1$ from $x(x-1)$.

In standard form this becomes $y''' - \frac{3x}{x(x-1)}y'' + \frac{6x^2}{x(x-1)}y' - \frac{\cos x}{x(x-1)}y = \frac{\sqrt{x+5}}{x(x-1)}$.

These functions will be continuous when $x \neq 0$ and $x \neq 1$. We also know that $x \geq -5$ from the last term.

The intervals are $(-5, 0)$, $(0, 1)$ and $(1, \infty)$ in which all the x_0 can have an element from.

If we let the left-hand side of equation in the standard form define the differential operator L,

$$L[y] = \frac{\mathrm{d}^n y}{\mathrm{d}x^n} + p_1\frac{\mathrm{d}^{n-1}y}{\mathrm{d}x^{n-1}} + \cdots + p_n y = (D^n + p_1 D^{n-1} + \cdots + p_n)[y]$$

then the standard form equation can be expressed in the operator form

$$L[y](x) = g(x)$$

Keep in mind that L is a linear operator - that is, it satisfies

$$L[y_1 + y_2 + \cdots + y_m] = L[y_1] + L[y_2] + \cdots + L[y_m]$$
$$L[cy] = cL[y]$$

where c is any constant.

Definition: Wronksian

Let $f_1, \ldots, f_n$ be any n functions that are $(n-1)$ times differentiable.

The function

$$W[f_1, f_2, \ldots, f_n] = \begin{vmatrix} f_1 & f_2 & \ldots & f_n \\ f_1' & f_2' & \ldots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \ldots & f_n^{(n-1)} \end{vmatrix}$$

is called the Wronksian of $f_1, \ldots f_n$.

Theorem 6.2

Let $y_1, \ldots y_n$ be n solutions on (a, b) of

$$y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \cdots + p_n(x)y(x) = 0$$

where $p_1, \ldots, p_n$ are continuous on (a, b). If at some point x_0 in (a, b) these solutions satisfy

$$W[y_1, \ldots, y_n](x_0) \neq 0$$

then every solution of the above equation on (a, b) can be expressed in the form

$$y(x) = C_1 y_1(x) + \cdots + C_n y_n(x)$$

where $C_1, \ldots, C_n$ are constants.

Definition: Linear Dependence of Functions

The M functions $f_1, f_2, \ldots, f_m$ are said to be linearly dependent on an interval I if at least one of them can be expressed as a linear combination of the others on I; equivalently, they are linearly dependent if there exist constants $c_1, c_2, \ldots, c_m$, not all zero, such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_m f_m(x) = 0$$

for all x in I. Otherwise, they are said to be linearly independent on I.

Example

Show that the functions $f_1(x) = e^x$, $f_2(x) = e^{-2x}$, and $f_3(x) = 3e^x - 2e^{-2x}$ are linearly dependent on $(-\infty, \infty)$.

We can see that $f_3 = 3f_1 - 2f_2$. We can see that f_3 is a linear combination of the other two functions.

We have a set of constants, not all zero that $c_1 f_1 + c_2 f_2 + c_3 f_3 = 0$ from $3f_1 - 2f_2 - 1f_3$, so the set of $\{f_1, f_2, f_3\}$ is linearly dependent.

If you do the Wronksian of the functions: $W[f_1, f_2, f_3]$, we get 0 which eans that it is linearly dependent. The process of writing the Wronksian takes a lot of paper, so it is easier likely to do the $c_1 f_1 + c_2 f_2 +$

$\cdots + c_n f_n = 0$ method.

To prove that functions $f_1, f_2, \ldots, f_m$ are linearly independent, a convenient approach is to assume the equation defined in the linear dependence definition holds and show that this forces $c_1 = c_2 = \cdots = c_m = 0$.

Example

Show that the functions $f_1(x) = x, f_2(x) = x^2$, and $f_3(x) = 1 - 2x^2$ are linearly independent on $(-\infty, \infty)$.

Assume $c_1 f_1 + c_2 f_2 + c_3 f_3 = 0$. If we can show this, then we can show its independence.

From this we will get $c_1 x + c_2 x^2 + c_3(1 - 2x^2) = 0$.

If we let $x = 0$, we get $c_3 = 0$.

If we let $x = 1$, we get $c_1 + c_2 - c_3 = 0$ and if we let $x = -1$, we get $-c_1 + c_2 - c_3 = 0$.

From this we see that $c_1 + c_2 = 0$ and $-c_1 + c_2 = 0$.

The functions are linearly independent when c_1, c_2 and c_3 are equal to 0, so x, x^2, and $1 - 2x^2$ are linearly independent.

There are other ways to do this as well.

Theorem 6.3

If $y_1, y_2, \ldots, y_n$ are n solutions to $y^{(n)} + p_1 y^{(n-1)} + \cdots + p_n y = 0$ on the interval (a, b), with $p_1, p_2, \ldots, p_n$ continuous on (a, b), then the following statements are equivalent

1. $y_1, y_2, \ldots, y_n$ are linearly dependent on (a, b).
2. The Wronksian $W[y_1, y_2, \ldots, y_n](x_0)$ is zero at some point x_0 in (a, b).
3. The Wronksian $W[y_1, y_2, \ldots, y_n](x)$ is identically zero on (a, b).

The contrapositives of these statements are also equivalent:

1. $y_1, y_2, \ldots, y_n$ are linearly independent on (a, b).
2. The Wronksian $W[y_1, y_2, \ldots, y_n](x_0)$ is nonzero at some point x_0 in (a, b).
3. The Wronksian $W[y_1, y_2, \ldots, y_n](x)$ is never zero on (a, b)

Whenever the last 3 are met, $\{y_1, y_2, \ldots, y_n\}$ is called a fundamental solution set for linear independence theorem on (a, b).

It is useful to keep in mind the following sets consist of functions that are linearly independencet on every open interval (a, b):

$$\{1, x, x^2, \ldots, x^n\}$$

$$\{1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots, \cos nx, \sin nx\}$$

$$\{e^{\alpha_1 x}, e^{\alpha_2 x}, \ldots, e^{\alpha_n x}\}$$

where α_i are distinct constants.

Theorem 6.4

Let $y_p(x)$ be a particular solution to the nonhomogeneous equation

$$y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \cdots + p_n(x)y(x) = g(x)$$

on the interval (a,b) with $p_1, p_2, \ldots, p_n$ continuous on (a,b), and let $\{y_1, \ldots, y_n\}$ be a fundamental solution set for the corresponding homogeneous equation

$$y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \cdots + p_n(x)y(x) = 0$$

Then every solution of the original nonhomogeneous equation on the interval (a, b) can be expressed in the form

$$y(x) = y_p(x) + C_1 y_1(x) + \cdots + C_n y_n(x)$$

Example

Find a general solution on the interval $(-\infty, \infty)$ to

$$L[y] = y''' - 2y'' - y' + 2y = 2x^2 - 2x - 4 - 24e^{-2x}$$

given that $y_{p_1}(x) = x^2$ is a particular solution to $L[y] = 2x^2 - 2x - 4, y_{p_2}(x) = e^{-2x}$ is a particular solution to $L[y] = -12e^{-12x}$, and that $y_1(x) = e^{-x}, y_2(x) = e^x$, and $y_3(x) = e^{2x}$ are solutions to the corresponding homogeneous equation.

We know that $\{e^{-x}, e^x, e^{2x}\}$ is a fundamental solution set for homogeneous equations so we have $C_1 e^{-x} + C_2 e^x + C_3 e^{2x}$.

We know that $L[x^2] = 2x^2 - 2x - 4$ and $L[e^{-2x}] = -12e^{-2x}$. From the former, we have $L[2e^{-2x}] = -24e^{-2x}$.

We know that $Ly_p = 2x^2 - 2x - 4 - 24e^{-2x}$. We also know that $L[x^2 - 2e^{-2x}] = L[x^2] - 2L[e^{-2x}] = 2x^2 - 2x - 4 - 24e^{-2x}$.

The solution of the nonhomogeneous equation is $x^2 - 2e^{-2x}$.

The general solution is therefore $y(x) = x^2 - 2x^{-2x} + C_1 e^{-2x} + C_2 e^x + C_3 e^{2x}$.

6.2 Homogeneous Linear Equations with Constant Coefficients

Consider the homogeneous linear nth-order differential equation with constant coefficients

$$a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \cdots + a_1 y'(x) + a_0 y(x) = 0$$

e^{rx} is a solution to the equation, provided r is a root of the auxiliary (or characteristic equation)

$$P(r) = a_n r^n + a_{n-1} r^{n-1} + \cdots + a_0 = 0$$

Distinct real roots: If the roots $r_1, r_2, \ldots, r_n$ of the auxiliary equation are real and distinct, then the n solutions to the first equation defined are

$$y_1(x) = e^{r_1 x}, \quad y_2(x) = e^{r_2 x} \quad , \ldots, \quad , y_n(x) = e^{r_n x}$$

Example

Find a general solution to

$$y''' - 2y'' - 5y' + 6y = 0$$

Using the auxiliary equation we get $r^3 - 2r^2 - 5r + 6 = 0$ from this.

From algebra, we know that the possible roots are $\pm 1, \pm 2, \pm 3, \pm 6$.

Let's assume $r = 1$ is a solution. From synthetic division, we see that $r = 1$ is a root. Now we can see that $(r - 1)(r^2 - r - 6)$ is a solution.

Factoring this gives $(r - 1)(r - 3)(r + 2)$.

The general solution is $y = C_1 e^x + C_2 e^{3x} + C_3 e^{-2x}$.

Looking at complex roots: If $\alpha + i\beta(\alpha, \beta$ real) is a complex root of the auxiliary equation, then so is its complex conjugate $\alpha - i\beta$. If we accept complex-valued functions as solutions, then both $e^{(\alpha + i\beta)x}$ and

$e^{(\alpha-i\beta)x}$ are solutions to the original homogeneous linear equation. The real-valued functions (which are linearly independent) corresponding to the complex roots $\alpha \pm i\beta$ are

$$e^{\alpha x}\cos(\beta x), e^{\alpha x}\sin(\beta x)$$

Example

Find a general solution to

$$y''' + y'' + 3y' - 5y = 0$$

The auxiliary equation is $r^3 + r^2 + 3r - 5$.

The possible roots are $\pm 1, \pm 5$.

We know that 1 works from synthetic division and we get $(r-1)(r^2+2r+5)$.

From $r^2 + 2r + 5$, we get that $-1 \pm 2i$ are the roots of this.

We get $c_1 e^x + c_2 e^{-x}\cos 2x + c_3 e^{-x}\sin 2x$.

If r_1 is a root of multiplicity m, then the m linearly independent solutions are

$$e^{r_1 x}, \quad xe^{r_1 x}, \quad x^2 e^{r_1 x}, \ldots, x^{m-1}e^{r_1}x$$

If $\alpha + i\beta$ is a repeated complex root of multiplicity m, then the $2m$ linearly independent real-valued solutions are

$$e^{\alpha x}\cos(\beta x), xe^{\alpha x}\cos(\beta x), \ldots, x^{m-1}e^{\alpha x}\cos(\beta x)$$
$$e^{\alpha x}\sin(\beta x), xe^{\alpha x}\sin(\beta x), \ldots, x^{m-1}e^{\alpha x}\sin(\beta x)$$

Example

Find a general solution to

$$y^{(4)} - y^{(3)} - 3y'' + 5y' - 2y = 0$$

The auxiliary equation is $r^4 - r^3 - 3r^2 + 5r - 2 = 0$.

The possible roots are $\pm 1, \pm 2$.

We know that $r = 1$ works from plugging in. Using synthetic division, we get $(r-1)(r^3 - 3r + 2)$.

From the $r^3 - 3r + 2$ term, we can factor this to $(r-1)(r^2 + r - 2)$.

The auxiliary equation ends up being $(r-1)^3(r+2)$.

The general solution ends up being $c_1 e^x + C_2 xe^x + C_3 x^2 e^x + C_4 e^{-2x}$.

Example

Find a general solution to

$$y^{(4)} - 8y^{(3)} + 26y'' - 40y' + 25y = 9$$

The auxiliary equation is $r^4 - 8r^3 + 26r^2 - 40r + 25 = 0$.

Let's assume we are told that $r_1 = 2 + i$ and $r_2 = 2 - i$.

This means that $(r - (2 + i))(r - (2 - i)) = r^2 - 4r + 5$ is a factor.

Dividing $r^4 - 8r^3 + 26r^2 - 40r + 25$ from this gives us $r^2 - 4r + 5$.

We know the roots are $2 + i, 2 - i, 2 + i, 2 - i$.

Since $2 + i$ and $2 - i$ have multiplicity of two, then the solution is $y = C_1 e^{2x} \cos x + C_2 e^{2x} \sin x + C_3 x e^{2x} \cos x + C_4 x e^{2x} \sin x$.

6.3 Undetermined Coefficients and the Annihilator Method

Previously we used the Method of Undetermined Coefficients to find a particular solution to a nonhomogeneous linear second-order constant coefficient equation

$$L[y] = (aD^2 + bD + c)[y] = f(x)$$

when $f(x)$ had a particular form (a product of a polynomial, an exponential, and a sinusoid) by observing a solution form y_p must resemble f. We also had to make accomodations when y_p was a solution to the homogeneous equation $L[y] = 0$.

The annihilator method uses the observation that suitable types of nonhomogeneities $f(x)$ are themselves solutions to homogeneous differential equations with constant coefficients.

1. Any nonhomogeneous term of the form $f(x) = e^{rx}$ satisfies $(D - r)[f] = 0$
2. Any nonhomogeneous term of the form $f(x) = x^k e^{rx}$ satisfies $(D-r)^m[f] = 0$ for $k = 0, 1, \ldots, m-1$.
3. Any nonhomogeneous term of the form $f(x) = \cos \beta x$ or $\sin \beta x$ satisfies $(D^2 + \beta^2)[f] = 0$
4. Any nonhomogeneous term of the form $f(x) = x^k e^{\alpha x} \cos \beta x$ or $x^k e^{\alpha x} \sin \beta x$ satisfies $[(D - \alpha)^2 + \beta^2]^m[f] = 0$ for $k = 0, 1, \ldots, m - 1$.

We have that D^n annihilates polynomial of degree $n - r$.

We have that $D - r$ annihilates e^{rx}.

We have that $(D - r)^k$ annihilates $x^{k-1} e^{rx}$

We have that $D^2 - 2\alpha D + (\alpha^2 + \beta^2)$ annihilates $e^{\alpha x} \cos \beta x, e^{\alpha x} \sin \beta x$.

If have a power of x^{k-1} to the above, then raise the above to the power of k to annihilate this. If we just have $\cos \beta x$ or $\sin \beta x$, then the operator becomes $D^2 + \beta^2$.

