1 Symmetric Matrices and Quadratic Forms

1.1 Diagonalization of Symmetric Matrices

A symmetric matrix is a matrix A such that $A^T = A$. Such a matrix is necessarily square. Its main diagonal entries are arbitrary, but its other entries occur in pairs - in opposite sides of the main diagonal.

Theorem 1.1

If A is symmetric, then any two eigenvectors from different eigenspaces are orthogonal.

Theorem 1.2

An $n \times n$ matrix A is orthogonally diagonalizable if and only if A is a symmetric matrix.

The set of eigenvalues of a matrix A is sometimes called the spectrum of A, and the spectral theorem can describe the eigenvalues.

Theorem 1.3

An $n \times n$ symmetric matrix A has the following properties:

- A has n real eigenvalues, counting multiplicities.
- The dimension of the eigenspace for each eigenvalue λ equals the multiplicity of λ as a root of the characteristic equation.
- The eigenspaces are mutually orthogonal, in the sense that eigenvectors corresponding to different eigenvalues are orthogonal.
- A is orthogonally diagonalizable.

1.2 Quadratic Forms

A quadratic form on $\mathbb{R}^n$ is a function Q defined on $\mathbb{R}^n$ whose value at a vector $\mathbf{x}$ in $\mathbb{R}^n$ can be computed by an expression of the form $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$, where A is an $n \times n$ symmetric matrix. The matrix A is called the matrix of the quadratic form.

If $\mathbf{x}$ represents a variable vector in $\mathbb{R}^n$, then a change of variable is an equation of the form

$$\mathbf{x} = P\mathbf{y}$$

where P is an invertible matrix and $\mathbf{y}$ is a new variable vector in $\mathbb{R}^n$. Here $\mathbf{y}$ is the coordinate vector of $\mathbf{x}$ relative to the basis of $\mathbb{R}^n$ determined by the columns of P.

Theorem 1.4

Let A be an $n \times n$ symmetric matrix. Then there is an orthogonal change of variable, $\mathbf{x} = P\mathbf{y}$, that transforms the quadratic form $\mathbf{x}^T A\mathbf{x}$ into a quadratic form $\mathbf{y}^T D\mathbf{y}$ with no cross-product term.

Definition

A quadratic form Q is

- positive definite if $Q(\mathbf{x}) > 0$ for all $\mathbf{x} \neq 0$
- negative definite if $Q(\mathbf{x}) < 0$ for all $\mathbf{x} \neq 0$
- indefinite if $Q(\mathbf{x})$ assumes both positive and negative values.

The same as above can be generalized to eigenvalues for a quadratic form $\mathbf{x}^T A\mathbf{x}$ if the eigenvalues of A are positive, negative, or has both respectively.