13

Vectors and the Geometry of Space

Chapter Preview We now make a significant departure from previous chapters by stepping out of the *xy*-plane $(\mathbb{R}^2)$ into three-dimensional space $(\mathbb{R}^3)$. The fundamental concept of a *vector*—a quantity with magnitude and direction—is introduced in two and three dimensions. We then develop the algebra associated with vectors (how to add, subtract, and combine them in various ways), and we define two fundamental operations for vectors: the dot product and the cross product. The chapter concludes with a brief survey of basic objects in three-dimensional geometry, namely lines, planes, and elementary surfaces.

13.1 Vectors in the Plane

Imagine a raft drifting down a river, carried by the current. The speed and direction of the raft at a point may be represented by an arrow (**Figure 13.1**). The length of the arrow represents the speed of the raft at that point; longer arrows correspond to greater speeds. The orientation of the arrow gives the direction in which the raft is headed at that point. The arrows at points A and C in Figure 13.1 have the same length and direction, indicating that the raft has the same speed and heading at these locations. The arrow at B is shorter and points to the left of the rock, indicating that the raft slows down as it nears the rock.

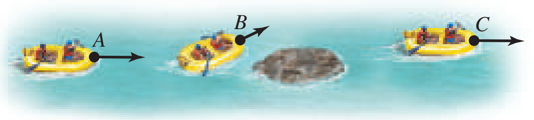
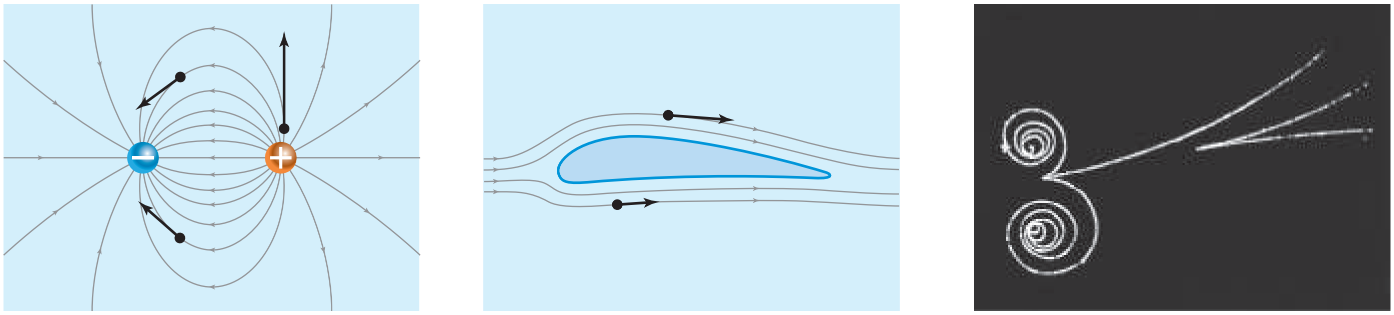


Figure 13.1

Basic Vector Operations

The arrows that describe the raft's motion are examples of *vectors*—quantities that have both *length* (or *magnitude*) and *direction*. Vectors arise naturally in many situations. For example, electric and magnetic fields, the flow of air over an airplane wing, and the velocity and acceleration of elementary particles are described by vectors (**Figure 13.2**). In this section, we examine vectors in the *xy*-plane; then we extend the concept to three dimensions in Section 13.2.

The vector whose *tail* is at the point P and whose *head* is at the point Q is denoted $\overrightarrow{PQ}$ (**Figure 13.3**). The vector $\overrightarrow{QP}$ has its tail at Q and its head at P. We also label vectors with single boldface characters such as $\mathbf{u}$ and $\mathbf{v}$.



Electric field vectors due to two charges

Velocity vectors of air flowing over an airplane wing

Tracks of elementary particles in a cloud chamber are aligned with the velocity vectors of the particles.

Figure 13.2

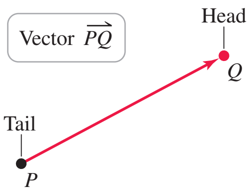


Figure 13.3

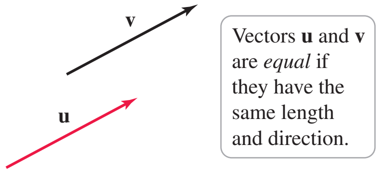


Figure 13.4

Two vectors **u** and **v** are *equal*, written $\mathbf{u} = \mathbf{v}$, if they have equal length and point in the same direction (**Figure 13.4**). An important fact is that equal vectors do not necessarily have the same location. *Any* two vectors with the same length and direction are equal.

Not all quantities are represented by vectors. For example, mass, temperature, and price have magnitude, but no direction. Such quantities are described by real numbers and are called *scalars*.

➤ In this text, *scalar* is another word for *real number*.

➤ The vector **v** is commonly handwritten as $\vec{v}$. The zero vector is handwritten as $\vec{0}$.

Vectors, Equal Vectors, Scalars, Zero Vector

Vectors are quantities that have both length (or magnitude) and direction. Two vectors are **equal** if they have the same magnitude and direction. Quantities having magnitude but no direction are called **scalars**. One exception is the **zero vector**, denoted **0**: It has length 0 and no direction.