Definition

A linear differential operator A is said to annihilate a function f if

$$A[f](x) = 0$$

for all x. That is, A annihilates f if f is a solution to the homogeneous linear differential equation above on $(-\infty, \infty)$.

Example

Find a differential operator that annihilates

$$6xe^{-4x} + 5e^x \sin 2x$$

We know that $(D+4)^2$ will annihilate $6xe^{-4x}$.

We saw that the form that annihilates the other part of the equation is $D^2 - 2\alpha D + (\alpha^2 + \beta^2)$.

We know that $\alpha = 1$ and $\beta = 2$.

The operator that will annihilate that term is $D^2 - 2D + 5$, so this term annihilates $5e^x \sin 2x$.

The sum will be annihilated by multiplying $(D+4)^2$ and $(D^2 - 2D + 5)$.

Example

Find a general solution to

$$y'' - y = xe^x + \sin x$$

Method 1: Undetermined Coefficients

The homogeneous equation is $m^2 - 1$, so the solution to the homogeneous equation is $y_c = c_1 e^x + c_2 e^{-x}$

The form of the particular solution looks like $y_p = (Ax + B)e^x + C \sin x + D \cos x$.

Let's find the form of xe^x first.

We have that $y_p = (Ax + B)e^x$, then the derivative is $Ae^x + (Ax + B)e^x = Axe^x + (A + B)e^x$. The second derivative is $Ae^x + Axe^x + (A + B)e^x = Axe^x + (2A + B)e^x$.

Plugging this in gives $Axe^x + (2A + B)e^x - (Ax + B)e^x = xe^x$. We end up getting $2Ae^x = xe^x$.

Because of the overlap with the homogeneous equation, the particular solution is actually $y_p = x(Ax + B)e^x = (Ax^2 + Bx)e^x$.

The first derivative of this is $(2Ax + B)e^x + (Ax^2 + Bx)e^x = [Ax^2 + (2A + B)x + B]e^x$. The second derivative is $(2Ax + 2A + B)e^x + [Ax^2 + (2A + B)x + B]e^x = [Ax^2 + (4A + B)x + (2A + 2B)]e^x$.

Plugging this in gives $[Ax^2 + (4A + B)x + (2A + 2B)]e^x - (Ax^2 + Bx)e^x = xe^x$.

Simplifying this gives $4A = 1$ and $2A + 2B = 0$. From this we get $A = 1/4$ and $B = -1/4$.

The solution for $y_p = (1/4x^2 - 1/4x)e^x = x(\frac{1}{4}x - \frac{1}{4})e^x$.

Now we need to solve the other part of y_p.

Doing derivatives and plugging in stuff we get $C = -1/2$ and $D = 0$, so $y_p = x(\frac{1}{4}x - \frac{1}{4})e^x - \frac{1}{2}\sin x$.

Therefore $y = c_1 e^x + c_2 e^{-x} + x(-\frac{1}{4}x - \frac{1}{4})e^x - \frac{1}{2}\sin x$

Method 2: Annihilator Method We know that $(D-1)^2$ annihilates xe^x.

We know that for $\sin x$ the form is $D^2 - 2\alpha D + (\alpha^2 + \beta^2)$.

So $D^2 + 1$ annihilates $\sin x$.

$(D-1)^2(D^2+1)$ annihilates $xe^x + \sin x$.

Rewrite the equation using differential operator notation. We end up getting $(D^2 - 1)y = xe^x + \sin x$.

This gives $(D+1)(D-1)y = xe^x + \sin x$. Applying $(D-1)^2(D^2+1)$ to both sides, we get $(D+1)(D-1)^3(D^2+1)y = (D-1)^2(D+1)[xe^x + \sin x]$.

We get that $(D+1)(D-1)^3(D^2+1)y = 0$.

We would have $y = c_1 e^{-x} + c_2 e^x + c_3 xe^x + c_4 x^2 e^x + c_5 \sin x + c_6 \cos x$ as the general solution to the homogeneous equation.

The particular solution is exactly what we got in the same form using the annihilator method.

Belpw for me later *Exercise* Find a general solution to $y''' - 3y'' + 4y = xe^2x$ pls later anastasia come back

6.4 Method Of Variation of Parameters

The method of undetermined coefficients and the annihilator method work only for linear equations with constant coefficients and when the nonhomogeneous term is a solution to some homogeneous linear equation with constant coefficients. The method of variation of parameters discussed in chapter 4 generalizes to higher-order linear equations with variable coefficients.

Our goal is to find a solution to the standard form equation

$$L[y](x) = g(x)$$

where $L[y] = y^{(n)} + p_1 y^{(n-1)} + \cdots + p_n y$ and the coefficient functions $p_1, p_2, \ldots, p_n$ as well as g are continuous on (a, b).

A general solution to $L[y](x) = 0$ is $y_h(x) = C_1 y_1(x) + \cdots + C_n y_n(x)$.

In the method of variation of parameters, there exists a particular solution to the standard form equation of the form

$$y_p(x) = v_1(x)y_1(x) + \cdots + v_n(x)y_n(x)$$

The functions $v_1', v_2', \ldots, v_n'$ must satisfy the system

$$\begin{aligned} y_1 v_1' + \cdots + y_n v_n' &= 0 \\ \vdots \quad\ \vdots\ \vdots\ \vdots\ &\vdots \\ y_1^{(n-2)} v_1' + \cdots + y_n^{(n-2)} v_n' &= 0 \\ y_1^{(n-1)} v_1' + \cdots + y_n^{(n-1)} v_n' &= g \end{aligned}$$

Solving the system using Cramer's Rule, we find that $v_k'(x) = \frac{g(x)W_k(x)}{W[y_1,\ldots,y_n](x)}$ where $k = 1, \ldots, n.$ and where $W_k(x)$ is the determinant of a matrix obtained from the Wronksian $W[y_1, \ldots, y_n](x)$ by replacing the kth column by Col$[0, \ldots, 0, 1]$.

Example

Find a general solution to the Cauchy-Euler equation

$$x^3 y''' + x^2 y'' - 2xy' + 2y = x^3 \sin x, \qquad x > 0$$

From Cauchy Euler, we see that $y = x^r, y' = rx^{r-1}, y'' = r(r-1)x^{r-2}, y''' = r(r-1)(r-2)x^{r-3}$.

Plugging this in the homogeneous equation gives $x^3 \cdot r(r-1)(r-2)x^{r-3} + x^2 r(r-1)x^{r-2} - 2xrx^{r-1} + 2x^r = 0$.

This gives $x^r[r^3 - 3r^2 + 2r] + x^r[r^2 - r] - 2x^r[r] + 2x^r = 0$. Factoring this gives $x^r[r^3 - 3r^2 + 2r + r^2 - r - 2r + 2] = 0$.

Assuming $x \neq 0$, we get $r^3 - 2r^2 - r + 2 = 0$. Factoring this gives $(r-2)(r-1)(r+1) = 0$.

The general solution to the homogeneous equation is $y = c_1 x^2 + c_2 x^{-1} + c_3 x$.

From above we see that $y_1 = x^2, y_2 x^{-1}, y_3 = x$.

The particular solution will be of the form $y_p = v_1 x^2 + v_2 x^{-1} + v_3 x$.

Starting with the Wronksian of x^2, x^{-1}, x.

Before, we get $g(x) = \frac{x^3 \sin x}{x^3} = \sin x$. This comes from dividing the $x^3 \sin x$ by the leading coefficient.

The next determinant for v_1 is the same as the original Wronksian above, but the first column has $0, 0, \sin x$ instead of $x^2, 2x, 2$.

The determinant for v_2 is the same as the original, but the second column is replaced by $0, 0, \sin x$ instead of $x^{-1}, -x^{-2}, 2x^{-3}$.

The determinant for v_3 is the same as the original, but the third column is replaced by $0, 0 \sin x$ instea dof $x, 1, 0$.

For the original Wronksian, we get $x(4x^{-2} + 2x^{-2}) - 1(2x^{-1} - 2x^{-1}) = 6x^{-1} = W$.

For the Wronksian of v_1, we get $\sin x(x^{-1} + x^{-1}) = 2x^{-1} \sin x$.

For the Wronksian of v_2, we get $-\sin x(x^2 - 2x^2) = x^2 \sin x$.

For the Wronksian of v_3, we get $\sin x(-1 - 2) = -3 \sin x$.

So we get $v_1' = \frac{W_1}{W} = \frac{2x^{-1} \sin x}{6x^{-1}} = \frac{1}{3} \sin x$

We get $v_2' = \frac{W^2}{W} = \frac{x^2 \sin x}{6x^{-1}} = \frac{1}{6} x^3 \sin x$

We get $v_3' = \frac{W^3}{W} = \frac{-3 \sin x}{6x^{-1}} = -\frac{1}{2} x \sin x$.

Integrating, we get $v_1 = -\frac{1}{3} \cos x$.

For v_3, we have $-\frac{1}{2} \int x \sin x \mathrm{d}x$. Let $u = x$ and $\mathrm{d}v = \sin x$. From this, $v = -\cos x$ and $\mathrm{d}u = 1$.

Integrating by parts should give $v_3 = \frac{1}{2} x \cos x - \frac{1}{2} \sin x$.

For v_2, we get $u = x^3$ then $3x^2, 6x, 6, 0$. and for $\mathrm{d}v$ we get $\sin x, -\cos x, \sin x, -\cos x, \sin x$.

This is tabular integration by parts. We get $v_2 = \frac{1}{6}[-x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x]$.

Simplifying this gives $v_2 = -\frac{1}{6} x^3 \cos x + \frac{1}{2} x^2 \sin x + x \cos x - \sin x$.

We can now get y_p.

Yea, so y_p is simply $y_p = [-\frac{1}{3} \cos x]x^2 + [-\frac{1}{6} x^3 \cos x + \frac{1}{2} x^2 \sin x + x \cos x - \sin x]x^{-1} + [\frac{1}{2} x \cos x - \frac{1}{2} \sin x]x$.

Simplifying this gives $\cos x - x^{-1} \sin x$.

So the general solution is $y = \cos x - x^{-1} \sin x + c_1 x^2 + c_2 x^{-1} + c_3 x$.

7 Laplace Transforms

7.1 Definition of the Laplace Transform

Definition

Let $f(t)$ be a function on $[0, \infty)$. The Laplace transform of f is the function F defined by the integral

$$F(s) = \int_0^\infty e^{-st} f(t) \mathrm{d}t$$

The domain of $F(s)$ is all the values of s for which the integral above exists. The Laplace transform of f is denoted by both F and $\mathcal{L}\{f\}$.

Example

Determine the Laplace transform of the constant function $f(t) = 1, t \geq 0$.

Let $F(s) = \int_0^\infty e^{-st} 1 \mathrm{d}t = \int_0^\infty e^{-st} \mathrm{d}t$. This is equal to $-\frac{1}{s}e^{-st}$ with bounds ∞ and 0.

Remember this is an improper integral where we have $\lim_{b\to\infty} -\frac{1}{s}e^{-st}$ from 0 to b.

This gives $-\frac{1}{s}e^{-sb} - \frac{1}{s}e^{0}$ on the inside of the limit, so we get $\lim_{b\to\infty} \left[-\frac{1}{s}e^{-sb} + \frac{1}{s}\right]$.

The above equals $\lim_{b\to\infty} \left[-\frac{1}{s} \cdot \frac{1}{e^{rb}} + \frac{1}{s}\right]$.

The restriction is $s > 0$ because $\frac{1}{e^{sb}}$ has to be greater than 0.

Our result ends up being $\frac{1}{s}$.

$\mathcal{L}\{1\} = \frac{1}{s}$.

Example

Determine the Laplace transform of $f(t) = t$.

We have $\mathcal{L}\{t\} = \int_0^\infty e^{-st} t \mathrm{d}t = \lim_{b\to\infty} \left[\int_0^b e^{-st} t \mathrm{d}t\right]$.

Integrating by parts gives the inside equal to $-\frac{1}{s} \cdot t \cdot \frac{1}{e^{st}} - \frac{1}{s^2}e^{-st}$ with bounds 0 to b.

Plugging this in gives $\lim_{b\to\infty} -\frac{1}{s} \cdot \frac{b}{e^{sb}} - \frac{1}{s} \cdot \frac{1}{e^{sb}} + \frac{1}{s^2}$.

We see that $\frac{b}{e^{sb}}$ is indeterminate, so using L'Hopital's Rule, the derivative is $\frac{1}{se^{sb}}$ and the limit as b approaches ∞ gives this as 0.

We are left with $\frac{1}{s^2}$.

$\mathcal{L}\{t\} = \frac{1}{s^2}$.

We will see that $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$.

Example

Determine the Laplace transform of $f(t) = e^{at}$, where a is a constant.

The integral is $\int_0^\infty e^{-st} \cdot e^{at} \mathrm{d}t = \int_0^\infty e^{-(s-a)t} \mathrm{d}t$.

Integrating this gives $-\frac{1}{s-a}e^{-(s-a)t}$ evaluated from 0 to ∞.

As t goes to infinity, we get 0 and then we get $0 - \frac{-1}{s-a}e^{0} = \frac{1}{s-a}$.

So $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$.

If we were to find the Laplace of e^{5t}, from the above example it would be $\frac{1}{s-5}$.

Example

Find $\mathcal{L}\{\sin bt\}$, where b is a nonzero constant.

The integral this time is $\int_0^\infty e^{-st} \cdot \sin bt \mathrm{d}t$.

Integrating gives $-\frac{1}{s}\sin bte^{-st} + \frac{b}{s}\left[-\frac{1}{s}\cos bte^{-st} - \int -\frac{1}{s}e^{-st}(-b)\sin bt\mathrm{d}t\right]$.

(Do this example later)

Involves factoring Laplace stuff.

$\mathcal{L}\{\sin bt\} = \frac{b}{s^2+b^2}$.

Example

Determine the Laplace transform of

$$f(t) = \begin{cases} 2 & 0 < t < 5 \\ 0 & 5 < t < 10 \\ e^{4t} & t > 10 \end{cases}$$

To do this, you just do $\int_0^\infty e^{-st} f(t)\mathrm{d}t = \int_0^5 e^{-st} \cdot 2\mathrm{d}t + \int_5^{10} e^{-st} \cdot 0\mathrm{d}t + \int_1 0^\infty e^{-st} \cdot e^{4t}\mathrm{d}t$.

Evaluating this gives the laplace as $-\frac{2}{s}e^{-5s} + \frac{2}{s} + \frac{1}{s-4}e^{-(s-4)10}$

An important property of the Laplace transform is its linearity. That is, the Laplace transform $\mathcal{L}$ is a linear operator.

Theorem 7.1

Let f, f_1, and f_2 be functions whose Laplace transforms exist for $s > \alpha$ and let c be a constant. Then, for $s > \alpha$,

$$\mathcal{L}\{f_1 + f_2\} = \mathcal{L}\{f_1\} + \mathcal{L}\{f_2\}$$
$$\mathcal{L}\{cf\} = c\mathcal{L}\{f\}$$

Exercise Determine $\mathcal{L}\{11 + 5e^{4t} - 6\sin 2t\}$.

