1 Relations

The most direct way to express a relationship between elements of two sets is to use ordered pairs made up of two related elements. For this reason, sets of ordered pairs are called binary relations.

Definition

Let A and B be sets. A binary relation from A to B is a subset of $A \times B$.

In other words, a binary relation from A to B is a set R of ordered pairs, where the first element of each ordered pair comes from A and the second element comes from B. We use the notation $a\ R\ b$ to denote that $(a, b) \in R$ and a $a \not R\ b$ to denote that $(a, b) \notin R$. Moreover, when (a, b) belongs to R, a is said to be related to b by R.

Binary relations represent relationships between the elements of two sets. We will introduce n-ary relations, which express relationships among elements of more than two sets.

Definition

A relation on a set A is a relation from A to A.

In other words, a relation on a set A is a subset of $A \times A$.

Definition

A relation R on a set A is called reflexive if $(a, a) \in R$ for every element $a \in A$.

Definition

A relation R is called symmetric if $(b, a) \in R$ whenever $(a, b) \in R$, for all $a, b \in A$. A relation R on a set A such that for all $a, b \in A$, if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$ is called antisymmetric.

Definition

A relation R on a set A is called transitive if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, for all $a, b, c \in A$.

Definition

Let R be a relation from a set A to a set B and S a relation from B to a set C. The composite of R and S is the relation consisting of ordered pairs (a, c), where $a \in A$, $c \in C$, and for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. We denote the composite of R and S by $S \circ R$.

Definition

Let R be a relation on the set A. The powers $R^n.n = 1, 2, 3, \ldots,$ are defined recursively by

$$R^1 = R \text{ and } R^{n+1} = R^n \circ R$$

Theorem 1.1

The relation R on a set A is transitive if and only if

$$R^n \subseteq R \text{ for } n = 1, 2, 3, \ldots$$