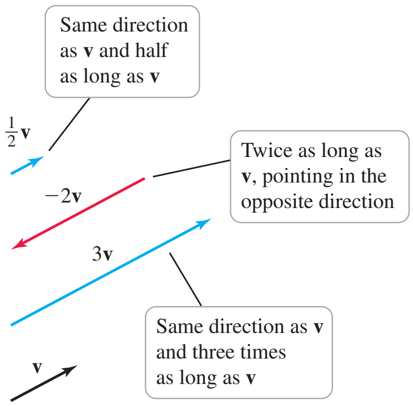


Figure 13.5

Scalar Multiplication

A scalar c and a vector **v** can be combined using scalar-vector multiplication, or simply *scalar multiplication*. The resulting vector, denoted $c\mathbf{v}$, is called a *scalar multiple* of **v**. The length of $c\mathbf{v}$ is $|c|$ multiplied by the length of **v**. The vector $c\mathbf{v}$ has the same direction as **v** if $c > 0$. If $c < 0$, then $c\mathbf{v}$ and **v** point in opposite directions. If $c = 0$, then the product $0\mathbf{v} = \mathbf{0}$ (the zero vector).

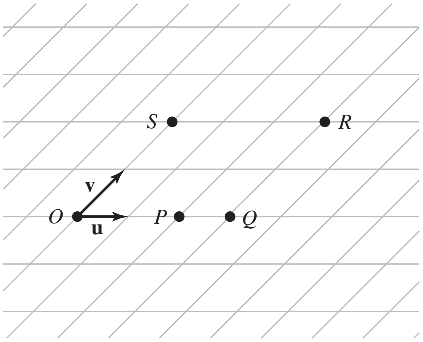
For example, the vector $3\mathbf{v}$ is three times as long as **v** and has the same direction as **v**. The vector $-2\mathbf{v}$ is twice as long as **v**, but it points in the opposite direction. The vector $\frac{1}{2}\mathbf{v}$ points in the same direction as **v** and has half the length of **v** (**Figure 13.5**). The vectors **v**, $3\mathbf{v}$, $-2\mathbf{v}$, and $\frac{1}{2}\mathbf{v}$ are examples of *parallel vectors*: Each one is a scalar multiple of the others.

DEFINITION Scalar Multiples and Parallel Vectors

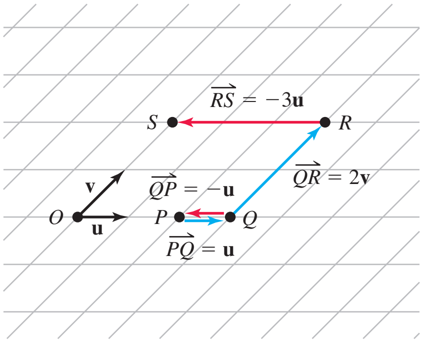
Given a scalar c and a vector **v**, the **scalar multiple** $c\mathbf{v}$ is a vector whose length is $|c|$ multiplied by the length of **v**. If $c > 0$, then $c\mathbf{v}$ has the same direction as **v**. If $c < 0$, then $c\mathbf{v}$ and **v** point in opposite directions. Two vectors are **parallel** if they are scalar multiples of each other.

➤ For convenience, we write $-\mathbf{u}$ for $(-1)\mathbf{u}$, $-c\mathbf{u}$ for $(-c)\mathbf{u}$, and $\mathbf{u}/c$ for $\frac{1}{c}\mathbf{u}$.

QUICK CHECK 1 Describe the length and direction of the vector $-5\mathbf{v}$ relative to $\mathbf{v}$. ◀



(a)



(b)

Figure 13.6

QUICK CHECK 2 Sketch the sum $\mathbf{v}_a + \mathbf{w}$ in Figure 13.7 if the direction of $\mathbf{w}$ is reversed. ◀

Notice that $0\mathbf{v} = \mathbf{0}$ for all vectors $\mathbf{v}$. It follows that *the zero vector is parallel to all vectors*. While it may seem counterintuitive, this result turns out to be a useful convention.

EXAMPLE 1 **Parallel vectors** Using Figure 13.6a, write the following vectors in terms of $\mathbf{u}$ or $\mathbf{v}$.

a. $\overrightarrow{PQ}$ **b.** $\overrightarrow{QP}$ **c.** $\overrightarrow{QR}$ **d.** $\overrightarrow{RS}$

SOLUTION

a. The vector $\overrightarrow{PQ}$ has the same direction and length as $\mathbf{u}$; therefore, $\overrightarrow{PQ} = \mathbf{u}$. These two vectors are equal even though they have different locations (Figure 13.6b).

b. Because $\overrightarrow{QP}$ and $\mathbf{u}$ have equal length but opposite directions, $\overrightarrow{QP} = (-1)\mathbf{u} = -\mathbf{u}$.

c. $\overrightarrow{QR}$ points in the same direction as $\mathbf{v}$ and is twice as long as $\mathbf{v}$, so $\overrightarrow{QR} = 2\mathbf{v}$.

d. $\overrightarrow{RS}$ points in the direction opposite that of $\mathbf{u}$ with three times the length of $\mathbf{u}$. Consequently, $\overrightarrow{RS} = -3\mathbf{u}$.

Related Exercise 15 ◀

Vector Addition and Subtraction

To illustrate the idea of vector addition, consider a plane flying horizontally at a constant speed in a crosswind (Figure 13.7). The length of vector $\mathbf{v}_a$ represents the plane's *airspeed*, which is the speed the plane would have in still air; $\mathbf{v}_a$ points in the direction of the nose of the plane. The wind vector $\mathbf{w}$ points in the direction of the crosswind and has a length equal to the speed of the crosswind. The combined effect of the motion of the plane and the wind is the *vector sum* $\mathbf{v}_g = \mathbf{v}_a + \mathbf{w}$, which is the velocity of the plane relative to the ground.

Figure 13.8 illustrates two ways to form the vector sum of two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ geometrically. The first method, called the **Triangle Rule**, places the tail of $\mathbf{v}$ at the head of $\mathbf{u}$. The sum $\mathbf{u} + \mathbf{v}$ is the vector that extends from the tail of $\mathbf{u}$ to the head of $\mathbf{v}$ (Figure 13.8b).