A function $f(t)$ on $[a, b]$ is said to have a jump discontinuity at $t_0 \in (a, b)$ if $f(t)$ is discontinuous at t_0, but the one-sided limits

$$\lim_{t \to t_0^-} f(t) \quad \text{and} \quad \lim_{t \to t_0^+} f(t)$$

exist as finite numbers.

Definition

A function $f(t)$ is said to be piecewise continuous on a finite interval $[a, b]$ if $f(t)$ is continuous at every point in $[a, b]$, except possibly for a finite number of points at which $f(t)$ has a jump discontinuity.

A function $f(t)$ is said to be piecewise continuous on $[0, \infty)$ if $f(t)$ is piecewise continuous on $[0, N]$ for all $N > 0$.

In contrast, the function $f(t) = 1/t$ is not piecewise continuous on any interval containing the origin, since it has an "infinite jump" at the origin.

A function that is piecewise continuous on a finite interval is not necessarily integrable over that interval. However, piecewise continuity on $[0, \infty)$ is not enough to guarantee the existence (as a finite number) of the improper integral over $[0, \infty)$; we also need to consider the growth of the integrand for large t. The Laplace transform of a piecewise continuous function exists, provided the function does not grow "faster than an exponential".

Definition

A function $f(t)$ is said to be of exponential order α if there exist positive constants T and M such that

$$|f(T)| \leq Me^{\alpha t}$$

for all $t \geq T$.

Theorem 7.2

If $f(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order α, then $\mathcal{L}\{f\}(s)$ exists for $s > a$.

Here are common Laplace transforms:

- $\mathcal{L}\{1\} = \frac{1}{s}$
- $\mathcal{L}\{t\} = \frac{1}{s^2}$
- $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$
- $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$
- $\mathcal{L}\{\sin bt\} = \frac{b}{s^2+b^2}$
- $\mathcal{L}\{\cos bt\} = \frac{s}{s^2+b^2}$

7.2 Properties of the Laplace Transform

Theorem 7.3

If the Laplace transform $\mathcal{L}\{f\}(s) = F(s)$ exists for $s > \alpha$, then

$$\mathcal{L}\{e^{\alpha t} f(t)\}(s) = F(s - a)$$

for $s > \alpha + a$

Example

Determine the Laplace transform of $e^{\alpha t} \sin bt$

We know the Laplace of $\sin bt$ is equal to $\frac{b}{s^2+b^2}$.

Multiplying by $e^{\alpha t}$ just shifts it $F(s - \alpha) = \frac{b}{(s-\alpha)^2+b^2}$

Theorem 7.4

Let $f(t)$ be continuous on $[0, \infty)$ and $f'(t)$ be piecewise continuous on $[0, \infty)$, with both of exponential order α. Then for $s > \alpha$,

$$\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0)$$

Theorem 7.5

Let $f(t), f'(t), \ldots, f^{(n-1)}(t)$ be continuous on $[0, \infty)$ and let $f^{(n)}(t)$ be piecewise continuous on $[0, \infty)$, with all these functions of exponential order α. Then, for $s > \alpha$,

$$\mathcal{L}\{f^{(n)}\}(s) = s^n \mathcal{L}\{f\}(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$$

Example

Using the above theorems and the fact that $\mathcal{L}\{\sin bt\}(s) = \frac{b}{s^2+b^2}$, determine $\mathcal{L}\{\cos bt\}$

We know that $f'(t) = b \cos bt$ from this. So $\mathcal{L}\{b \cos bt\} = s\mathcal{L}\{\sin bt\} - f(0)$.

We know that $b\mathcal{L}\{\cos bt\} = s\mathcal{L}\{\sin bt\}$, since $f(0) = 0$.

So simplifying gives the Laplace transform as $\frac{s}{s^2+b^2}$

Example

Prove the following identity for continous functions $f(t)$ (assuming the transforms exist):

$$\mathcal{L}\left\{\int_0^t f(\tau)\mathrm{d}\tau\right\}(s) = \frac{1}{s}\mathcal{L}\{f(t)\}(s)$$

We know $g(t) = \int_0^t f(\tau)\mathrm{d}\tau$. From this we know $g'(t) = f(t)$.

We get that $\mathcal{L}\{g'(t)\} = s\mathcal{L}\{g(t)\} - g(0)$. and that $\mathcal{L}\{f(t)\} = s\mathcal{L}\{\int_0^t f(\tau)\mathrm{d}\tau\}$.

We also know $g(0) = 0$.

So the Laplace of the function is equal to $\frac{1}{s}\mathcal{L}\{f(t)\}$.

Theorem 7.6

Let $F(s) = \mathcal{L}\{f\}(s)$ and assume $f(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order α. Then, for $s > \alpha$,

$$\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \frac{\mathrm{d}^n F}{\mathrm{d}s^n}(s)$$

Example

Determine $\mathcal{L}\{t \sin bt\}$.

We know $f(t) = \sin bt$ and that $n = 1$.

This is equal to $(-1)^1 \frac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}\{\sin bt\}$.

We end up getting $-\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{b}{s^2+b^2}\right)$.

We end up getting $\frac{2bs}{(s^2+b^2)^2}$.

Here are some basic properties of Laplace Transforms

- $\mathcal{L}\{f + g\} = \mathcal{L}\{f\} + \mathcal{L}\{g\}$.
- $\mathcal{L}\{cf\} = c\mathcal{L}\{f\}$ for any constant c.
- $\mathcal{L}\{e^{at} f(t)\}(s) = \mathcal{L}\{f\}(s - a)$
- $\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0)$

- $\mathcal{L}\{f''(s)\} = s^2\mathcal{L}\{f\}(s) - sf(0) - f'(0)$
- $\mathcal{L}\{f^{(n)}\}(s) = s^n\mathcal{L}\{f\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$
- $\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}s^n}(\mathcal{L}\{f\}(s))$

7.3 Inverse Laplace Transform

Example

Solve the initial value problem

$$y'' - y = -t \qquad y(0) = 0, \qquad y'(0) = 1$$

We can say that $\mathcal{L}\{y'' - y\} = \mathcal{L}\{-t\}$.

Using properties we know that $\mathcal{L}\{y''\} - \mathcal{L}\{y\} = -\mathcal{L}\{t\}$

This is equal to $s^2\mathcal{L}\{y\} - sy(0) - y'(0) = \mathcal{L}\{y\} = -\frac{1}{s^2}$.

Now plugging in $\mathcal{L}\{y(t)\} = Y(s)$, we get $s^2Y(s)1 - Y(s) = -\frac{1}{s^2}$

Simplifying gives $Y(s)(s^2 - 1) = \frac{s^2-1}{s^2}$.

We see that $Y(s) = \frac{1}{s^2}$. This is the Laplace of t, so $y(t) = t$.

Definition

Given a function $F(s)$, if there is a function $f(t)$ that is cintinuous on $[0, \infty)$ and satisfies

$$\mathcal{L}\{f\} = F$$

then we say that $f(t)$ is the inverse Laplace transform of $F(s)$ and employ the notation $f = \mathcal{L}^{-1}\{F\}$.

Example

Determine $\mathcal{L}\{F\}$ for $F(s) = \frac{2}{s^2}$.

The Inverse Laplace transform of this is t^2.

Determine it for $F(s) = \frac{3}{s^2+9}$.

This is $\sin 3t$ from the definition.

Determine it for $\frac{s-1}{s^2-2s+5}$.

This simplifies to $\frac{s-1}{(s-1)^2+4} = F(s-1)$. This is the same as $\cos 2t$ but shifted by 1. The Inverse Laplace transform ends up being $e^t \cos 2t$.

Theorem 7.7

Assume that $\mathcal{L}^{-1}\{F\}, \mathcal{L}^{-1}\{F_1\}$, and $\mathcal{L}^{-1}\{F_2\}$ exist and are continuous on $[0, \infty)$ and let c be any constant. Then

$$\mathcal{L}^{-1}\{F_1 + F_2\} = \mathcal{L}^{-1}\{F_1\} + \mathcal{L}^{-1}\{F_2\}$$
$$\mathcal{L}^{-1}\{cF\} = c\mathcal{L}^{-1}\{F\}$$

Example

Determine $\mathcal{L}^{-1}\left\{\frac{5}{s-6}-\frac{6s}{s^2+9}+\frac{3}{2s^2+8s+10}\right\}$.

The first two terms of this gives $5e^6t - 6\cos 3t$.

For the last term, We see that $\frac{1}{2(s^2+4s+5)}$ lets us put $\frac{3}{2}$ in the front and we can complete the square for this for the denominator to give $\frac{1}{(s+2)^2+1}$.

The last term ends up being $\frac{3}{2}e^{-2t}\sin t$.

Exercise Determine $\mathcal{L}^{-1}\{\frac{5}{s+2}^4\}$

Exercise Determine $\mathcal{L}^{-1}\{\frac{3s+2}{s^2+2s+10}\}$.

Method of Partial Fractions - A rational function of the form $\frac{P(s)}{Q(s)}$, where $P(s)$ and $Q(s)$ are polynomials with the degree of P less than the degree of Q has a partial fraction expansion whose form is based on the linear and quadratic factors of $Q(s)$. We consider the three cases:

1. Nonrepeated linear factors
2. Repeated linear factors
3. Quadratic factors

Nonrepeated Linear Factors - If $Q(s)$ can be factored into a product of distinct linear factors, $Q(s) = (s-r_1)(s-r_2)\dots(s-r_n)$, where the r_i's are all distinct real numbers, then the partial fraction expansion has the form

$$\frac{P(s)}{Q(s)} = \frac{A_1}{s-r_1} + \frac{A_2}{s-r_2} + \cdots + \frac{A_n}{s-r_n}$$

where the A_i's are real numbers.

Example

Determine $\mathcal{L}^{-1}\{F\}$, where $F(s) = \frac{7s-1}{(s+1)(s+2)(s-3)}$.

The decomposition is equal to $\frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s-3}$.

Solving for A, B, C gives $2, -3, 1$ respectively.

We end up getting $\frac{2}{s+1} + \frac{-3}{s+2} + \frac{1}{s-3}$. This gives us $2e^{-t} - 3e^{-2t} + e^{3t}$.

Repeated Linear Factors - Let $s-r$ be a factor of $Q(s)$ and suppose $(s-r)^m$ is the highest power of $s-r$ that divides $Q(s)$. Then the portion of the partial fraction expansion of $P(s)/Q(s)$ that corresponds to the term $(s-r)^m$ is

$$\frac{A_1}{s-r} + \frac{A_2}{(s-r^2)} + \cdots + \frac{A_m}{(s-r)^m}$$

where the A_i's are real numbers.

Example

Determine $\mathcal{L}\left\{\frac{s^2+9s+2}{(s-1)^2(s+3)}\right\}$.

We end up getting $\frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+3}$.

Solving for A, B, C gives $2, 3, -1$ respectively.

This gives $2e^t + 3t^t t - e^{-3t}$.

Quadratic Factors - Let $(s-\alpha)^2 + \beta^2$ be a quadratic factor of $Q(s)$ that cannot be reduced to linear factors with real coefficients. Suppose m is the highest power of $(s-\alpha)^2 + \beta^2$ that divides $Q(s)$. THen the portion of the partial fraction expansion that corresponds to $(s-\alpha)^2 + \beta^2$ is

$$\frac{C_1 s + D_1}{(s-\alpha)^2 + \beta^2} + \frac{C_2 s + D_2}{[(s-\alpha)^2 + \beta^2]^2} + \cdots + \frac{C_m s + D_m}{[(s-\alpha)^2 + \beta^2]^m}$$

When looking up Laplace transforms, the following equivalent form is more convenient

$$\frac{A_1(s-\alpha)+\beta B_1}{(s-\alpha)^2+\beta^2}+\frac{A_2(s-\alpha)\beta B_2}{[(s-\alpha)^2+\beta^2]^2}+\cdots+\frac{A_m(s-\alpha)+\beta B_m}{[(s-\alpha)^2+\beta^2]^m}$$

Example

Determine $\mathcal{L}^{-1}\left\{\frac{2s^2+10s}{(s^2-2s+5)(s+1)}\right\}$.

The partial fraction is $\frac{As+B}{(s^2-2s+5)}+\frac{C}{s+1}$.

Solving the system gives $A, B, C = 3, 5, -1$.

So we are now finding the Laplace transform of $\frac{3s+5}{(s-1)^2+4}0\frac{1}{s+1}$.

The first term of this can be rewritten as $\frac{3(s-1)+8}{(s-1)^2+4}$.

The transform ends up being $3e^t\cos 2t+4e^t\sin 2t-e^{-t}$.

7.4 Solving Initial Value Problems

Method of Laplace Transforms

To solve initial value problems:

- Take the Laplace transforms of both sides of the equation
- Use the properties of the Laplace transform and the initial conditions to obtain an equation for the Laplace transform of the solution and then solve this equation for the transform
- Determine the inverse Laplace transform of the solution by looking it up in a table or by using a suitable method (such as partial fractions) in combination with the table.

Example

Solve the initial value problem

$$y''-2y'+5y=-8e^{-t} \qquad y(0)=2, \qquad y'(0)=12$$

This is equal to $\mathcal{L}\{y''\}-2\mathcal{L}\{y'\}+5\mathcal{L}\{y\}=-8\mathcal{L}\{e^{-t}\}$.

This ends up being $s^2\mathcal{L}\{y\}-sy(0)-y'(0)-2[s\mathcal{L}\{y\}-y(0)]+5\mathcal{L}\{y\}=-8\frac{1}{s+1}$.

We know that $\mathcal{L}\{y\}=Y(s)$.

So $Y(s)[s^2-2s+5]-2s-12+4=\frac{-8}{s+1}$.

This is $Y(s)(s^2-2s+5)=2s+8-\frac{8}{s+1}$.

This ends up being $Y(s)=\frac{2s}{s^2-2s+5}+\frac{8}{s^2-2s+5}-\frac{8}{(s+1)(s^2-2s+5)}$.

Simplifying ends up getting $\frac{2s^2+10s}{(s+1)(s^2-2s+5)}$.

Doing partial fraction decomposition gives $\frac{3s+5}{s^2-2s+5}+\frac{-1}{s+1}=\frac{3(s-1)+8}{(s-1)^2+4}+\frac{-1}{s+1}$.

The Inverse Laplace of this is $3e^t\cos 2t+4e^t\sin 2t-e^{-t}$.

Exercise Solve the initial value problem

$$y''+4y'-5y=te^t \qquad y(0)=1 \qquad y'(0)=0$$

Example

Solve the initial value proiblem

$$w''(t) - 2w'(t) + 5w(t) = -8e^{\pi - t} \qquad w(\pi) = 2 \qquad w'(\pi) = 12$$

Let's introduce a new function $y(t) = w(t + \pi)$.

