1 Determinants

1.1 Introduction to Determinants

Definition of the Determinant: given an $n \times n$ matrix $A = [a_{ij}]$, the determinant is det A $= a_{11}a_{22} - a_{12}a_{21}$.

For $n \geq 2$, the determinant of an A is the sum of n terms of the form $+/- a_{1j}A_{1j}$ with the plus and minus signs alternating, where the entries $a_{11}, a_{12}, \ldots, a_{1n}$ are from the first row of A.

The (i, j)-cofactor $= C_{ij} = (-1)^{i+j}$ det A$_{ij}$ s.t. det A $= a_{11}C_{11} + a_{12} + C_{12} + \cdots + a_{1n} + C_{1n}$.

Theorem 1.1

The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column/

The expansion across the ith row using the cofactors is: det A $= a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}Cin$

The expansion across the jth column is: det A $= a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + n_{nj}C_{nj}$.

This theorem is helpful for computing determinants of a matrix that contains many zeros. For example, if 1 row contains many zeros, than a cofactor expansion across that row will be easier to calculate.

Theorem 1.2

If a is a triangular matrix, then det A is the product of the entries on the main diagonal of A.

1.2 Properties of Determinants

"The secret of determinants lies in how they change when row operations are performed"

Theorem 1.3

Let A be a square matrix.

- A multiple of one row of A is added to another row to produce a matrix B, then det $B =$ det A
- if two rows of A are interchanged to produce B then det $B =$ -det A
- if 1 row of A is multiplied by k to produce B, then det $B = k$det A

we can use a strategy to reduce a matrix to echelon form and then use the fact that the determinant of a triangular matrix is the product of its diagonal entries.

Theorem 1.4

A square matrix A is invertible if and only if det A is not zero. If A is an $n \times n$ matrix, then det $A^T =$ det A.

Theorem 1.5

If A and B are $n \times n$ matrices, then det $AB =$ (det A)(det B)

1.3 Cramer's Rule, Volume, and Linear Transformations

Cramer's Rule: Cramers rule is needed for a variety of theoretical calculations. However the formula is inefficient for hand calculations except for 2×2.

For any $n \times n$ matrix A and any **b** in $\mathbf{R}^n$ let $A_1(\mathbf{b})$ be the matrix obtained from A by replacing the ith column by **b**, $A_i(\mathbf{b}) = [\mathbf{a}_1 \mathbf{a}_2 \ldots \mathbf{a}_{i-1} \mathbf{b} \mathbf{a}_{i+1} \ldots \mathbf{a}_n]$

Theorem 1.6

Let A be an intertible invertible $n \times n$ matrix. For any **b** in $\mathbf{R}^n$, the unique solution **x** of $A\mathbf{x} = \mathbf{b}$ has entries given by: $\mathbf{x}_i = (\det A_i(\mathbf{B})/(\det A))$ $i = 1, 2, \ldots, n$.

Application to engineering: A number of important engineering problems can be analyzed by Laplace transformations. This approach converts an appropriate system of linear differential equations into a system of linear algebraic equations whose coefficient involve a parameter.

Formula for A^{-1}: Cramer's Rule leads to a general formula for the inverse of an $n \times n$ matrix.

Theorem 1.7

Let A be an invertible $n \times n$ matrix. Then A^{-1} is given by: (Google This)

Determinants as Area of Volume

Theorem 1.8

If A is a 2×2 matrix, the area of the parallelogram determined by the columns of A is —det A—. If A is a 3×3 matrix, the volume of the paralellepiped determined by the columns is —det A—.

Theorem 1.9

Let $T : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ be the linear the transformation obtained by a 2×2 matrix A. If S is a parallelogram in $\mathbf{R}^2$ then the area of $T(S)$ is equal to —det A— times the area of S.

If T is determined by a 3×3 matrix A, and S is a paralellepiped in $\mathbf{R}^3$, then the volume of $T(S)$ is equal to the area of —det A— times the volume of S.