When $\mathbf{u}$ and $\mathbf{v}$ are not parallel, another way to form $\mathbf{u} + \mathbf{v}$ is to use the **Parallelogram Rule**. The *tails* of $\mathbf{u}$ and $\mathbf{v}$ are connected to form adjacent sides of a parallelogram; then the remaining two sides of the parallelogram are sketched. The sum $\mathbf{u} + \mathbf{v}$ is the vector that coincides with the diagonal of the parallelogram, beginning at the tails of $\mathbf{u}$ and $\mathbf{v}$ (Figure 13.8c). Both the Triangle Rule and the Parallelogram Rule produce the same vector sum $\mathbf{u} + \mathbf{v}$.

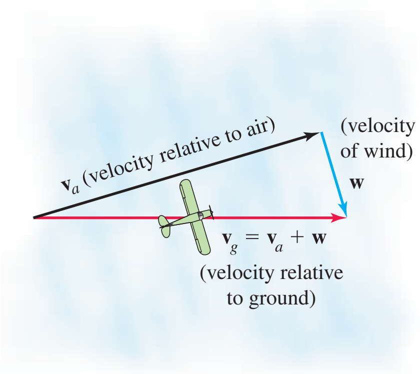
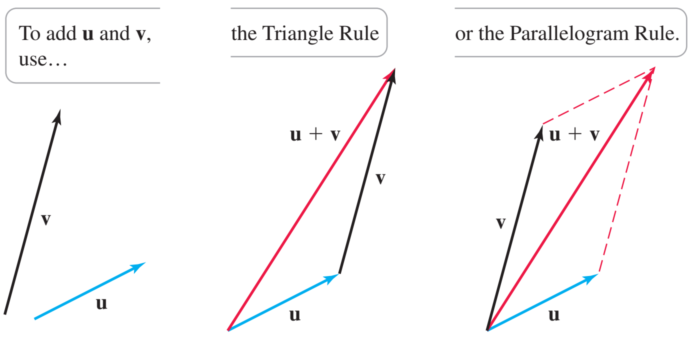


Figure 13.7



(a) (b) (c)

Figure 13.8

QUICK CHECK 3 Use the Triangle Rule to show that the vectors in Figure 13.8 satisfy $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$. ◀

The difference $\mathbf{u} - \mathbf{v}$ is defined to be the sum $\mathbf{u} + (-\mathbf{v})$. By the Triangle Rule, the tail of $-\mathbf{v}$ is placed at the head of $\mathbf{u}$; then $\mathbf{u} - \mathbf{v}$ extends from the tail of $\mathbf{u}$ to the head of $-\mathbf{v}$ (Figure 13.9a). Equivalently, when the tails of $\mathbf{u}$ and $\mathbf{v}$ coincide, $\mathbf{u} - \mathbf{v}$ has its tail at the head of $\mathbf{v}$ and its head at the head of $\mathbf{u}$ (Figure 13.9b).

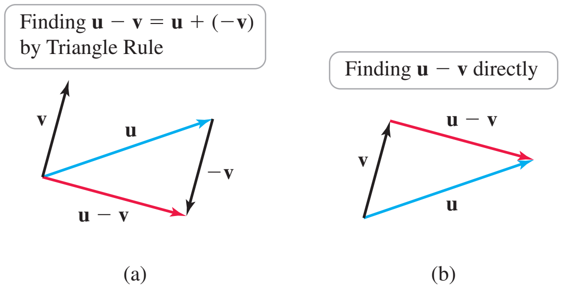


Figure 13.9

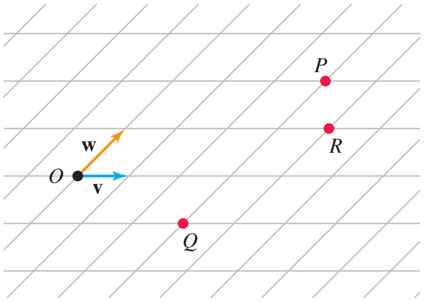


Figure 13.10

EXAMPLE 2 **Vector operations** Use Figure 13.10 to write the following vectors as sums of scalar multiples of $\mathbf{v}$ and $\mathbf{w}$.

a. $\overrightarrow{OP}$ **b.** $\overrightarrow{OQ}$ **c.** $\overrightarrow{QR}$

SOLUTION

a. Using the Triangle Rule, we start at O, move three lengths of $\mathbf{v}$ in the direction of $\mathbf{v}$ and then two lengths of $\mathbf{w}$ in the direction of $\mathbf{w}$ to reach P. Therefore, $\overrightarrow{OP} = 3\mathbf{v} + 2\mathbf{w}$ (Figure 13.11a).

b. The vector $\overrightarrow{OQ}$ coincides with the diagonal of a parallelogram having adjacent sides equal to $3\mathbf{v}$ and $-\mathbf{w}$. By the Parallelogram Rule, $\overrightarrow{OQ} = 3\mathbf{v} - \mathbf{w}$ (Figure 13.11b).

c. The vector $\overrightarrow{QR}$ lies on the diagonal of a parallelogram having adjacent sides equal to $\mathbf{v}$ and $2\mathbf{w}$. Therefore, $\overrightarrow{QR} = \mathbf{v} + 2\mathbf{w}$ (Figure 13.11c).