Replace t with $t + \pi$ in this equation and we get $w''(t + \pi) - 2w'(t + \pi) + 5w(t + \pi) = -8e^{\pi-(t+\pi)}$.

Substituting the derivatives gives $y''(t) - 2y'(t) + 5y(t) = -8e^{-t}$.

This basically comes out to $y = 3e^t \cos 2t + 4e^t \sin 2t - e^{-t}$.

Replacing everything with $t - \pi$ gives $3e^{t-\pi} \cos 2(t - \pi) + 4e^{t-\pi} \sin 2(t - \pi) - e^{-(t-\pi)} = y(t - \pi)$.

This gives $w(t) = 3e^{t-\pi} \cos 2t + 4e^{t-\pi} \sin 2t - e^{-(t-\pi)}$.

7.5 Transforms of Discontinuous Functions

Definition

The unit step function $u(t)$ is defined to by

$$u(t) := \begin{cases} 0, & t < 0, \\ 1, & 0 < t \end{cases}$$

Example

Graph $u(t), u(t - a)$, and $Mu(t - a)$.

The graph of $u(t)$ is just as given above.

The graph of $u(t - a)$ is just a horizontal shift.

The graph of $Mu(t - a)$ will just have the one with 1 multiplied by M

Definition

The rectangular window function $\prod_{a,b}(t)$ is defined by

$$\prod_{a,b}(t) := u(t - a) - u(t - b) = \begin{cases} 0, & t < a \\ 1, & a < t < b \\ 0, & b < t \end{cases}$$

Example

Write the function

$$f(t) = \begin{cases} 3 & t < 2 \\ 1 & 2 < t < 5 \\ t & 5 < t < 8 \\ t^2/10 & 8 < t \end{cases}$$

In terms of window and step functions.

This is $3\prod_{0,2}(t) + 1\prod_{2,5}(t) + t\prod_{5,8}(t) + \frac{t^2}{10}u(t - 8)$.

Also this can be written as $3u(t) - 2u(t-2) + (t-1)u(t-5) + (\frac{t^2}{10} - t)u(t-8)$.

$$\mathcal{L}\{u(t-a)\}(s) = \frac{e^{-as}}{s}$$

Theorem 7.8

Let $F(s) = \mathcal{L}\{f\}(s)$ exist for $s > \alpha \geq 0$. If a is a positive constant, then

$$\mathcal{L}\{f(t-a)u(t-a)\}(s) = e^{-\alpha s}F(s)$$

and, conversely, an inverse Laplace transform of $e^{-as}F(s)$ is given by

$$\mathcal{L}^{-1}\{e^{-asF(s)}\}(t) = f(t-a)u(t-a)$$

$$\mathcal{L}\{g(t)u(t-a)\}(s) = e^{-as}\mathcal{L}\{g(t+a)\}(s)$$

Example

Determine the Laplace transform of $t^2u(t-1)$.

$a =$ from here, and $g(t) = t^2$.

We take $\mathcal{L}\{g(t)u(t-1)\} = e^{-s} \cdot \mathcal{L}\{g(t+1)\}$.

Replacing $g(t)$ gives that $t^2 + 2t + 1$ for the inside, so the Answer ends up being $e^{-s} \cdot [\frac{2!}{s^3} + \frac{2}{s^2} + \frac{1}{s}]$.

Example

Determine $\mathcal{L}\{(\cos t)u(t-\pi)\}$.

This has $a = \pi$. So we can see that We are doing $e^{-\pi s}\mathcal{L}\{g(t+\pi)\}$.

$g(t) = \cos t$, so $g(t+\pi) = \cos(t+\pi) = \cos t \cos \pi - \sin t \sin \pi = -\cos t$.

So the Laplace is $e^{-\pi s} \cdot -1 \cdot \frac{s}{s^2+1}$.

Exercise Determine $\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2}\right\}$ and sketch its graph.

Example

The current I in an LC series circuit is governed by the initial value problem

$$I'' + 4I(t) = g(t) \qquad I(0) = 0 \qquad I'(0) = 0$$

where

$$g(t) = \begin{cases} 1 & 0 < t < 1 \\ -1 & 1 < t < 2 \\ 0 & 2 < t \end{cases}$$

Determine the current as a function of time t.

$g(t) = 1\prod_{0,1} + -1\prod_{1,2} = 1[u(t-0) - u(t-1)] - 1[u(t-1) - u(t-2)]$. This is equal to $g(t) = 1u(t-0) - 2u(t-1) + u(t-2)$.

This simplifies to $1 - 2u(t-1) + u(t-2)$

The Laplace of the initial value problem is $s^2\mathcal{L}\{I\} - sI(0) - I'(0) + 4\mathcal{L}\{I\} = \mathcal{L}\{1 - 2u(t-1) + u(t-2)\}$

We end up getting $(s^2+4)\mathcal{L}\{I\} = \frac{1}{s} - \frac{2e^{-s}}{s} + \frac{e^{-2s}}{s}$.

We get that $\mathcal{L}\{I\} = \frac{1}{s(s^2+4)} - 2e^{-s}\left[\frac{1}{s(s^2+4)}\right] + e^{-2s}\left[\frac{1}{s(s^2+4)}\right]$.

Using partial fraction decomposition of $\frac{1}{s(s^2+4)}$ gives $\frac{1}{4}\cdot\frac{1}{s} + -\frac{1}{4}\cdot\frac{s}{s^2+4}$.

If we call what we got above to be $F(s)$, we get $F(s) - 2e^{-s}F(s) + e^{-2s}F(s)$.

The inverse of what we have is $I = \mathcal{L}^{-1}\{F(s)\} - 2\mathcal{L}^{-1}\{e^{-s}F(s)\} + \mathcal{L}^{-1}\{e^{-2s}F(s)\}$.

Doing Laplace stuff gives $I = \frac{1}{4} - \frac{1}{4}\cos 2t - 2\left[\frac{1}{4} - \frac{1}{4}\cos 2(t-1)\right]u(t-1) + \left[\frac{1}{4} - \frac{1}{4}\cos 2(t-2)\right]u(t-2)$.

7.6 Transforms of Periodic and Power Functions

Definition

A function $f(t)$ is said to be periodic of period T $(\neq 0)$ if

$$f(t+T) = f(t)$$

for all t in the domain of f.

To specificy a periodic function, it is sufficient to give its values over one period.

The square wave function can be epxressed as

$$f(t) = \begin{cases} 1, & 0 < t < 1 \\ -1, & 1 < t < 2 \end{cases}$$

and $f(t)$ has period 2.

For convenience, we introduce a notation for a "windowed" version of a periodic function (using a rectangular window whose width is the period T)

$$f_T(t) := f(T)\prod_{0,T}(t) = f(t)[u(t) - u(t-T)] = \begin{cases} f(t), & 0 < t < T \\ 0, & \text{otherwise} \end{cases}$$

Theorem 7.9

If f has period T and is piecewise continuous on $[0, T]$, then the Laplace transform $F(s) = \int_0^\infty e^{-st} f(t)\mathrm{d}t$ and $F_T(s) = \int_0^T e^{-st} f(t)\mathrm{d}t$ are related by

$$F_T(s) = F(s)[1 - e^{-sT}]$$

or

$$F(s) = \frac{F_T(s)}{1 - e^{-st}}$$

Example

Determine $\mathcal{L}\{f\}$, where f is the square wave function.

The function of the step function gives

$$f_T(t) = 1 \prod_{0,1} + -1 \prod_{1,2} = u(t) - 2u(t-1) + u(t-2)$$

The Laplace of this gives $\frac{e^0}{s} - \frac{2e^{-s}}{s} + \frac{e^{-2s}}{s} = \frac{1-2e^{-s}+e^{-2s}}{s}$

$F(s)$ is just $\frac{F_T(s)}{1-e^{-2s}} = \frac{1-e^{-s}}{s(1+e^{-s})}$.

7.7 Convolution

Definition

Let $f(t)$ and $g(t)$ be piecewise continuous on $[0, \infty)$. The convolution of $f(t)$ and $g(t)$, denoted $f * g$, is defined by

$$(f * g)(t) := \int_0^t f(t - v)g(v)\mathrm{d}v$$

Example

Find the convolution of t and t^2.

Let $f(t) = t$ and $g(t) = t^2$

$t * t^2 = \int_0^t (t - v) \cdot v^2 \mathrm{d}v$

So let's integrate. We get $\frac{tv^3}{3} - \frac{v^4}{4}$. Putting in the bounds gives $\frac{t^4}{12}$.

Theorem 7.10

Let $f(t), g(t)$, and $h(t)$ be piecewise continuous on $[0, \infty)$. Then

- $f * g = g * f$
- $f * (g + h) = (f * g) + (f * h)$
- $(f * g) * h = f * (g * h)$
- $f * 0 = 0$

Theorem 7.11

Let $f(t)$ and $g(t)$ be piecewise continuous on $[0, \infty)$ and of exponential order α and set $F(s) = \{f\}(s)$ and $G(s) = \mathcal{L}\{g\}(s)$. Then

$$\mathcal{L}\{f * g\}(s) = F(s)G(s)$$

or, equivalently,

$$\mathcal{L}^{-1}\{F(s)G(s)\}(t) = (f * g)(t)$$

Example

Use the convolution theorem to solve the initial value problem

$$y'' + y = g(t) \qquad y(0) = 0 \qquad y'(0) = 0$$

where $g(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order.

We can get that $\mathcal{L}\{y''\} + \mathcal{L}\{y\} = G(s)$ from the problem.

Doing the Laplace transform gives $s^2Y(s) - sy(0) - y'(0) + Y(s) = G(s)$.

This simplifies to $(s^2 + 1)Y(s) = G(s)$.

So $Y(s) = \frac{1}{s^2+1} \cdot G(s)$.

Taking the Laplace transform of both sides gives us $y(t) = \mathcal{L}\{\frac{1}{s^2+1}G(s)\}$.

The right side is just $\sin t * g(t)$.

We know that $y(t) = \int_0^t \sin(t - v)g(v)\mathrm{d}v$ from this.

Example

Use the convolution theorem to find $\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\}$.

From the Convolution Theorem, we find that $\mathcal{L}\{F(s)G(s)\} = f(t) * g(t)$.

From that definition, the laplace is $\sin t * \sin t$.

This is $\int_0^t \sin(t - v) \cdot \sin v\mathrm{d}v$.

Note that $\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$.

So applying this, we get that $\frac{1}{2}\int_0^t \cos[t - v - v] - \cos[t - v + v]\mathrm{d}v$.

This is equal to $\frac{1}{2}\int \cos[-(2v - t)] - \cos t\mathrm{d}v$.

Remember that $\cos(-A) = \cos A$.

So we end up getting $\frac{1}{2}\int \cos(2v - t) - \cos t\mathrm{d}v$.

Integrating gives $\frac{1}{2}[\frac{1}{2}\sin(2v - t) - v\cos t]$ from 0 to t.

Simplifying this gives you $\frac{\sin t - t\cos t}{2}$

Example

Solve the integro-differential equation

$$y'(t) = 1 - \int_0^t y(t - v)e^{-2v}\mathrm{d}v \qquad y(0) = 1$$

The integral in the expression is just a convolution.

The integral is $y * e^{-2t}$.

The Laplace transform of both sides results in $\mathcal{L}\{y'(t)\} = \mathcal{L}\{1\} - \mathcal{L}\{y(t) * e^{-2t}\}$.

So this is $sY(s) - y(0) = \frac{1}{s} - \mathcal{L}\{y(t)\} \cdot \mathcal{L}\{e^{-2t}\}$.

This is $sY(s) - 1 = \frac{1}{s} - Y(s) \cdot \frac{1}{s+2}$.

$(s + \frac{1}{s+2})Y(s) = 1 + \frac{1}{s}$.

We end up getting $\frac{s^2+2s+1}{s+2}Y(s) = 1 + \frac{1}{s}$.

Factoring and solving for $Y(s)$ gives $\frac{s+2}{(s+1)^2} \cdot \frac{s+1}{s}$.

This gives us $\frac{s+2}{s(s+1)}$.

Doing the partial fraction decomposition gives us $2 = A$ and $1 = -B$.

So we end up getting $\frac{2}{s} - \frac{1}{s+1}$. Taking the inverse laplace transform of both sides gives us $2 - e^{-t}$.

7.8 Impulses and the Dirac Delta Function

Definition

The Dirac delta function $\delta(t)$ is characterized by the following two properties:

$$\delta(t) = \begin{cases} 0, & t \neq 0, \text{“infinite”} \quad t = 0 \end{cases}$$

and

$$\int_{-\infty}^{\infty} f(t)\delta(t)\mathrm{d}t = f(0)$$

for any function $f(t)$ that is continuous on an open interval containing $t = 0$.

By shifting the argument of $\delta(t)$, we have $\delta(t - a) = 0.t \neq a$, and

$$\int_{-\infty}^{\infty} f(T)\delta(t - a)\mathrm{d}t = f(a)$$

for any function $f(t)$ that is continuous on an interval containing $t = a$.

When $t_0 = 0$, we derive from the limiting properties of the $\mathcal{F}_n$'s of a “function” δ that satisfies the first equation of this topic and the integral condition

$$\int_{-\infty}^{\infty} \delta(t)\mathrm{d}t = 1$$

The Laplace transform of the Dirac Delta function can be equickly derived from the property given above from shifting the argeumtn. Since $\delta(t - a) = 0$ for $t \neq a$, then setting $f(t) = e^{-st}$ in that function, we find for $a \geq 0$

$$\int_{0}^{\infty} \delta(t - a)\mathrm{d}t = \int_{-\infty}^{\infty} e^{-st}\delta(t - a)\mathrm{d}t = e^{-as}$$

Thus, for $a \geq 0$,

$$\mathcal{L}\{\delta(t - a)\}(s) = e^{-as}$$

Example

Use the Laplace transform to solve the initial value-value problem

$$y' + y = \delta(t - 1), \qquad y(0) = 2$$

Taking the Laplace of both sides gives $sY(s) - y(0) + Y(s) = e^{-s}$.

Now we see that $Y(s) = \frac{1}{s+1}e^{-s} + \frac{2}{s+1}$.

This becomes $e^{-(t-1)}u(t-1) + 2e^{-t}$.

To write this as a piecewise function we can write this as $y(t) = \begin{cases} 2e^{-t} & 0 < t < 1 \\ e^{-t-1} + 2e^{-t} & t > 1 \end{cases}$.

Example

A mass attached to a spring is released from rest 1 m below the equilibrium position for the mass-spring system and begins to vibrate. After π seconds, the mass is struck by a hammer exerting an impulse on the mass. The system is governed by the symbolic initial value problem

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 9x = 3\delta(t-\pi); \qquad x(0) = 1, \qquad \frac{\mathrm{d}x}{\mathrm{d}t}(0) = 0$$

where $x(t)$ denotes the displacement from equilibrium at time t. Determine $x(t)$.

