1 Counting

1.1 The Basics of Counting

Definition

The Product Rule: Suppose that a procedure can be broken down into a sequence of two tasks. If there are n_1 ways to do the first task and for each of these ways of doing the first task, there are n_2 ways to do the second task, then there are $n_1 n_2$ ways to do the procedure.

Definition

The Sum Rule: If a task can be done either in one of n_1 ways or in one of n_2 ways, where none of the set of n_1 ways is the same as any of the set of n_2 ways, then there are $n_1 + n_2$ ways to do the task.

Definition

The Subtraction Rule: If a task can be done in either n_1 ways or n_2 ways, then the number of ways to do the task if $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.

Definition

The Division Rule: There are n/d ways to do a task if it can be done using a procedure that can be carried out in n ways, and for every way w, exactly d of the n ways correspond to way w.

1.2 The Pigeonhole Principle

Theorem 1.1

The Pigeonhole Principle: If k is a positive integer and $k+1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

Corollary 1.2

A function f from a set with $k+1$ or more elements to a set with k elements is not one-to-one.

Theorem 1.3

The Generalized Pigeonhole Principle: If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

Theorem 1.4

Every sequence of $n^2 + 1$ distinct real numbers contains a subsequence of length $n+1$ that is either strictly increasing or strictly decreasing.

1.3 Permuations and Combinations

Theorem 1.5

If n is a positive integer and r is an integer with $1 \leq r \leq n$, then there are

$$P(n,r) = n(n-1)(n-2)\cdots(n-r+1)$$

r-permutations of a set with n distinct elements.

Corollary 1.6

If n and r are integers with $0 \leq r \leq n$, then $P(n,r) = \frac{n!}{(n-r)!}$.

Theorem 1.7

The number of r-combinations of a set with n elements, where n is a nonnegative integer and r is an integer with $0 \leq r \leq n$, equals

$$C(n,r) = \frac{n!}{r!(n-r)!}$$

Corollary 1.8

Let n and r be nonnegative integers with $r \leq n$. Then $C(n,r) = C(n,n-r)$.

Definition

A combinatorial proof of an identity is a proof that uses counting arguments to prove that both sides of the identity count the same objects but in different ways or a proof that is based on showing that there is a bijection between the sets of objects counted by the two sides of the identity. These two types of proofs are called double counting proofs and bijective proofs, respectively.

1.4 Binomial Coefficients and Identities

The number of r-combinations from a set with n elements is often denoted by $\binom{n}{r}$. This number is also called a binomial coefficient because these numbers occur as coefficients in the expansion of powers of binomial expressions such as $(a+b)^n$.

The binomial theorem gives the coefficients of the expansion of powers of binomial expressions. A binomial expression is simply the sum of two terms, such as $x+y$.

Theorem 1.9

Let x and y be variables, and let n be a nonnegative integer. Then

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n$$

We can prove some useful identities from this.

Corollary 1.10

Let n be a nonnegative integer. Then

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Corollary 1.11

Let n be a positve integer. Then

$$\sum_{k=0}^{n} (-k)^k \binom{n}{k} = 0$$

Corollary 1.12

Let n be a nonnegative integer. Then

$$\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n$$

The binomial coefficients satisfy many different identities. We introduce one of the most important of these now.

Theorem 1.13

Let n and k be positive integers with $n \geq k$. Then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

Theorem 1.14

Let m, n, and r be nonnegative integers with r not exceeding either m or n. Then

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k} \binom{n}{k}$$

Corollary 1.15

If n is a nonnegative integer, then

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$$

Theorem 1.16

Let n and r be nonnegative integers with $r \leq n$. Then

$$\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r}$$

1.5 Generalized Permuations and Combinations

Theorem 1.17

The number of r-permutations of a set of n objects with repetition allowed is n^r.

Theorem 1.18

There are $C(n+r-1, r) = C(n+r-1, n-1)$ r-combinations from a set with n elements when repetition of elements is allowed.

TABLE 1 **Combinations and Permutations With and Without Repetition.**

Type	*Repetition Allowed?*	*Formula*
r-permutations	No	$\dfrac{n!}{(n-r)!}$
r-combinations	No	$\dfrac{n!}{r!\,(n-r)!}$
r-permutations	Yes	n^r
r-combinations	Yes	$\dfrac{(n+r-1)!}{r!\,(n-1)!}$

Theorem 1.19

The number of different permutations of n objects, where there are n_1 indistinguishable objects of type 1, n_2 indistinguishable objects of type 2, ..., and n_k indistinguishable objects of type k, is

$$\frac{n!}{n_1!n_2!\cdots n_k!}$$

Theorem 1.20

The number of ways to distribute n distinguishable objects into k distinguishable boxes so that n_i objects are placed into box i, $i = 1, 2, \ldots, k$ equals

$$\frac{n!}{n_1!n_2!\cdots n_k!}$$

There are $C(n+r-1, n-1)$ ways to place r indistinguishable objects into n distinguishable boxes.