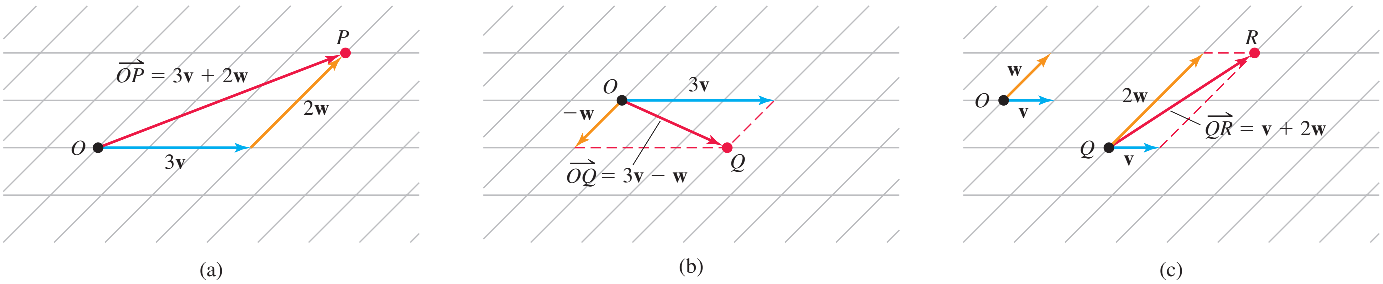


Figure 13.11

Related Exercises 17–18 ◀

Vector Components

So far, vectors have been examined from a geometric point of view. To do calculations with vectors, it is necessary to introduce a coordinate system. We begin by considering a vector $\mathbf{v}$ whose tail is at the origin in the Cartesian plane and whose head is at the point (v_1, v_2) (Figure 13.12a).

➤ Round brackets (a, b) enclose the *coordinates* of a point, while angle brackets $\langle a, b \rangle$ enclose the *components* of a vector. Note that in component form, the zero vector is $\mathbf{0} = \langle 0, 0 \rangle$.

DEFINITION Position Vectors and Vector Components

A vector $\mathbf{v}$ with its tail at the origin and head at the point (v_1, v_2) is called a **position vector** (or is said to be in **standard position**) and is written $\langle v_1, v_2 \rangle$. The real numbers v_1 and v_2 are the x- and y-**components** of $\mathbf{v}$, respectively. The position vectors $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ are **equal** if and only if $u_1 = v_1$ and $u_2 = v_2$.

There are infinitely many vectors equal to the position vector $\mathbf{v}$, all with the same length and direction (Figure 13.12b). It is important to abide by the convention that $\mathbf{v} = \langle v_1, v_2 \rangle$ refers to the position vector $\mathbf{v}$ *or to any other vector equal to* $\mathbf{v}$.

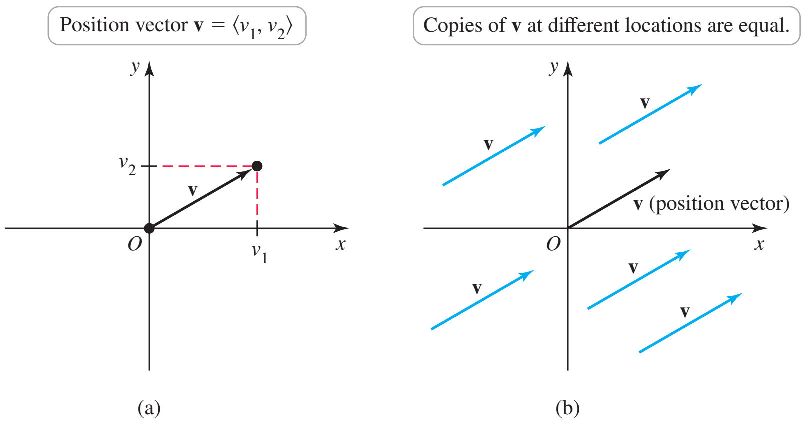


Figure 13.12

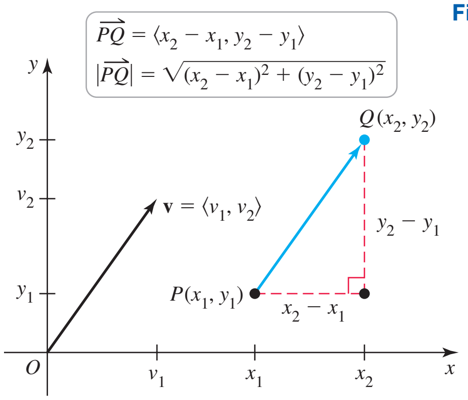


Figure 13.13

Now consider the vector $\overrightarrow{PQ}$ equal to $\mathbf{v} = \langle v_1, v_2 \rangle$, but not in standard position, with its tail at the point $P(x_1, y_1)$ and its head at the point $Q(x_2, y_2)$. The x-component of $\overrightarrow{PQ}$ is the difference in the x-coordinates of Q and P, or $x_2 - x_1$. The y-component of $\overrightarrow{PQ}$ is the difference in the y-coordinates, $y_2 - y_1$ (**Figure 13.13**). Therefore, $\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1 \rangle = \langle v_1, v_2 \rangle = \mathbf{v}$.

As already noted, there are infinitely many vectors equal to a given position vector. All these vectors have the same length and direction; therefore, they are all equal. In other words, two arbitrary vectors are equal if they are equal to the same position vector. For example, the vector $\overrightarrow{PQ}$ from $P(2, 5)$ to $Q(6, 3)$ and the vector $\overrightarrow{AB}$ from $A(7, 12)$ to $B(11, 10)$ are equal because they both equal the position vector $\langle 4, -2 \rangle$.

Magnitude

The magnitude of a vector is simply its length. By the Pythagorean Theorem and Figure 13.13, we have the following definition.

QUICK CHECK 4 Given the points $P(2, 3)$ and $Q(-4, 1)$, find the components of $\overrightarrow{PQ}$. ◄

➤ Just as the absolute value $|p - q|$ gives the distance between the points p and q on the number line, the magnitude $|\overrightarrow{PQ}|$ is the distance between the points P and Q. The magnitude of a vector is also called its **norm**.