Doing the Laplace of the problem gives $s^2x(s) - s + 9x(s) = 3e^{-\pi s}$.

So we have $x(s) = \frac{s}{s^2+9} + \frac{3}{s^2+9}e^{-\pi s}$.

From this the inverse Laplace is $\cos(3t) + -\sin(3t)u(t-\pi)$.

7.9 Solving Linear Systems with Laplace Transforms

Example

Solve the initial value problem

$$\begin{aligned} x'(t) - 2y(t) &= 4t \qquad & x(0) &= 4 \\ y'(t) + 2y(t) - 4x(t) &= -4t - 2 \qquad & y(0) &= -5 \end{aligned}$$

Doing the Laplace of everything gives $sX(s) - x(0) - 2Y(s) = 4 \cdot \frac{1}{s^2}$ for the top equation and $sY(s) - y(0) + 2Y(s) - 4X(s) = -4 \cdot \frac{1}{s^2} - 2 \cdot \frac{1}{s}$ for the second equation.

After substituting we get

$$\begin{aligned} sX(s) - 2Y(S) &= \frac{4}{s^2} + 4 \\ -4X(s)(s+2)Y(s) &= -\frac{4}{s^2} - \frac{2}{s} - 5 \end{aligned}$$

By eliminating y, we get $X(s) = \frac{4s-2}{(s^2+2s-8)} = \frac{4s-2}{(s+4)(s-2)}$.

This is equivalent to $\frac{3}{s+4} + \frac{1}{s-2}$.

This gives us $x(t) = 3e^{-4t} + e^{2t}$.

We know from the problem that $y(t) = \frac{x'(t)-4t}{2}$.

So substituting values gives us $y(t) = \frac{1}{2}[-12e^{-4t} + 2e^{2t}] - 2t = -6e^{-4t} + e^{2t} - 2t$.

Example

Solve the initial value problem

$$x_1'' + 10x_1 - 4x_2 = 0$$
$$-4x_1 + x_2'' + 4x_2 = 0$$

subject to $x_1(0) = 0, \qquad x_1'(0) = 1, \qquad x_2(0) = 0, \qquad x_2'(0) = -1.$

The top equation's laplace transformation is $s_2x_1(s) - sx_1(0) - x_1'(0) + 10x_1(s) - 4x_2(s) = 0.$

The bottom equation becomes $-4x_1(s) + s^2x_2(s) - sx_2(0) - x_2'(0) + 4x_2(s) = 0.$

Solving the system of equations for $x_2(s)$ gives us $\frac{-s^2-6}{(s^2+12)(s^2+2)} = \frac{-2/5}{s^2+2} + \frac{-3/5}{s^2+12}.$

The Laplace gives $x_2(t) = -\frac{\sqrt{2}}{5}\sin(\sqrt{2}t) - \frac{\sqrt{3}}{10}\sin(2\sqrt{3}t).$

Doing the derivatives gives us $x_1 = -\frac{\sqrt{2}}{10}\sin(\sqrt{2}t) + \frac{\sqrt{3}}{5}\sin(2\sqrt{3}t).$

8 Series Solutions of Differential Equations

8.1 Introduction: The Taylor Polynomial Approximation

The best tool for numerically approximating a function $f(x)$ near a particular point x_0 is the Taylor polynomial.

The formula for the Taylor polynomial of degree n centered at x_0, approximating a function $f(x)$ possessing n derivatives x_0 is given by

$$p_n(x) = f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\frac{f'''(x_0)}{3!}(x-x_0)^3+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n = \sum_{j=0}^{n}\frac{f^{(j)}(x_0)}{j!}(x-x_0)^j$$

Example

Find the first four Taylor polynomials for e^x, expanded around $x_0 = 0$.

$p_n(x)$ is written as $f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3$.

Since we know the derivatives of $f(x) = e^x$ is just $e(x)$, $f^{(j)}(0) = 1$ for all of them.

This simplifies to $1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$.

The Taylor polynomial p_n is just the $(n+1)$st partial sum of the Taylor series

$$\sum_{j=0}^{\infty}\frac{f^{(j)}(x_0)}{j!}(x-x_0)^j$$

Example

Determine the fourth-degree Taylor polynomials matching the function $\cos x$ at $x_0 = 2$

So using what was previously given we have $f(2) + f'(2)(x-2) = \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3 + \frac{f^{(4)}}{4!}(x-2)^4$.

Filling in the $f^{(j)}$ values gives us $p_4(x) = \cos 2 - \sin 2(x-2) - \frac{\cos 2}{2}(x-2)^2 + \frac{\sin 2}{6}(x-2)^3 + \frac{\cos 2}{24}(x-2)^4$

Example

Find the first few Taylor polynomials approximating the solution around $x_0 = 0$ of the initiai value problem

$$y'' = 3y' + x^{7/2}y \qquad y(0) = 10 \quad y'(0) = 5$$

In general, this is just $y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \cdots + \frac{y^{(n)}(0)}{n!}x^n$.

Since we are given the problem, we know that $y''(0) = 3y'(0) + 0 = 15$.

As we continue taking derivatives with respect to x, we get $y''' = 3y'' + \frac{7}{3}x^{7/3}y + x^{7/3}y'$, and plugging in the numbrs gives us $y'''(0) = 45$.

Calculating the 4th derivative gives us 135.

The fifth derivative is no longer defined.

Example

Determine the Taylor polynomial of degree 3 for the solution to the initial value problem

$$y' = \frac{1}{x+y+1} \qquad y(0) = 0$$

Finding $y'(0)$ gives us 1, and finding $y''(0)$ gives us -2, and $y'''(0) = 10$.

We can estimate the accuracy to which a Taylor polynomial $p_n(x)$ approximates its target function $f(x)$ for x near x_0. The error $\epsilon_n(x)$ measures the accuracy of the approximation,

$$\epsilon_n(x) = f(x) - p_n(x)$$

and can be estimated by $\epsilon_n(x) = \frac{f^{(n+1)}(\aleph)}{(n+1)!}(x-x_0)^{n+1}$, where $\aleph$ is guaranteed to lie between x_0 and x if the $(n+1)$st derivative of f exists and is continuous on an interval containing x_0 and x.

8.2 Power Series and Analytic Functions

A power series about the point x_0 is an expression of the form

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \dots$$

where x is a variable and the a_n's are constants.

A power series is convergent at a specified value of x if its sequence of partial sums $\{S_N(x)\}$ converges, that is

$$\lim_{N\to\infty} S_N(x) = \lim_{N\to\infty} \sum_{n=0}^{N} a_n(x-x_0)^n$$

If the limit does not exist at x, then the series is said to be divergent.

Every power series has an interval of convergence. The interval of convergence is the set of all real numbers x for which the series converges. The center of the interval of convergence is the center x_0 of the series. Within its interval of convergence a power series converges absolutely. In other words, if x is in the interval of convergence and is not an endpoint of the interval, then the series of absolute values

$$\sum_{n=0}^{\infty} |a_n(x-x_0)^n|$$

converges.

Theorem 8.1

For each power series, there is a number $\rho \quad (0 \le \rho < \infty)$, called the radius of convergence of the power series, such that the series converges absolutely for $|x-x_0| < \rho$ and diverges for $|x-x_0| > \rho$. If the series converges for all values of x, then $\rho = \infty$. WHen the series converges only at x_0, then $\rho = 0$.

Theorem 8.2

If, for n large, the coefficients a_n are nonzero and satisfy

$$\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = L \qquad (0 \le L \le \infty)$$

then the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ is $\rho = L$.

Example

Determine the interval and radius of convergence of

$$\sum_{n=0}^{\infty} \frac{(-2)^n}{n+1}(x-3)^n$$

From the ratio test, the radius of convergence is $\rho = \frac{1}{2}$.

The interval of convergence is $|x-3| < \frac{1}{2}$.

So the interval is $-5/2 < x < 7/2$.

For $7/2$, it converges, so $-5/2 < x \le 7/2$.

Theorem 8.3

If $\sum_{n=0}^{\infty} a_n(x-x_0)^n = 0$ for all x in some open interval, then each coefficient a_n equals zero.

Theorem 8.4

If the series $f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$ has a positive radius of convergence ρ, then f is differentiable in the interval $|x-x_0| < \rho$ and termwise differentiation gives the power series for the derivative:

$$f'(x) = \sum_{n=1}^{\infty} na_n(x-x_0)^{n-1} \qquad \text{for} \qquad |x-x_0| < \rho$$

Furthermore, termwise integration gives the power series for the integral of f:

$$\int f(x)\mathrm{d}x = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-x_0)^{n+1} + C \qquad \text{for} \qquad |x-x_0| < \rho$$

Example

Starting with the geometric series for $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{n=0}^{\infty} x^n \qquad -1 < x < 1$ find a power series for each of the following functions.

(a) $\frac{1}{1+x^2}$

Replace x with $-x^2$ and we get the power series equal to

$1 - x^2 + x^4 - x^6 + x^8 + \cdots = \sum_{n=0}^{\infty}(-1)^n x^{2n} \qquad -1 < x < 1.$

(b) $\frac{1}{(1-x)^2}$

This becomes $1 + 2x + 3x^2 + \cdots = \sum_{n=1}^{\infty} nx^{n-1}$

(c) $\arctan x$ This becomes $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$

Example

Express the series $\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$ as a series where the generic term is x^k instead of x^{n-2}.

Let $k = n-2$, so $n = k+2$.

Plugging this in gives us $\sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k$.

Example

Show that $x^3 \sum_{n=0}^{\infty} n^2(n-2)a_n x^n = \sum_{n=3}^{\infty}(n-3)^2(n-5)a_{n-3}x^n$.

Let $k = n+3$, so $n = k-3$.

Doing stuff gives you the answer of $\sum_{n=3}^{\infty}(n-3)^2(n-5)a_{n-3}x^n$.

Exercise Show that the identity $\sum_{n=1}^{\infty} n a_{n-1}x^{n-1} + \sum_{n=2}^{\infty} b_n x^{n+1} = 0$ implies that $a_0 = a_1 = a_2 = 0$ and $a_n = -\frac{b_{n-1}}{(n+1)}$ for $n \geq 3$.

Definition

A function f is said to be analytic at x_0 if, in an open interval about x_0, this function is the sum of a power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ that has a positive radius of convergence.

A polynomial is analytic at every x_0. A rational function $P(x)/Q(x)$ where $P(x)$ and $Q(x)$ are polynomials without a common factor, is analytic except at those x_0 for which $Q(x_0) = 0$. The elementary functions $e^x, \sin x, \cos x$ are analytic for all x while $\ln x$ is analytic for $x > 0$. Familiar representations are

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}$$

$$\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n$$

where the first three are valid for all x, whereas the last is valid for x in $(0, 2]$.

8.3 Power Series Solutions to Linear Differential Equations

Definition

A point x_0 is called an ordinary point if both $p = a_1/a_2$ and $q = a_0/a_2$ are analytic at x_0. If x_0 is not an ordinary point, it is called a singular point of the equation.

Example

Determine all the singular points of

$$xy'' + x(1-x)^{-1}y' + (\sin x)y = 0$$

The form of this is $y'' + \frac{1}{1-x}y' + \frac{\sin x}{x}y = 0$.

$p(x) = \frac{1}{1-x}$ can be represented as a power series as well as $q(x) = \frac{\sin x}{x}$.

The only singular point is at $x = 1$.

Example

Find a power series solution about $x = 0$ to

$$y' + 2xy = 0$$

We are substituting around $y = \sum_{n=0}^{\infty} a_n x^n$. The derivative is $y' = \sum_{n=1}^{\infty} n \cdot a_n x^{n-1}$.

Substituting this in gives $\sum_{n=1}^{\infty} na_n x^{n-1} + 2x \sum_{n=0}^{\sum} a_n x^n = 0$.

When we are trying to get x^1 in the summations, we get $a_1 + \sum_{n=2} na + nx^{n-1} + \sum_{n=0} 2a_n x^{n+1} = 0$.

Simplifying this gives us $a_1 + \sum_{k=1}[(k+1)a_{k+1} + 2a_{k-1}]x^k = 0$.

We have $a_{k+1} = \frac{-2a_{k-1}}{k+1}$.

From the expanded form of y we have $a_0 x_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots$.

We already know $a_1 = 0$.

We can keep finding the formulas, $a_2 = \frac{-2}{2} a_0$, $a_4 = \frac{-2}{4} \cdot \frac{-2}{2} a_0$ and $a_6 = \frac{-2}{6} \cdot \frac{-2}{4} \cdot \frac{-2}{2} a_0$, and the odd k will result in 0.

We have $y = a_0 + \frac{-2}{2} a_0 x^2 + \frac{(-2)^2}{4 \cdot 2} a_0 x^4 + \frac{(-2)^3}{6 \cdot 4 \cdot 2} a_0 x^6 + \dots + \frac{(-3)^n}{2 \cdot n!} x^{2n}$.

We can also write this as $y = a_0 \sum_{n=0} \frac{(-1)^n}{n!} x^{2n}$, which ends up being $a_0 e^{-x^2}$.

Example

Find a general solution to

$$2y'' + xy' + y = 0$$

in the form of a power series about the ordinary point $x = 0$.

We have $y'' + \frac{x}{2} y' + \frac{1}{2} y = 0$.

There are no singular points here, so all points are ordinary.

We will find this with $y = \sum_{n=0}^{\infty} a_n x^n$ and $y' = \sum_{n=1} a_n n x^{n-1}$ and $y'' = \sum_{n=2} a_n n(n-1) x^{n-2}$.

Plugging this in gives $2 \sum_{n=2} a_n n(n-1) x^{n-2} + x \sum_{n=1} a_n n x^{n-1} + \sum_{n=0} a_n x^n = 0$.

This will simplify to $4a_2 + a_0 + \sum_{k=1}[2a_{k+2}(k+2)(k+1) + (k+1)a_k]x^k = 0$.

The recurrence formula ends up being $a_{k+2} = \frac{-a_k}{2(k+2)}$.

Let's look at $k = 1, k = 2, k = 3, k = 4$ until we find a pattern.

We also know $a_2 = -\frac{1}{4} a_0$.

We have that $a_3 = \frac{-a_1}{2 \cdot 3}, a_4 = -\frac{a_2}{2 \cdot 4}, a_5 = -\frac{a_3}{2 \cdot 5}, a_6 = -\frac{a_4}{2 \cdot 6}$.

We can write a_4 in terms of a_0 as $-\frac{1}{2 \cdot 4} \cdot -\frac{1}{4} a_0$ and $a_6 = -\frac{2 \cdot 6}{\cdot} - \frac{1}{2 \cdot 4} \cdot -\frac{1}{4} a_0$.

