1 Matrix Algebra

1.1 Matrix Operations

Sums and Scalar Multiples of Matrices: if A and B are $m \times n$ matrices, $A + B$ is the $m \times n$ whose columns are the sums of the corresponding columns in A and B, the scalar multiple rA is the matrix whose columns are r times the corresponding columns in A.

Theorem 1.1

Matrix addition and scalar multiplication: Let A, B, and C be matrices of the same size, and let r and s be scalars.

1. $A + B = B + A$
2. $(A + B) + C = A + (B + C)$
3. $A + 0 = A$
4. $r(A + B) = rA = rB$
5. $(r + s)A = rA = sA$
6. $r(sA) = (rs)A$

Matrix Multiplication: if A is an $m \times n$ matrix and B is an $n \times p$ matrix with columns $\mathbf{b}_1, \mathbf{b}_2, \ldots \mathbf{b}_p$, then the product AB is the $m \times p$ matrix whose columns are $A\mathbf{b}_1, A\mathbf{b}_2, \ldots, A\mathbf{b}_p$, i.e, $AB = A[\mathbf{b}_1\mathbf{b}_2 \ldots \mathbf{b}_p] = [A\mathbf{b}_1 A\mathbf{b}_2 \ldots A\mathbf{b}_p]$. Matrix multiplication corresponds to composition of linear transformations.

- An efficient Matrix Multpilcation: if the product AB is defined, then the entry in row i and column j of AB is the sum of the products of corresponding entries from row i of A and column j of B. If $(AB)_{ij}$ denotes the (i, j)th entry in AB, and if A is $m \times n$, then $(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$.

Properties of Matrix Multpilcation: Let A be $m \times n$ and let B, C have sizes such that the sums and products are defined:

1. $A(BC) = (AB)C$ associative law
2. $A(B + C) = AB + AC$ left distributive law
3. $(B + C)A = BA + CA$ right distributive law
4. $r(AB) = (rA)B = A(rB)$ for any scalar r
5. $I_mA = A = AI_n$ Identity matrix for multiplication

- Matrix mutiplication is not commutative. In general AB does not equal BA.
- Cancellation laws do not hold for matrix multiplication.
- If $AB = 0$, you cannot conclude either $A = 0$ or $B = 0$.

Powers of a Matrix: If A is $n \times n$ and k is a positive integer, A^k denote the product of k copies of A.

The Transpose of a Matrix: given an $m \times n$ matrix A, the transpose of A is the $n \times m$ matrix whose columns are formed from the corresponding rows of A.

Theorem 1.2: Transpose

Let A, B denote matrices whose sizes are appropriate for the following:

1. $(A^T)^T = A$

2. $(A+B)^T = A^T + B^T$
3. for any scalar r, $(rA)^T = r(A)^T$
4. $(AB)^T = B^T A^T$

1.2 The Inverse of a Matrix

The Matrix Inverse is the matrix analogue of the multiplicative inverse of in real numbers.

- Invertible: an $n \times n$ matrix A is said to be invertible if there is an $n \times n$ matrix A^{-1} such that $A^{-1}A = AA^{-1} = I_n$. In this case A^{-1} is said to be the unique inverse of A.

Notice: because matrix multipilcation is not commutative, both equations are needed.

Singular Matrix: A matrix that is not invertible is a single matrix. An invertible matrix is nonsingular.

Theorem 1.3

Inverse of a 2×2: Let A be the 2×2 matrix shown. If $ab - dc$ is not zero, then A is invertible with A^{-1} as shown:

$$A = \begin{bmatrix} a & d \\ c & d \end{bmatrix} \qquad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Determinant: $\det A = ad - bc$. The theorem says that a 2×2 matrix is invertible iff $\det A$ is not zero.

Theorem 1.4

If A is an invertible matrix, then for each $\mathbf{b}$ in $\mathbf{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has a unique soultion $\mathbf{x} = A^{-1}\mathbf{b}$.