> **DEFINITION Magnitude of a Vector**
>
> Given the points $P(x_1, y_1)$ and $Q(x_2, y_2)$, the **magnitude**, or **length**, of $\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1 \rangle$, denoted $|\overrightarrow{PQ}|$, is the distance between P and Q:
>
> $$|\overrightarrow{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$
>
> The magnitude of the position vector $\mathbf{v} = \langle v_1, v_2 \rangle$ is $|\mathbf{v}| = \sqrt{v_1^2 + v_2^2}$.

EXAMPLE 3 Calculating components and magnitude Given the points $O(0, 0)$, $P(-3, 4)$, and $Q(6, 5)$, find the components and magnitude of the following vectors.

a. $\overrightarrow{OP}$ **b.** $\overrightarrow{PQ}$

SOLUTION

a. The vector $\overrightarrow{OP}$ is the position vector whose head is located at $P(-3, 4)$. Therefore, $\overrightarrow{OP} = \langle -3, 4 \rangle$ and its magnitude is $|\overrightarrow{OP}| = \sqrt{(-3)^2 + 4^2} = 5$.

b. $\overrightarrow{PQ} = \langle 6 - (-3), 5 - 4 \rangle = \langle 9, 1 \rangle$ and $|\overrightarrow{PQ}| = \sqrt{9^2 + 1^2} = \sqrt{82}$.

Related Exercise 19 ◄

Vector Operations in Terms of Components

We now show how vector addition, vector subtraction, and scalar multiplication are performed using components. Suppose $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$. The vector sum of $\mathbf{u}$ and $\mathbf{v}$ is $\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2 \rangle$. This definition of a vector sum is consistent with the Parallelogram Rule given earlier (**Figure 13.14**).

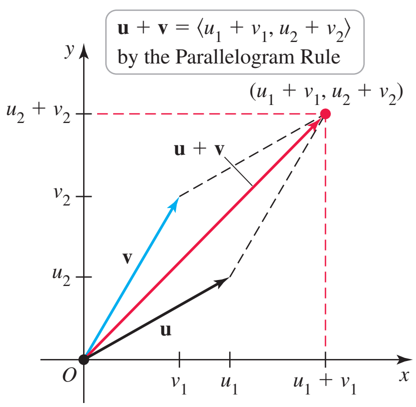


Figure 13.14

For a scalar c and a vector $\mathbf{u}$, the scalar multiple $c\mathbf{u}$ is $c\mathbf{u} = \langle cu_1, cu_2 \rangle$; that is, the scalar c multiplies each component of $\mathbf{u}$. If $c > 0$, $\mathbf{u}$ and $c\mathbf{u}$ have the same direction (Figure 13.15a). If $c < 0$, $\mathbf{u}$ and $c\mathbf{u}$ have opposite directions (Figure 13.15b). In either case, $|c\mathbf{u}| = |c||\mathbf{u}|$ (Exercise 83).

Notice that $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$, where $-\mathbf{v} = \langle -v_1, -v_2 \rangle$. Therefore, the vector difference of $\mathbf{u}$ and $\mathbf{v}$ is $\mathbf{u} - \mathbf{v} = \langle u_1 - v_1, u_2 - v_2 \rangle$.

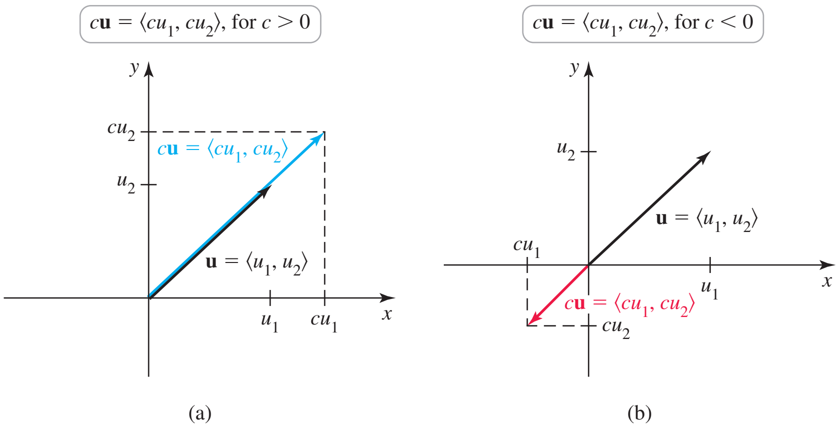


Figure 13.15

> **DEFINITION Vector Operations in $\mathbb{R}^2$**
>
> Suppose c is a scalar, $\mathbf{u} = \langle u_1, u_2 \rangle$, and $\mathbf{v} = \langle v_1, v_2 \rangle$.
>
> $\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2 \rangle$ Vector addition
>
> $\mathbf{u} - \mathbf{v} = \langle u_1 - v_1, u_2 - v_2 \rangle$ Vector subtraction
>
> $c\mathbf{u} = \langle cu_1, cu_2 \rangle$ Scalar multiplication

➤ Recall that $\mathbb{R}^2$ (pronounced *R-two*) stands for the *xy*-plane or the set of all ordered pairs of real numbers.

EXAMPLE 4 **Vector operations** Let $\mathbf{u} = \langle -1, 2 \rangle$ and $\mathbf{v} = \langle 2, 3 \rangle$.

a. Evaluate $|\mathbf{u} + \mathbf{v}|$.

b. Simplify $2\mathbf{u} - 3\mathbf{v}$.

c. Find two vectors half as long as $\mathbf{u}$ and parallel to $\mathbf{u}$.