With these patterns we can write this as $a_{2n+1} = \frac{(-1)^n}{2^n[(2n+1) \cdot \ldots 1]}$.

Ok we know $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a^4 x^4 + a^5 x^5 + \dots$.

So we get this is equal to $a_0 + a_1 x - \frac{1}{4} a_0 x^2 - \frac{1}{6} a_1 x^3 + \frac{1}{32} a_0 x^4 + \frac{1}{60} a_1 x^5$.

This is a linear combination of a_0 and a_1.

Example

Find the first few terms in a power series expansion about $x = 0$ for a general solution to

$$(1 + x^2)y'' - y' + y = 0$$

Yea, a lot of stuff happen.

If you do previous steps of changing the indices and writing out the power series, we get

$[2a_2 - a_1 + a_0] + [6a_3 - 2a_2 + a_1]x + \sum_{k=2}[(k+2)(k+1)a_{k+2} + (k+1)a_{k+1} + (k^2 - k + 1)a_k]x^k = 0$

And then we can find $a_{k+2} = \frac{-(k+1)a_{k+1} - (k^2-k+1)a_k}{(k+2)(k+1)}$.

We also know $a_2 = \frac{1}{2}(a_1 - a_0)$ and $a_3 = \frac{1}{6}(2a_2 - a_1) =$ $frac{-}a_0 6$.

Doing many many steps gives you $y = a_0 + -\frac{1}{2}a_0x^2 - \frac{1}{6}a_0x^3 + \frac{1}{12}a_0x^4 + \frac{3}{40}a_0x^5 - \frac{17}{720}a_0x^6$ for the case of when $a_1 = 0$.

When $a_0 = 0$, then the equation just becomes $a_1[x + \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{40}x^5 + \frac{1}{20}x^6 + \ldots]$.

8.4 Equations with Analytic Coefficients

We start by stating a basic existence theorem for the equation

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0$$

Theorem 8.5

Suppose x_0 is an ordinary point for the equation. THen this equation has two linearly independent analytic solutions of the form

$$y(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$$

Moreover, the radius of convergence of any power series solution of the form given by the above is at least as large as the distance from x_0 to the nearest singular point (real or complex-valued) of the original equation.

Example

Find a minimum value for the radius of convergence of a power series solution about $x = 0$ to

$$2y'' + xy' + y = 0$$

So we have $y'' + \frac{x}{2}y' + \frac{1}{2}y = 0$.

There are no singular points, so the radius of convergence is $\rho = \infty$

Example

Find a minimum value for the radius of convergence of a power series solution about $x = 0$ to

$$(1 + x^2)y'' - y' + y = 0$$

This is $y'' - \frac{1}{1+x^2}y' + \frac{1}{1+x^2}y = 0$.

The singular points are $\pm i$.

The distance from 0 is 1, so $\rho = 1$.

Example

Find the first few terms in a power series expansion about $x = 1$ for a general solution to

$$2y'' + xy' + y = 0$$

Also determine the radius of convergence of the series.

We can let $t = x - 1$, and $x = 1$ and $t = 0$.

So we can get $y(x) = \sum_{n=0}^{\infty} a_n(x-1)^n$, so $Y(t) = y(x) = y(t+1)$.

We have $2\frac{\mathrm{d}^2Y}{\mathrm{d}t^2} + (t+1)\frac{\mathrm{d}Y}{\mathrm{d}t} + Y = 0$.

Substituting some of this stuff in gives $2\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + (t+1)\sum_{n=1}^{\infty} na_n t^{n-1} + \sum_{n=0}^{\infty} a_n t^n = 0$.

We need to break off some stuff, to simplify the sums.

We get $(4a_2 + a_1 + a_0)t^0 + \sum_{k=1} 2(k+2)(k+1)a_{k+2}t^k + \sum_{k=1} ka_k t^k + \sum_{k=1}(k+1)a_{k+1}t^k + \sum_{k=1} a_k t^k$.

We can get $a_{k+2} = \frac{-a_k - a_{k+1}}{2(k+2)}$.

We know of course that $Y(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \dots$.

We also know it's a linear combination, so $Y(t) = a_0(1 - \frac{1}{4}t^2 + \frac{1}{24}t^3 + \dots) + a_1(t - \frac{1}{4}t^2 - \frac{1}{8}t^3 + \dots)$

And just substitute $t = x - 1$ into the above to solve it.

8.5 Method of Frobenius

Definition

A singular point x_0 of

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0$$

is said to be a regular singular point if both $(x - x_0)p(x)$ and $(x - x_0)^2 q(x)$ are analytic at x_0. Otherwise x_0 is called an irregular singular point.

Example

Classify the singular points of the equation

$$(x^2 - 1)^2 y''(x) + (x + 1)y'(x) - y(X) = 0$$

Rewriting this gives you $y'' + \frac{(x+1)}{(x+1)^2(x-1)^2}y' - \frac{1}{(x+1)^2(x-1)^2}y = 0$.

The singular points are $x = 1$ and $x = -1$.

$x = 1$ is an irregular singular point because it is not analytic for both $p(x)$ and $q(x)$. -1 is a regular singular point.

Definition

If x_0 is a regular singular point of $y'' + py' + qy = 0$, then the indical equation for this point is

$$r(r-1) + p_0 r + q_0 = 0$$

where

$$p_0 := \lim_{x \to x_0} (x - x_0)p(x), \qquad q_0 := \lim_{x \to x_0} (x - x_0)^2 q(x)$$

The roots of the indicial equation are called the exponents (indices) of the singularity x_0.

Example

Find the indical equation and the exponents of the singularity $x = -1$ of

$$(x^2 - 1)^2 y''(x) + (x + 1)y'(x) - y(x) = 0$$

In standard form we have $y'' + \frac{(x+1)}{(x+1)^2(x-1)^2} y' - \frac{1}{(x+1)^2(x-1)^2} y = 0$.

We have $(x + 1)p(x) = \frac{1}{(x-1)^2}$ and $(x + 1)^2 q(x) = \frac{-1}{(x-1)^2}$.

The limits are $1/4$ and $-1/4$ respectively from this.

So the indical equation becomes $r(r-1) + p_0 r + q_0 = 0$ or $r(r-1) + \frac{1}{4}r - \frac{1}{4} = 0$ or $r^2 - \frac{3}{4}r - \frac{1}{4} = 0$

Factoring gives $(4r + 1)(r - 1)$, and the indical roots are $r = -1/4$ and $r = 1$.

Example

Find a series expansion about the regular singular point $x = 0$ for a solution to

$$(x + 2)x^2 y''(x) - xy'(x) + (1 + x)y(x) = 0, \qquad x > 0$$

Finding the indical roots gives us $p_0 = -1/2$, and $q_0 = 1/2$.

The indicial equation is $2r^2 - 3r + 1 = 0$, so the indicial roots are $r = 1/2$ and $r = 1$.

Now expand about $r = 1$.

We get $(x + 2)x^2 \sum a_n(n + 1)nx^{n-1} - x \sum a_n(n + 1)x^n + (1 + x) \sum a_n x^{n+1} = 0$.

Do some simplification to get $\sum n = 0 a_n(n + 1)nx^{n+2} + \sum_{n=1} 2a_n(n + 1)nx^{n+1} - \sum_{n=1} a_n(n + 1)x^{n+1} \sum_{n=0} a_n x^{n+2}$.

Writing them to start all at the same index and combining gives you $\sum_{k=2}[a_{k-2}(k - 1)(k - 2) + 2a_{k-1}k(k - 1) - a_{k-1}(k - 1) + a_{k-2}]x^k = 0$.

Finding the recurrence formula gives $a_{k-1} = \frac{-(k^2-3k+3)}{(2k-1)(k-1)} a_{k-2}$.

Putting k values into the formula gives you $y = x - \frac{1}{3}x^2 + \frac{1}{10}x^3 - \frac{1}{30}x^4 + \ldots$.

Theorem 8.6

If x_0 is a regular singular point, then there exists at least one series solution, where $r = r_1$ is the larger root of the associated indicial equation. Moreoever, this series converges for all x such that $0 < x - x_0 < R$, where R is the distance from x_0 to the nearest other singular point (real or complex).

Example

Find as eries solution about the regular singular point $x = 0$ of

$$x^2y''(x) - xy'(x) + (1-x)y(x) = 0, \qquad x > 0$$

We have $x = 0$ is a regular singular point from writing this in general form.

Writing the indicial equation gives us $r = 1$.

Writing the summations gives you $\sum_{n=0} a_n(n+1)nx^{n+1} - \sum_{n=0} a_n(n+1)x^{n+1} + \sum_{n=0} a_n x^{n+1} - \sum_{n=0} a_n x^{n+2} = 0$.

Simplify this to get $a_{k-1} = \frac{a_{k-2}}{(k=1)^2}$.

You end up getting $y = x + x^2 + \frac{1}{4}x^3 + \frac{1}{36}x^4 + \ldots$.

9 Matrix Methods for Linear Systems

9.1 Introduction to Matrix Methods

The product of a matrix and a column vector is defined to be the collection of dot products of the rows of the matrix with the vector, arranged as a column vector:

$$\begin{bmatrix} \text{row } \# \ 1 \\ \text{row } \# \ 2 \\ \vdots \\ \text{row } \# \ 3 \end{bmatrix} [v] = \begin{bmatrix} \text{row } \# \ 1 \cdot v \\ \text{row } \# \ 2 \cdot v \\ \vdots \\ \text{row } \# \ 3 \cdot v \end{bmatrix}$$

where the vector v has n components; the dot product of two n-dimensional vectors is computed in the obvious way:

$$[a_1 \quad a_2 \quad \cdots \quad a_n] \cdot [x_1 \quad x_2 \quad \cdots \quad x_n] = a_1x_1 + a_2x_2 + \cdots + a_nx_n$$

Example

Express the system as a matrix equation.

$$\begin{aligned} x_1' &= 2x_1 + t^2x_2 + (4t + e^t)x_4 \\ x_2' &= (\sin t)x_2 + (\cos t)x_3 \\ x_3' &= x_1 + x_2 + x_3 + x_4 \\ x_4' &= 0 \end{aligned}$$

This is simply written as

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{bmatrix} = \begin{bmatrix} 2 & t^2 & 0 & (4t+e^t) \\ 0 & \sin t & \cos t & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

In general, if a system or differential equation is expresses as

$$\begin{aligned} x_1' &= a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n \\ x_2' &= a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n \\ &\vdots \\ x_n' &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n \end{aligned}$$

it is said to be a linear homogeneous system in normal form. The matrix form of such a system is

$$\mathbf{X}' = \mathbf{AX}$$

where $\mathbf{A}$ is the coefficient matrix

$$\mathbf{A} = \mathbf{A}(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \ldots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \ldots & a_{2n}(t) \\ \vdots & \vdots & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \ldots & a_{nn}(t) \end{bmatrix}$$

and $\mathbf{X}$ is the solution vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Example

Express the differential equation for the undamped, unforced mass-spring oscillator

$$my'' + ky = 0$$

as an equivalent system of first-order equations in normal form, expressed in matrix notation.

We have that $y' = v$ and $v' = -\frac{k}{m}y$.

So we can write this as $\begin{bmatrix} y \\ v \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -k/m & 0 \end{bmatrix} \begin{bmatrix} y \\ v \end{bmatrix}$.

We can write this as $a_n x_n' + a_{n-1}x_n + \cdots + a_1 x_2 + a_0 x_1 = 0$.

Which can be rewritten as $x_n' = -\frac{a_0}{a_n}x_1 - \frac{a_1}{a_n}x_2 - \cdots - \frac{a_{n-1}}{a_n}x_n$.

Using this can make it easy to get to matrix notation

Example

A coupled mass-spring oscillator is governed by the system

$$2\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + 6x - 2y = 0$$
$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 2y - 2x = 0$$

Let $x_1 = x$, $x_2 = x'$, $x_3 = y$, $x_4 = y'$.

This gives us $x_1' = x_2$, $x_2' = -3x_1 + x_3$, $x_3' = x_4$, $x_4' = x_1 - 2x_3$.

So the matrix form can be easily answered from that.

9.2 Review 1: Linear Algebraic Equations

A set of equations of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

(where the a_{ij}'s and b_i's are given constants) is called a linear system of n algebraic equations in the n unknowns $x_1, x_2, \ldots x_n$.

The Gauss-Jordan elimination algorithm uses elimination to uncouple the system making the values of the unknowns apparent.

Example

Solve the system

$$\begin{aligned} 2x_1 + 6x_2 + 8x_3 &= 16 \\ 4x_1 + 15x_2 + 19x_3 &= 38 \\ 2x_1 \qquad + 3x_3 &= 6 \end{aligned}$$

Solving the coefficient matrix gives you $(0, 0, 2) = (x_1, x_2, x_3)$.

Exercise Solve the system

$$\begin{aligned} x_1 + 2x_2 + 4x_3 + x_4 &= 0 \\ -x_1 - 2x_2 - 2x_3 \qquad &= 1 \\ -2x_1 - 4x_2 - 8x_3 + 2x_4 &= 4 \\ x_1 + 4x_2 + 2x_3 \qquad &= -3 \end{aligned}$$

Example

Solve the system

$$\begin{aligned} 2x_1 + 4x_2 + x_3 &= 8 \\ 2x_1 + 4x_2 \qquad &= 6 \\ -4x_1 - 8x_2 + x_3 &= -10 \end{aligned}$$

We will end up getting $x_1 + 2x_2 = 3$ and $x_3 = 2$, and x_2 has infinite solutions, and is called a free variable.

So $x_2 = t, x_1 = -2t + 3$, and $x_3 = 2$.

Exercise Solve the system

$$\begin{aligned} x_1 - x_2 + 2x_3 + 2x_4 &= 0 \\ 2x_1 - 2x_2 + 4x_3 + 3x_4 &= 1 \\ 3x_1 - 3x_2 + 6x_3 + 9x_4 &= -3 \\ 4x_1 - 4x_2 + 8x_3 + 8x_4 &= 0 \end{aligned}$$

9.3 Review 2: Matrices and Vectors

A matrix is a rectangular array of numbers arranged in rows and columns. An $m \times n$ matrix, that is, a matrix with m rows and n columns is usually denoted by

$$A := \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

Where the element in the ith row and jth column is a_{ij}. The notation $[a_{ij}]$ is used to designate A.

A square matrix has the same number of rows and columns. A diagonal matrix is a square matrix with only zero entries off the main diagonal. A column matrix, or vector, is an $n \times 1$ matrix. An $m \times n$ matrix whose entries are all zero is called a zero matrix. Matrices are denoted by boldfaces capital letters and vectors by boldfaced lower case letters.

The sum of two $m \times n$ matrices is given by

$$\mathbf{A} + \mathbf{B} = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$$

To multiply a matrix by a scalar (number), multiply each element in the matrix by that number:

$$r\mathbf{A} = r[a_{ij}] = [ra_{ij}]$$

The notation $-\mathbf{A}$ stands for $(-1)\mathbf{A}$.

