1 Sets, Functions, Sequences, Sums, and Matrices

1.1 Sets

Sets are used to group objects together.

A set is an unordered collection of objects, called elements or members of the set. A set contains its elements. We write $a \in A$ to denote a is in an element of set A. The notation $a \notin A$ denotes a is not an element of set A.

Here are some sets to remember:

- $\mathbb{N}$ is the set of all natural numbers
- $\mathbb{Z}$ is the set of all integers
- $\mathbb{Z}^+$ is the set of all positive integers
- $\mathbb{Q}$ is the set of all rational numbers
- $\mathbb{R}$ is the set of all real numbers
- $\mathbb{R}^+$ is the set all positive real numbers
- $\mathbb{C}$ is the set of all complex numbers

Two sets are equal only if they contain the same elements.

An empty set is notated as Ø.

A set with one element is a singleton set.

Set A is a subset of set B and set B is the superset of set A if every element of A is also an element of B. To indicate A is a subset of B we write $A \subseteq B$. For the equivalent superset, we write $B \supseteq A$.

For every nonempty set S, there is a guarantee to have at least two subsets, the empty set and the set S itself.

When we want to say that A is a subset of B, but $A \neq B$, we can write $A \subset B$.

If there are n distinct elements in a set S, then the set is finite and n is the cardinality of S. The cardinality of S is denoted as $|S|$.

Otherwise, the set is infinite if it is not finite.

Definition

Given a set S, the power set of S is the set of all subsets of the set S. The power set of S is defined as $\mathcal{P}(S)$.

The power set of a set has 2^n elements. Because sets are unordered, we need to represent ordered collections using ordered n-tuples.

Definition

The ordered n-tuple $(a_1, a_2, \cdots, a_n)$ is the ordered collection that has a_1 as its first element,

1.2 Set Operations

If we let A and B be sets, the union of the sets, $A \cup B$, is the set that contains those elements that are either in A or B, or in both.

The intersection of the sets, $A \cap B$, is the set containing those elements in both A and B.

Two sets are called disjoint if the intersection of the sets is an empty set.

The difference of sets A and B, or $A - B$ is the set containing those elements that are in A but not in B. It is also called the complement of B with respect to A.

Definition

Let U be the universal set. The complement of set A denoted as $\overline{A}$ is the complement of A with respect to U. Therefore the complement of the set A is $U - A$.

Much like the last chapter, there are some set identities and properties

TABLE 1 **Set Identities.**

Identity	*Name*
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

Credits to Rosen again.

The union of a collection of sets is the set that contains those elements that are members of at least one set in the collection.

The intersection of a collection of sets is the set that contains those elements that are members of all sets in the collection.

1.3 Functions

Definition

Let A and B be empty sets. A function f from A to B is an assignment of exactly one element of B to each element of A. We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A. If f is a function from A to B, we write $f : A \to B$.

If f is a function from A to B, we say A is the domain of f and B is the codomain of f. Also if $f(a) = b$, we can say b is the image of a and a is a preimage of b. The range, or image, is the set of all images of elements of A. Also, if f is a function from A to B, we say that f maps from A to B.

A function is called real-valued if its codomain is the set of real numbers and integer-valued if the codomain is the set of integers.

Some functions never assign the same value to two different domain elements. These are called one-to-one functions, or injective functions.

A function is called surjective or onto when the range and codomain are equal.

If a function is both surjective and injective, then it is bijective.

Only a one-to-one function can be invertible because the inverse of a one-to-one function exists.

1.4 Sequences and Summations

Sequences are ordered lists of elements. The terms of a sequence can be specified by providing a formula for each term of the sequence.

A sequence is used to represent an ordered list. We use the notation a_n to denote the image of the integer n. We call a_n a term of the sequence.

A geometric progression is a sequence in the following form:

$$a, ar, ar^2, \cdots, ar^n, \cdots$$

where the initian term a and common ratio r are real numbers.

An arithmetic progression is a seuqnece in the form:

$$a, a+d, a+2d, \cdots, a+nd, \cdots$$

where the initial term a and the common difference d are real numbers.

A recurrence relation for the sequence a_n is an equation that expresses a_n in terms of one or more of the previous terms in the sequence.

A sequence is called a solution of a recurrence relation if its terms satisfy the recurrence relation.

Definition

The Fibonacci sequence, $f_0, f_1, f_2, \cdots$, is defined by the initial conditions $f_0 = 0, f_1 = 1$, and the recurrence relation:

$$f_n = f_{n-1} + f_{n-2}$$

for $n = 2, 3, 4, \cdots$.

Now we introduce the summation notation.

We use the notation $\sum_{j=m}^{n} a_j$ to represent $a_m + a_{m+1} + \cdots + a_n$.

Here j is the index of summation and is abitrary.

Sums of terms of geometric progressions commonly arise.

$$\sum_{j=0}^{n} ar^j = \frac{ar^{n+1} - a}{r - 1}$$

when $r \neq 1$ and $(n+1)a$ when $r = 1$.

Here is some formulae for commonly occurring sums:

TABLE 2 Some Useful Summation Formulae.

Sum	*Closed Form*
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1} - a}{r - 1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x \rvert < 1$	$\frac{1}{1-x}$
$\sum_{k=1}^{\infty} kx^{k-1}, \lvert x \rvert < 1$	$\frac{1}{(1-x)^2}$

Credits to Rosen again.