Theorem 1.5

1. If A is an invertible matrix, then A^{-1} is invertible and $(A^{-1})^{-1} = A$
2. If A and B are $n \times n$ invertible matrices, then so is AB and $(AB)^{-1} = B^{-1}A^{-1}$. Generalization: the product of $n \times n$ invertible matrices is invertible, and the inverse is the product of the their inverse in the reverse order.
3. If A is an invertible matrix, then so is A^T, and $(A^T)^{-1} = (A^{-1})^T$

Theorem 1.6

An $n \times n$ matrix is invertible iff it is row equivalent to I_n, and any sequence of elementary row operations that reduces A to I_n also transforms I_n to A^{-1}.

1.3 Characterizations of Invertible Matrices

The Invertible Matrix Theorem: let A be an $n \times n$ matrix. Then the following statements are equivalent.

- A is an invertible matrix.
- A is row equivalent to the $n \times n$ identity matrix.
- A has n pivot positions
- the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution
- the columns of A form a linearly independent set
- the linear transform $\mathbf{x} \to A\mathbf{x}$ is one-to-one

- the equation $A\mathbf{x} = \mathbf{b}$ has at least 1 soln for each $\mathbf{b}$ in $\mathbf{R}^n$
- the columns of A span $\mathbf{R}^n$
- the linear transformation $\mathbf{x} \to A\mathbf{x}$ maps $\mathbf{R}^n$ onto $\mathbf{R}^n$
- there is an $n \times n$ matrix C such that $CA = I$
- there is an $n \times n$ matrix D such that $AD = I$
- A^T is an invertible matrix

Note that this only applies to square matrices.

Theorem 1.7: Inverse Transformation

Let $T : \mathbf{R}^n \to \mathbf{R}^n$ be a linear transformation and let A be the standard matrix for T. Then T is invertible if and only if A is an invertible matrix. In that case, the linear transformation S given by $S(\mathbf{x}) = A^{-1}\mathbf{x}$ is the unique solution satisfying $S(T(\mathbf{x})) = \mathbf{x}$ and $T(S(\mathbf{x})) = \mathbf{x}$ for all $\mathbf{x}$ in $\mathbf{R}^n$.

Recall that matrix multipilcation corresponds to composition of linear transformations. When a matrix A is invertible, the equation $A^{-1}A\mathbf{x} = \mathbf{x}$ can be viewed as a statement about linear transformations. A linear transformation $T : \mathbf{R}^n \to \mathbf{R}^n$ is said to be invertible if there exists a function $S : \mathbf{R}^n \to \mathbf{R}^n$ such that $S(T(\mathbf{x})) = \mathbf{x}$ and $T(S(\mathbf{x})) = \mathbf{x}$ for all $\mathbf{x}$ in $\mathbf{R}^n$.

1.4 Matrix Factorizations

A factorization of a matrix A is an equation that expresses A as a product of two or more matrices.

Whereas matrix multiplication involves a synthesis of data (combining the effects of two or more linear transformations into a single matrix), matrix factorization is an analysis of data.

The LU factorization:

At first assume that A is an $m \times n$ matrix that can be row reduced to echelon form, without row interchanges. Then A can be written in the form $A = LU$, where L is an $m \times m$ lower triangular matrix with 1's on the diagonal and U is an $m \times n$ echelon form of A.

Suppose A can be reduced to an echelon form U using only row replacements that add a multiple of one row to another below it. In this case, there exist unit lower triangular elementary matrices, $E_1, \ldots, E_p$ such that $E_p \cdots E_1 A = U$.

Then $A = (E_p \cdots E_1)^{-1} U = LU$ where $L = (E_p \cdots E_1)^{-1}$.

It can be shown that products and inverses of unit lower triangular matrices are also unit lower triangular. Thus L is unit lower triangular.

Algorithm:

- Reduce A to an echelon form U by a sequence of row replacement operations, if possible.
- Place entries in L such that the same sequence of row operations reduces L to I.