Properties of Matrix Addition and Scalar Multiplication

- $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$

- $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$
- $\mathbf{A} + \mathbf{0} = \mathbf{A}$
- $\mathbf{A} + (-\mathbf{A}) = \mathbf{0}$
- $r(\mathbf{A} + \mathbf{B}) = r\mathbf{A} + r\mathbf{B}$
- $(r + s)\mathbf{A} = r\mathbf{A} + s\mathbf{A}$
- $r(s\mathbf{A}) = (rs)\mathbf{A} = s(r\mathbf{A})$

Exercise Perform the indicated operation: $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Exercise Perform the indicated operation: $3\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

The product of a matrix A and a column vector x is the column vector composed of dot products of the rows of A with x. AB is only defined when the number of columns of A matches the number of rows of B.

Exercise Perform the indicated operation: $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

Exercise Perform the indicated operation: $\begin{bmatrix} 1 & 0 & 1 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & x \\ -1 & -1 & y \\ 4 & 1 & z \end{bmatrix}$

Properties of Matrix Multiplication

- $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$
- $(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$
- $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$
- $(r\mathbf{A})\mathbf{B} = r(\mathbf{A[}B]) = \mathbf{A}(r\mathbf{B})$

Let A be an $m \times n$ matrix and let x and y be $n \times 1$ vectors. Then Ax is an $m \times 1$ vector so we can think of multiplication by A as defining an operator that maps $n \times 1$ vectors into $m \times 1$ vectors. Multiplication by A defines a linear operator since $\mathbf{A}(\mathbf{x} + \mathbf{y}) = \mathbf{Ax} + \mathbf{Ay}$ and $\mathbf{A}(\mathbf{rx}) = \mathbf{rAx}$.

Examples of linear operations are:

1. Stretching or contracting the components of a vector by constant factors
2. rotating a vector through some angle about a fixed axis
3. reflecting a vector in a plane mirror

We express the linear system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

In matrix notation as $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{A}$ is the coefficient matrix, $\mathbf{x}$ is the vector of unknowns, and $\mathbf{B}$ is the vector of constants occurring on the right-hand side:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

If $b = 0$, the system $Ax = b$ is said to be homogeneous.

The matrix obtained from A by interchaing its rows and columns is called the transpose of A and is denoted by A^T.

Exercise Find A^T if $\mathbf{A} = \begin{bmatrix} 1 & 2 & 6 \\ -1 & 2 & -1 \end{bmatrix}$.

There is a multiplicative identity in matrix algebra, namely, a square diagonal matrix **I** with ones down the main diagonal. Multiplying **I** on the right or left by any other matrix (with compatible dimensions) reproduces the latter matrix.

Exercise Demonstrate the identity property for $\mathbf{A} = \begin{bmatrix} 1 & 2 & 6 \\ -1 & 2 & -1 \end{bmatrix}$

Some square matrices **A** can be paired with other square matrices **B** having the property that $\mathbf{BA} = \mathbf{I}$. When this happens,

1. **B** is the unique matrix satisfying $\mathbf{BA} = \mathbf{I}$ and
2. **B** also satisfies $\mathbf{AB} = \mathbf{I}$.

In such a case, **B** is the inverse of **A** and write $\mathbf{B} = \mathbf{A}^{-1}$. A matrix that has no inverse is said to be singular.

When an inverse for the coefficient matrix A in a system of linear equations $A\mathbf{x} = \mathbf{b}$, the solution can be calculated directly by $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$.

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. The matrix A is invertible if and only if $ad - bc \neq 0$. If $ad - bc = 0$, then A does not have a multiplicative inverse.

Exercise If $A = \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}$, solve $Ax = b$ where $b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Finding the Inverse of a Matrix. Row operations include

- Interchanging two rows of the matrix
- Multiplying a row of the matrix by a nonzero scalar
- Adding a scalar multiple of one row of the matrix to another row

If the $n \times n$ matrix **A** has an inverse, $\mathbf{A}^{-1}$ can be found by performing row operations on the $n \times 2n$ matrix $[\mathbf{A}|\mathbf{I}]$ obtained by writing **A** and **I** side by side. If the procedure produces a new matrix in the form $[\mathbf{I}|\mathbf{B}]$, then $\mathbf{A}^{-1} = \mathbf{B}$.

Exercise Find the inverse of $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix}$

For a 2×2 matrix **A**, the determinant of **A**, denoted by $\det \mathbf{A}$ or $|\mathbf{A}|$, is defined by

$$\det \mathbf{A} := \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_21$$

The determinant of a 3×3 matrix A can be defined in terms of its cofactor expansion about the first row

$$\det \mathbf{A} := \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Exercise Find the determinant $\begin{vmatrix} 2 & 4 \\ 1 & 1 \end{vmatrix}$

Exercise Find the determinant $\begin{vmatrix} 1 & 2 & 1 \\ 0 & 3 & 5 \\ 2 & 1 & -1 \end{vmatrix}$

Theorem 9.1

Let **A** be an $n \times n$ matrix. The following statements are equivalent:

- **A** is singular (does not have an inverse).
- The determinant of **A** is zero.
- $\mathbf{Ax} = \mathbf{0}$ has nontrivial solutions ($\mathbf{x} \neq 0$)
- The columns (rows) of **A** form a linearly dependent set.

The columns of A are linearly dependent means there exist scalars $c_1, \ldots, c_n$ not all zero, such that

$$c_1 a_1 + c_2 a_2 + \cdots + c_n a_n = 0$$

where a_j is the vector forming the jth column of **A**.

If **A** is a singular square matrix ($\det \mathbf{A} = 0$) then $\mathbf{Ax} = \mathbf{0}$ has infinitely many solutions.

If **A** is singular, $\mathbf{Ax} = \mathbf{b}$ either has no solutions or it has infinitely many of the form

$$x = x_p + x_h$$

where x_p is a particular solution to $\mathbf{Ax} = \mathbf{b}$ and x_h is any of the infinite solutions to $\mathbf{Ax} = \mathbf{0}$, the homogeneous system.

Exercise In a previous section, we saw the system

$$\begin{bmatrix} 2 & 4 & 1 \\ 2 & 4 & 0 \\ -4 & -8 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \\ -10 \end{bmatrix}$$

has solutions $x_1 = 3 - 2s, x_2 = s, x_3 = 2$ where $-\infty < s < \infty$.

1. Write the solution in matrix notation and identify x_p and x_h.
2. Verify $\det \mathbf{A} = 0$
3. Give the identity that exhibits the linear dependence of the columns of **A**.

If **A** is a nonsingular square matrix (i.e., **A** has an inverse and $\det \mathbf{A} \neq 0$), then $\mathbf{Ax} = \mathbf{0}$ has $\mathbf{x} = \mathbf{0}$ as its only solution and the unique solution to $\mathbf{Ax} = \mathbf{b}$ is $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$.

If the entries $a_{ij}(t)$ in a matrix $\mathbf{A}(t)$ are functions of the variable t, then $\mathbf{A}(t)$ is a matrix function of t. Similarly, if the entries $x_i(t)$ of a vector $\mathbf{x}(t)$ are functions of t, then $\mathbf{x}(t)$ is a vector function of t.

A matrix $\mathbf{A}(t)$ is said to be continuous at t_0 if each entry $a_{ij}(t)$ is continuous at t_0. $\mathbf{A}(t)$ is differentiable at t_0 if each entry $a_{ij}(t)$ is differentiable at t_0.

$$\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t}(t_0) = \mathbf{A}'(t_0) := [a'_{ij}(t_0)]$$

$$\int_a^b \mathbf{A}(t)\mathrm{d}t := \left[\int_a^b a_{ij}(t)\mathrm{d}t\right]$$

Exercise Let $\mathbf{A}(t) = \begin{bmatrix} t^2+1 & \cos t \\ e^t & 1 \end{bmatrix}$

1. Find: $\mathbf{A}'(t)$
2. Find: $\int_0^1 \mathbf{A}(t)\mathrm{d}t$

Differentiation Formulas for Matrix Functions:

- $\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{CA}) = \mathbf{C}\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t}$ (**C** a constant matrix)
- $\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{A}+\mathbf{B}) = \frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} + \frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t}$
- $\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{AB}) = \mathbf{A}\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t} + \frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t}\mathbf{B}$

Exercise Show that $\mathbf{x}(t) = \begin{bmatrix} \cos\omega t \\ \sin\omega t \end{bmatrix}$ is a solution of the matrix differential equation $\mathbf{x}' = \mathbf{Ax}$ where $\mathbf{A} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}$

9.4 Linear Systems in Normal Form

A system of n linear differential equations is in normal form if it is expressed as

$$\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t)$$

where $x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$, $f(t) = \begin{bmatrix} f_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$ and $A(t) = \begin{bmatrix} a_{ij}(t) \end{bmatrix}$ is an $n \times n$ matrix.

A system is called homogeneous when $\mathbf{f}(t) = 0$; otherwise it is called nonhomogeneous. When the elements of A are all constants, the system is said to have constant coefficients.

The initial value problem for the normal system is the problem of finding a differentiable vector function $\mathbf{x}(t)$ that satisfies the system on an interval I and also satisfies the initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$ where t_0 is a given point of I and $\mathbf{x}_0$ is a given vector.

Theorem 9.2

Suppose $\mathbf{A}(t)$ and $\mathbf{f}(t)$ are continuous on an open interval I that contains the point t_0. Then, for any choice of the initial vector $\mathbf{x}_0$, there exists a unique solution $\mathbf{x}(t)$ on the whole interval I to the initial value problem

$$\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t), \qquad \mathbf{x}(t_0) = \mathbf{x}_0$$

Definition

The m vector functions $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are said to be linearly dependent on an interval I if there exist constants $c_1, \ldots, c_n$, not all zero, such that

$$c_1\mathbf{x}_1(t) + \cdots + c_n\mathbf{x}_n(t) = \mathbf{0}$$

for all t in I. If the vectors are not linearly dependent, they are said to be linearly independent on I.

Example

Show that the vector functions $x_1(t) = \begin{bmatrix} e^t \\ 0 \\ e^t \end{bmatrix}$, $x_2(t) = \begin{bmatrix} 3e^t \\ 0 \\ 3e^t \end{bmatrix}$, and $x_3 = \begin{bmatrix} t \\ 1 \\ 0 \end{bmatrix}$ are linearly dependent on $(-\infty, \infty)$.

They are dependent because let $c_1 = -3, c_2 = 1, c_3 = 0$ to get $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Example

Show that the vector functions $x_1(t) = \begin{bmatrix} e^{2t} \\ 0 \\ e^{2t} \end{bmatrix}$, $x_2(t) = \begin{bmatrix} e^{2t} \\ e^{2t} \\ -e^{2t} \end{bmatrix}$, and $x_3(t) = \begin{bmatrix} e^t \\ 2e^t \\ e^t \end{bmatrix}$ are linearly independent on $(-\infty, \infty)$.

The only way we can get $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is if the three constants are 0.

A set of vector functions $x_1(t), x_2(t), \ldots, x_n(t)$ each having n components is linearly independent on an interval I if we can find one point t_0 in I where the determinant $\det \begin{bmatrix} x_1(t_0) & \ldots & x_n(t_0) \end{bmatrix}$ is not zero. We call this detemrinant the Wronksian. (This was previously defined)

Theorem 9.3

Let $\mathbf{x}_1, \dots, \mathbf{x}_n$ be n linearly independent solutions to the homogeneous system

$$\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t)$$

on the interval I, where $\mathbf{A}(t)$ is an $n \times n$ matrix function continuous on I. Then every solution to the above on I can be expressed in the form

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + \cdots + c_n\mathbf{x}_n(t)$$

A set of solutions $\{x_1, \dots x_n\}$ that are linearly independent on I is called a fundamental solution set for the homogeneous system on I. The linear combination written with arbitrary constants, is referred to as the general solution to the homogeneous system.

If we take the vectors in a fundamental solution set and let them form the columns of a matrix $\mathbf{X}(t)$.

$$\mathbf{X}(t) = \begin{bmatrix}\mathbf{x}_1(t) & \mathbf{x}_2(t) & \dots & \mathbf{x}_n(t)\end{bmatrix} = \begin{bmatrix} x_{1,1}(t) & x_{1,2}(t) & \dots & x_{1,n}(t) \\ x_{2,1}(t) & x_{2,2}(t) & \dots & x_{2,n}(t) \\ \vdots & \vdots & & \vdots \\ x_{n,1}(t) & x_{n,2}(t) & \dots & x_{n,n}(t) \end{bmatrix}$$

Then the matrix $\mathbf{X}(t)$ is called a fundamental matrix for the homogeneous system.

Example

Verify that the set $S = \left\{ \begin{bmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix}, \begin{bmatrix} -e^{-t} \\ 0 \\ e^{-t} \end{bmatrix}, \begin{bmatrix} 0 \\ e^{-t} \\ -e^{-t} \end{bmatrix} \right\}$ is a fundamental solution set for the system $\mathbf{x}'(t) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$
$textbf x(t)$ on the interval $(-\infty, \infty)$ and find a fundamental matrix for the system. Determine a general solution for the system.

Testing the three matrices in the system gives the correct resulting vector, and finding the Wronksian shows us that the columns are linearly independent, so the general solution is

$$x = c_1 \begin{bmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix} + c_2 \begin{bmatrix} -e^{-t} \\ 0 \\ e^{-t} \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ e^{-t} \\ -e^{-t} \end{bmatrix}$$

Theorem 9.4

Let $\mathbf{x}_p$ be a particular solution to the nonhomogeneous system

$$\mathbf{x}'(t) = \mathbf{A}(t) + \mathbf{f}(t)$$

on the interval I, and let $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ be a fundamental solution set on I for the corresponding homogeneous system $\mathbf{x}(t) = \mathbf{A}(t)\mathbf{x}(t)$. Then every solution to the nonhomogeneous system on I can be expressed in the form

$$\mathbf{x}(t) = \mathbf{x}_p(t) + c_1\mathbf{x}_1(t) + \cdots + c_n\mathbf{x}_n(t)$$

where $c_1, \dots, c_n$ are constants.

Approach to Solving Normal Systems:

1. To determine a general solution to the $n \times n$ homogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x}$:
    - Find a fundamental solution set $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ that consists of n linearly independent solutions to the homogeneous system.

- Form the linear combination

$$\mathbf{x} = \mathbf{X}\mathbf{c} = c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n$$

where $\mathbf{c} = \text{col}(c_1, \ldots, c_n)$ is any constant vector and $\mathbf{X} = \begin{bmatrix}\mathbf{x}_1 & \ldots & \mathbf{x}_n\end{bmatrix}$ is the fundamental matrix, to obtain a general solution.