SOLUTION

a. Because $\mathbf{u} + \mathbf{v} = \langle -1, 2 \rangle + \langle 2, 3 \rangle = \langle 1, 5 \rangle$, we have

$|\mathbf{u} + \mathbf{v}| = \sqrt{1^2 + 5^2} = \sqrt{26}.$

b. $2\mathbf{u} - 3\mathbf{v} = 2\langle -1, 2 \rangle - 3\langle 2, 3 \rangle = \langle -2, 4 \rangle - \langle 6, 9 \rangle = \langle -8, -5 \rangle$

c. The vectors $\frac{1}{2}\mathbf{u} = \frac{1}{2}\langle -1, 2 \rangle = \langle -\frac{1}{2}, 1 \rangle$ and $-\frac{1}{2}\mathbf{u} = -\frac{1}{2}\langle -1, 2 \rangle = \langle \frac{1}{2}, -1 \rangle$ have half the length of $\mathbf{u}$ and are parallel to $\mathbf{u}$.

Related Exercises 26, 28, 30 ◄

Unit Vectors

A **unit vector** is any vector with length 1. Two useful unit vectors are the **coordinate unit vectors** $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$ (Figure 13.16). These vectors are directed along the coordinate axes and enable us to express all vectors in an alternative form. For example, by the Triangle Rule (Figure 13.17a),

$$\langle 3, 4 \rangle = 3\langle 1, 0 \rangle + 4\langle 0, 1 \rangle = 3\mathbf{i} + 4\mathbf{j}.$$

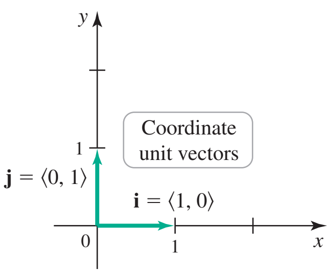


Figure 13.16

In general, the vector $\mathbf{v} = \langle v_1, v_2 \rangle$ (Figure 13.17b) is also written

$$\mathbf{v} = v_1\langle 1, 0 \rangle + v_2\langle 0, 1 \rangle = v_1\mathbf{i} + v_2\mathbf{j}.$$

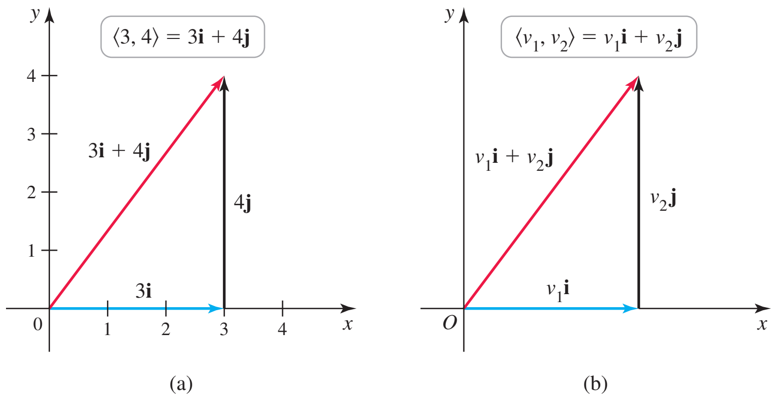


Figure 13.17

➤ Coordinate unit vectors are also called **standard basis vectors**.

$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|}$ and $-\mathbf{u} = -\frac{\mathbf{v}}{|\mathbf{v}|}$ have length 1.

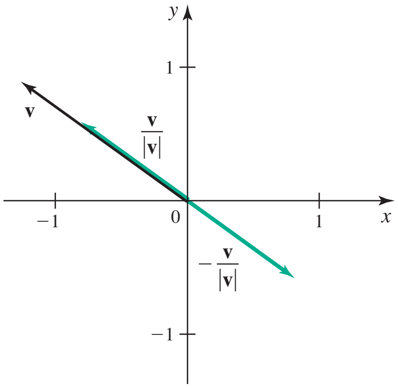


Figure 13.18

QUICK CHECK 5 Find vectors of length 10 parallel to the unit vector $\mathbf{u} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$. ◀

Given a nonzero vector $\mathbf{v}$, we sometimes need to construct a new vector parallel to $\mathbf{v}$ of a specified length. Dividing $\mathbf{v}$ by its length, we obtain the vector $\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|}$. Because $\mathbf{u}$ is a positive scalar multiple of $\mathbf{v}$, it follows that $\mathbf{u}$ has the same direction as $\mathbf{v}$. Furthermore, $\mathbf{u}$ is a unit vector because $|\mathbf{u}| = \frac{|\mathbf{v}|}{|\mathbf{v}|} = 1$. The vector $-\mathbf{u} = -\frac{\mathbf{v}}{|\mathbf{v}|}$ is also a unit vector with a direction opposite that of $\mathbf{v}$ (Figure 13.18). Therefore, $\pm\frac{\mathbf{v}}{|\mathbf{v}|}$ are unit vectors parallel to $\mathbf{v}$ that point in opposite directions.

To construct a vector that points in the direction of $\mathbf{v}$ and has a specified length $c > 0$, we form the vector $\frac{c\mathbf{v}}{|\mathbf{v}|}$. It is a positive scalar multiple of $\mathbf{v}$, so it points in the direction of $\mathbf{v}$, and its length is $\left|\frac{c\mathbf{v}}{|\mathbf{v}|}\right| = |c|\frac{|\mathbf{v}|}{|\mathbf{v}|} = c$. The vector $-\frac{c\mathbf{v}}{|\mathbf{v}|}$ points in the opposite direction and also has length c. With this construction, we can also write $\mathbf{v}$ as the product of its magnitude and a unit vector in the direction of $\mathbf{v}$:

$$\mathbf{v} = \underbrace{|\mathbf{v}|}_{\text{magnitude}} \cdot \underbrace{\frac{\mathbf{v}}{|\mathbf{v}|}}_{\text{direction}}.$$

EXAMPLE 5 **Magnitude and unit vectors** Consider the points $P(1, -2)$ and $Q(6, 10)$.

a. Find $\overrightarrow{PQ}$ and two unit vectors parallel to $\overrightarrow{PQ}$.

b. Find two vectors of length 2 parallel to $\overrightarrow{PQ}$.

c. Express $\overrightarrow{PQ}$ as the product of its magnitude and a unit vector.