2. To determine a general solution to the nonhomogeneous system $\mathbf{x}'(t) = \mathbf{A}\mathbf{x} + \mathbf{f}$:

- Find a particular solution $\mathbf{x}_p$ to the nonhomogeneous system.
- Form the sum of the particular solution and the general solution $\mathbf{X}\mathbf{c} = c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n$ to the corresponding homogeneous system in part 1,

$$\mathbf{x} = \mathbf{x}_p + \mathbf{X}\mathbf{c} = \mathbf{x}_p + c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n$$

to obtain a general solution to the given system.

9.5 Homogeneous Linear Systems with Constant Coefficients

We now define a procedure for obtaining a general solution for the homogeneous system

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$$

Definition

Let $\mathbf{A} = [a_{ij}]$ be an $n \times n$ constant matrix. The eigenvalues of $\mathbf{A}$ are those (real or complex) numbers r for which $(\mathbf{A} - r\mathbf{I}) = \mathbf{0}$ has at least one nontrivla solution $\mathbf{u}$. The corresponding nontrivial solutions $\mathbf{u}$ are called the eigenvectors of $\mathbf{A}$ associated with r.

Finding eigenvalues of a matrix A is equivalent to finding the zeroes of the polynomial $p(r) = \det(A - rI)$. The equation $\det(A - rI) = 0$ is called the characteristic equation of A and $p(r)$ is the characteristic polynomial of A.

Example

Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$

First find the characteristic equation, this is the determinant of $\begin{bmatrix} 2-r & -3 \\ 1 & -2-r \end{bmatrix}$. So the characteristic equation is $r^2 - 1$ and the eigenvalues are $r = -1$ and $r = 1$.

Now doing the procedure above, we find that for $r = -1$, the eigenvector is $u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and for $r = 1$, the eigenvector is $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

The collection of all eigenvectors associated with an eigenvalue forms a subspace when the zero vector is adjoined. These subspaces are called eigenspaces.

Exercise Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}$$

Theorem 9.5

Suppose the $n \times n$ constant matrix $\mathbf{A}$ has n linearly independent eigenvectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$. Let r_i be the eigenvalue corresponding to $\mathbf{u}_i$. Then

$$\{e^{r_1 t}\mathbf{u}_1, e^{r_2 t}\mathbf{u}_2, \ldots, e^{r_n t}\mathbf{u}_n\}$$

is a fundamental solution set (and $\mathbf{X}(t) = \begin{bmatrix} e^{r_1 t}\mathbf{u}_1 & e^{r_2 t}\mathbf{u}_2 & \ldots & e^{r_n t}\mathbf{u}_n \end{bmatrix}$ is a fundamental matrix) on $(-\infty, \infty)$ for the homogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x}$. Consequently, a general solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is

$$\mathbf{x}(t) = c_1 e^{r_1 t}\mathbf{u}_1 + c_2 e^{r_2 t}\mathbf{u}_2 + \cdots + c_n e^{r_n t}\mathbf{u}_n$$

where $c_1, \ldots, c_n$ are arbitrary constants.

Example

Find a general solution of $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$, where $\mathbf{A} = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$

We get the eigenvalues as ± 1 from this. Therefore, when we find the the general solution, we can see that this is

$$x(t) = c_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}$$

, and this can be written in different ways, but this is a way to write the general solution.

Theorem 9.6

If $r_1, \ldots, r_m$ are distinct eigenvalues for the matrix $\mathbf{A}$ and $\mathbf{u}_i$ is an eigenvector associated with r_i, then $\mathbf{u}_1, \ldots, \mathbf{u}_m$ are linearly independent.

Corollary 9.7

If the $n \times n$ constant matrix $\mathbf{A}$ has n distinct eigenvalues $r_1, \ldots, r_n$ and $\mathbf{u}_i$ is an eigenvector associated with r_i, then

$$\{e^{r_1 t}\mathbf{u}_1, \ldots, e^{r_n t}\mathbf{u}_n\}$$

is a fundamental solution set for the homogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

Exercise Solve the initial value problem $\mathbf{x}'(T) = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix} \mathbf{x}(t)$ where $\mathbf{x}(0) = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$.

Definition

A real symmetric matrix $\mathbf{A}$ is a matrix with real entries that satisfies $\mathbf{A}^T = \mathbf{A}$.

If A is an $n \times n$ real symmetric matrix, there always exist n linearly independent eigenvectors.

Example

Find a general solution of $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$, where $\mathbf{A} = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$.

From this, we can find the eigenvalues are $r = 3$ and $r = -3$.

The eigenvectos for $r = 3$ are $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and for c_3 it is $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$, and the general solution can be found from this.

Second Solution:

Suppose r_1 is an eigenvalue of multiplicity two and that there is only one eigenvector associated with this value. A second solution can be found of the form

$$x_2 = Kte^{r_1 t} + Pe^{r_1 t}$$

where $\mathbf{K} = \begin{bmatrix} k_1 \\ \vdots \\ k_n \end{bmatrix}$ and $\mathbf{P} = \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix}$ satisfy $(A - r_1 I)K = 0$ and $(A - r_1 I)P = K$.

Example

Find the general solution of $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$, where $\mathbf{A} = \begin{bmatrix} 3 & -18 \\ 2 & -9 \end{bmatrix}$.

The eigenvalue for this is $r = -3$ with multiplicity 2.

If we find the first eigenvector we get $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$, and we can set this equal to k.

Then we can use this to find the second eigenvector by doing $\begin{bmatrix} 6 & -18 \\ 2 & -6 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, and finding this eigenvector as $\begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$.

This shows that $x_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} te^{-3t} + \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} e^{-3t}$.

Therefore the general solution is $x = c_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{-3t}$ added on to c_2 times what x_2 is.

When the coefficient matrix A has only one eigenvalue associated with an eigenvalue r_1 of multiplicity three, we can find a second solution of the form

$$x_2 = Kte^{r_1 t} + Pe^{r_1 t}$$

and a third solution of the form

$$x_3 = K\frac{t^2}{2}e^{r_1 t} + Pte^{r_1 t} + Qe^{r_1 t}$$

where $\mathbf{K} = \begin{bmatrix} k_1 \\ \vdots \\ k_n \end{bmatrix}$, $\mathbf{P} = \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix}$, and $Q = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$ satfisfy

$$(A - r_1 I)K = 0$$
$$(A - r_1 I)P = K$$
$$(A - r_1 I)Q = P$$

Exercise Find the general solution $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$, where $\mathbf{A} = \begin{bmatrix} 2 & 1 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 2 \end{bmatrix}$.

9.6 Complex Eigenvalues

If the real matrix $\mathbf{A}$ has complex conjugate eigenvalues $\alpha \pm i\beta$ with corresponding eigenvectors $\mathbf{a} \pm i\mathbf{b}$, then two linearly independent real vector solutions to $\mathbf{x}'(t) = A\mathbf{x}(t)$ are

$$e^{\alpha t}\cos\beta t\mathbf{a} - e^{\alpha t}\sin\beta t\mathbf{b}$$

$$e^{\alpha t}\sin\beta t\mathbf{a} + e^{\alpha t}\cos\beta t\mathbf{b}$$

Example

Find a general solution of $\mathbf{x}'(t) = \begin{bmatrix} -1 & 2 \\ -1 & -3 \end{bmatrix} \mathbf{x}(t)$.

The eigenvalue for this is $-2 \pm i$.

The first vector gives you $e^{-2t}[\begin{bmatrix} 1 \\ -1 \end{bmatrix} \cos t - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \sin t]$ and the other gives $e^{-2t}[\begin{bmatrix} 1 \\ -1 \end{bmatrix} \sin t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cos t]$.

9.7 Nonhomogeneous Linear Systems

The Method of Undetermined Coefficients can be used to find a particular solution to the nonhomogeneous linear system

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{f}(t)$$

where A is an $n \times n$ constant matrix and the entries of $\mathbf{f}(t)$ are polynomials, exponential functions, sines and cosines, or finite sums and products of these functions.

Example

Find a general solution of $x'(t) = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} x(t) + t \begin{bmatrix} -9 \\ 0 \\ -18 \end{bmatrix}$.

The particular solution is $x_p = \begin{bmatrix} At+B \\ Ct+D \\ Et+F \end{bmatrix}$.

Solve the homogeneous equation now and it is $x_h = c_1 e^{3t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{-3t} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

We also can see now that $\begin{bmatrix} A \\ C \\ E \end{bmatrix} = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} At+B \\ Ct+D \\ Et+F \end{bmatrix} + \begin{bmatrix} -9t \\ 0 \\ -18t \end{bmatrix}$.

Multiplying the matrices gives $\begin{bmatrix} A \\ C \\ E \end{bmatrix} = \begin{bmatrix} At+B-2Ct-2D+2Et+2F-9t \\ -2At-2B+Ct+D+2Et+2F \\ 2At+2B+2Ct+2D+Et+F-18t \end{bmatrix}$.

We see that $A = t[A-2C+2E-9] + [B-2D+2F]$

We have also $C = t[-2A+C+2E] + [-2B+D+2F]$.

We can also see $E = t[2A+2C+e-18] + [2B+2D+F]$.

From these, we can write 6 equations to get the following matrix.

$$\begin{bmatrix} 1 & 0 & -2 & 0 & 2 & 0 & 9 \\ -1 & 1 & 0 & -2 & 0 & 2 & 0 \\ -2 & 0 & 1 & 0 & 2 & 0 & 0 \\ 0 & -2 & -1 & 1 & 0 & 2 & 0 \\ 2 & 0 & 2 & 0 & 1 & 0 & 18 \\ 0 & 2 & 0 & 2 & -1 & 1 & 0 \end{bmatrix}$$

Putting this in our calculator gives us the particular solution $x_p = \begin{bmatrix} 5t+1 \\ 2t \\ 4t+2 \end{bmatrix}$.

If $x_1, x_2, \ldots, x_n$ is a fundamental set of solutions of the homogeneous system $\mathbf{x}'(t) = A\mathbf{x}(t)$ on an interval

I, then it sgeneral solution on the interval is the linear combination $x = c_1x_1 + c_2x_2 + \cdots + c_nx_n$ or

$$\mathbf{X} = c_1\begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{pmatrix} + c_2\begin{pmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{n2} \end{pmatrix} + \cdots + c_n\begin{pmatrix} x_{1n} \\ x_{2n} \\ \vdots \\ x_{nn} \end{pmatrix} = \begin{pmatrix} c_1x_{11} + c_2x_{12} + \cdots + c_nx_{1n} \\ c_1x_{21} + c_2x_{22} + \cdots + c_nx_{2n} \\ \vdots \\ c_1x_{n1} + c_2x_{n2} + \cdots + c_nx_{nn} \end{pmatrix}$$

The solution can be written as $\mathbf{x}(t) = \mathbf{X}(t)\mathbf{C}$ where $\mathbf{C}$ is an $n \times 1$ column vector of arbitrary constants $c_1, c_2, \ldots, c_n$ and

$$\mathbf{X}(t) = \begin{bmatrix} \mathbf{x}_1(t) & \mathbf{x}_2(t) & \ldots & \mathbf{x}_n(t) \end{bmatrix} = \begin{bmatrix} x_{1,1}(t) & x_{1,2}(t) & \ldots & x_{1,n}(t) \\ x_{2,1}(t) & x_{2,2}(t) & \ldots & x_{2,n}(t) \\ \vdots & \vdots & & \vdots \\ x_{n,1}(t) & x_{n,2}(t) & \ldots & x_{n,n}(t) \end{bmatrix}$$

is the fundamental matrix of the system on ther interval.

Because a general solution to $\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t)$ is given by $\mathbf{x}(t) = \mathbf{X}(t)\mathbf{C}$ we seek a particular solution to the nonhomogeneous system $\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t)$ of the form $\mathbf{x}_p(t) = \mathbf{X}(t)\mathbf{v}(t)$ where $v(t) = \begin{bmatrix} v_1(t) \\ \vdots \\ v_n(t) \end{bmatrix}$ can be found by

$$\mathbf{v}(t) = \int \mathbf{X}^{-1}(t)\mathbf{f}(t)\mathrm{d}t$$

and

$$\mathbf{x}_p(t) = \mathbf{X}(t)\mathbf{v}(t) = \mathbf{X}(t)\int \mathbf{X}^{-1}(t)\mathbf{f}(t)\mathrm{d}t$$

Combining with the solution to the nonhomogeneous system gives the general solution

$$\mathbf{x}(t) = \mathbf{X}(t)C + \mathbf{X}(t)\int \mathbf{X}^{-1}(t)\mathbf{f}(t)\mathrm{d}t$$

Example

Find the solution to the initial value problem

$$\mathbf{x}'(t) = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} e^{2t} \\ 1 \end{bmatrix}, \qquad \mathbf{x}(0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Previously we found the homogeneous solution is $c_1\begin{bmatrix} 3 \\ 1 \end{bmatrix}e^t + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{-3t}$.

Now to find $v(t)$, remember we need to find $X(t)v(t)$ and $v(t) = \int X^{-1}(t)f(t)\mathrm{d}t$.

The inverse of $x(t)$ is $X^{-1} = \begin{bmatrix} \frac{1}{2}e^{-t} & -\frac{1}{2}e^{-t} \\ -\frac{1}{2}e^{t} & \frac{3}{2}e^{t} \end{bmatrix}$.

So $v(t) = \int \begin{bmatrix} \frac{1}{2}e^{-t} & -\frac{1}{2}e^{-t} \\ -\frac{1}{2}e^{t} & \frac{3}{2}e^{t} \end{bmatrix}\begin{bmatrix} e^{2t} \\ 1 \end{bmatrix}\mathrm{d}t$.

Integrating this out gives $v(t) = \begin{bmatrix} \frac{1}{2}e^{t} + \frac{1}{2}e^{-t} \\ -\frac{1}{2}e^{3t} + \frac{3}{2}e^{t} \end{bmatrix}$.

Now multiply this with $X^{-1}(t)$ to get $\begin{bmatrix} \frac{4}{3}e^{2t} + 3 \\ \frac{1}{3}e^{2t} + 2 \end{bmatrix}$ for x_p.

The general solution is $x = c_1\begin{bmatrix} 3 \\ 1 \end{bmatrix}e^t + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{-t} + \begin{bmatrix} \frac{4}{3} \\ \frac{1}{3} \end{bmatrix}e^{2t} + \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

Now plugging in the initial conditions gives $c_1 = -\frac{3}{2}$ and $c_2 = -\frac{5}{6}$.

Plugging in everything gives $x = \begin{bmatrix} -\frac{9}{2}e^t - \frac{5}{6}e^{-t} + \frac{4}{3}e^{2t} + 3 \\ -\frac{3}{2}e^t - \frac{5}{6}e^{-t} + \frac{1}{3}e^{2t} + 2 \end{bmatrix}$.