SOLUTION

a. $\overrightarrow{PQ} = \langle 6 - 1, 10 - (-2)\rangle = \langle 5, 12\rangle$, or $5\mathbf{i} + 12\mathbf{j}$. Because $|\overrightarrow{PQ}| = \sqrt{5^2 + 12^2} = \sqrt{169} = 13$, a unit vector parallel to $\overrightarrow{PQ}$ is

$$\frac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|} = \frac{\langle 5, 12\rangle}{13} = \left\langle \frac{5}{13}, \frac{12}{13} \right\rangle = \frac{5}{13}\mathbf{i} + \frac{12}{13}\mathbf{j}.$$

The unit vector parallel to $\overrightarrow{PQ}$ with the opposite direction is $\left\langle -\frac{5}{13}, -\frac{12}{13} \right\rangle$.

b. To obtain two vectors of length 2 that are parallel to $\overrightarrow{PQ}$, we multiply the unit vector $\frac{5}{13}\mathbf{i} + \frac{12}{13}\mathbf{j}$ by ± 2:

$$2\left(\frac{5}{13}\mathbf{i} + \frac{12}{13}\mathbf{j}\right) = \frac{10}{13}\mathbf{i} + \frac{24}{13}\mathbf{j} \quad \text{and} \quad -2\left(\frac{5}{13}\mathbf{i} + \frac{12}{13}\mathbf{j}\right) = -\frac{10}{13}\mathbf{i} - \frac{24}{13}\mathbf{j}.$$

QUICK CHECK 6 Verify that the vector $\left\langle \frac{5}{13}, \frac{12}{13} \right\rangle$ has length 1. ◄

c. The unit vector $\left\langle \frac{5}{13}, \frac{12}{13} \right\rangle$ points in the direction of $\overrightarrow{PQ}$, so we have

$$\overrightarrow{PQ} = |\overrightarrow{PQ}| \frac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|} = 13\left\langle \frac{5}{13}, \frac{12}{13} \right\rangle.$$

Related Exercises 34, 43, 45 ◄

Properties of Vector Operations

➤ The Parallelogram Rule illustrates the commutative property $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

When we stand back and look at vector operations, ten general properties emerge. For example, the first property says that vector addition is commutative, which means $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$. This property is proved by letting $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$. By the commutative property of addition for real numbers,

$$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2 \rangle = \langle v_1 + u_1, v_2 + u_2 \rangle = \mathbf{v} + \mathbf{u}.$$

The proofs of other properties are outlined in Exercises 78–81.

SUMMARY Properties of Vector Operations

Suppose $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors and a and c are scalars. Then the following properties hold (for vectors in any number of dimensions).

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$	Commutative property of addition
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$	Associative property of addition
3. $\mathbf{v} + \mathbf{0} = \mathbf{v}$	Additive identity
4. $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$	Additive inverse
5. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$	Distributive property 1
6. $(a + c)\mathbf{v} = a\mathbf{v} + c\mathbf{v}$	Distributive property 2
7. $0\mathbf{v} = \mathbf{0}$	Multiplication by zero scalar
8. $c\mathbf{0} = \mathbf{0}$	Multiplication by zero vector
9. $1\mathbf{v} = \mathbf{v}$	Multiplicative identity
10. $a(c\mathbf{v}) = (ac)\mathbf{v}$	Associative property of scalar multiplication

These properties allow us to solve vector equations. For example, to solve the equation $\mathbf{u} + \mathbf{v} = \mathbf{w}$ for $\mathbf{u}$, we proceed as follows:

$$\begin{aligned} (\mathbf{u} + \mathbf{v}) + (-\mathbf{v}) &= \mathbf{w} + (-\mathbf{v}) && \text{Add } -\mathbf{v} \text{ to both sides.} \\ \mathbf{u} + \underbrace{(\mathbf{v} + (-\mathbf{v}))}_{\mathbf{0}} &= \mathbf{w} + (-\mathbf{v}) && \text{Property 2} \\ \mathbf{u} + \mathbf{0} &= \mathbf{w} - \mathbf{v} && \text{Property 4} \\ \mathbf{u} &= \mathbf{w} - \mathbf{v}. && \text{Property 3} \end{aligned}$$

QUICK CHECK 7 Solve $3\mathbf{u} + 4\mathbf{v} = 12\mathbf{w}$ for $\mathbf{u}$. ◄

Applications of Vectors

Vectors have countless practical applications, particularly in the physical sciences and engineering. These applications are explored throughout the remainder of this text. For now, we present two common uses of vectors: to describe velocities and forces.

➤ *Velocity of the boat relative to the water* means the velocity (direction and speed) the boat has relative to someone traveling with the current.

Velocity Vectors Consider a motorboat crossing a river whose current is everywhere represented by the constant vector $\mathbf{w}$ (Figure 13.19); this means that $|\mathbf{w}|$ is the speed of the moving water and $\mathbf{w}$ points in the direction of the moving water. Assume the vector $\mathbf{v}_w$ gives the velocity of the boat relative to the water. The combined effect of $\mathbf{w}$ and $\mathbf{v}_w$ is the sum $\mathbf{v}_g = \mathbf{v}_w + \mathbf{w}$, which is the velocity of the boat that would be observed by someone on the shore (or on the ground).

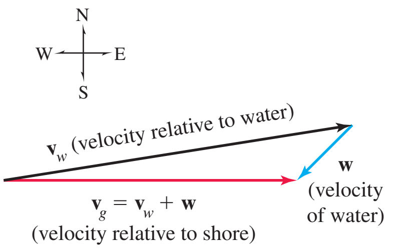


Figure 13.19

EXAMPLE 6 **Speed of a boat in a current** Suppose the water in a river moves southwest (45° west of south) at 4 mi/hr and a motorboat travels due east at 15 mi/hr relative to the shore. Determine the speed of the boat and its heading relative to the moving water (Figure 13.19).

SOLUTION To solve this problem, the vectors are placed in a coordinate system (**Figure 13.20**). Because the boat moves east at 15 mi/hr, the velocity relative to the shore is $\mathbf{v}_g = \langle 15, 0 \rangle$. To obtain the components of $\mathbf{w} = \langle w_x, w_y \rangle$, observe that $|\mathbf{w}| = 4$ and that the lengths of the sides of the 45–45–90 triangle in Figure 13.20 are

$$|w_x| = |w_y| = |\mathbf{w}| \cos 45° = \frac{4}{\sqrt{2}} = 2\sqrt{2}.$$

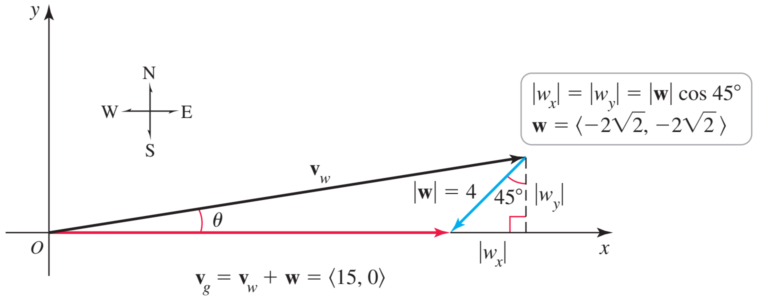
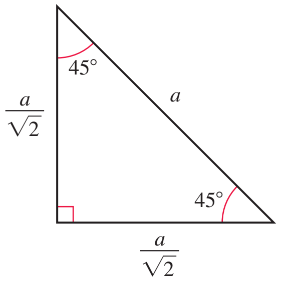


Figure 13.20

Given the orientation of $\mathbf{w}$ (southwest), $\mathbf{w} = \langle -2\sqrt{2}, -2\sqrt{2} \rangle$. Because $\mathbf{v}_g = \mathbf{v}_w + \mathbf{w}$ (Figure 13.20),

$$\begin{aligned} \mathbf{v}_w = \mathbf{v}_g - \mathbf{w} &= \langle 15, 0 \rangle - \langle -2\sqrt{2}, -2\sqrt{2} \rangle \\ &= \langle 15 + 2\sqrt{2}, 2\sqrt{2} \rangle. \end{aligned}$$

➤ Recall that the lengths of the legs of a 45–45–90 triangle are equal and are $1/\sqrt{2}$ times the length of the hypotenuse.



The magnitude of $\mathbf{v}_w$ is

$$|\mathbf{v}_w| = \sqrt{(15 + 2\sqrt{2})^2 + (2\sqrt{2})^2} \approx 18.$$

Therefore, the speed of the boat relative to the water is approximately 18 mi/hr.

The heading of the boat is given by the angle θ between $\mathbf{v}_w$ and the positive x-axis. The x-component of $\mathbf{v}_w$ is $15 + 2\sqrt{2}$ and the y-component is $2\sqrt{2}$. Therefore,

$$\theta = \tan^{-1}\left(\frac{2\sqrt{2}}{15 + 2\sqrt{2}}\right) \approx 9°.$$

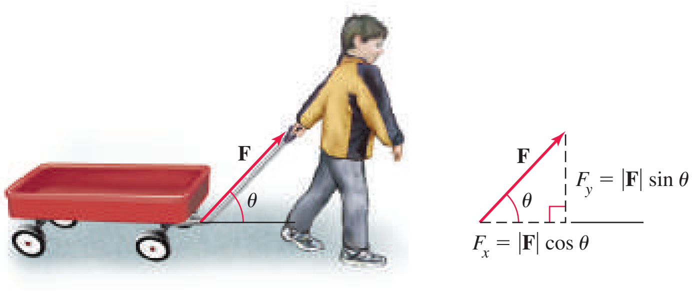
The heading of the boat is approximately 9° north of east, and its speed relative to the water is approximately 18 mi/hr.

Related Exercises 56–57 ◄

➤ The magnitude of $\mathbf{F}$ is typically measured in pounds (lb) or newtons (N), where $1\text{ N} = 1\text{ kg-m/s}^2$.

➤ The vector $\langle \cos\theta, \sin\theta \rangle$ is a unit vector. Therefore, any position vector $\mathbf{v}$ may be written $\mathbf{v} = \langle |\mathbf{v}| \cos\theta, |\mathbf{v}| \sin\theta \rangle$, where θ is the angle that $\mathbf{v}$ makes with the positive x-axis.

Force Vectors Suppose a child pulls on the handle of a wagon at an angle of θ with the horizontal (**Figure 13.21a**). The vector $\mathbf{F}$ represents the force exerted on the wagon; it has a magnitude $|\mathbf{F}|$ and a direction given by θ. We denote the horizontal and vertical components of $\mathbf{F}$ by F_x and F_y, respectively. From **Figure 13.21b**, we see that $F_x = |\mathbf{F}| \cos\theta$, $F_y = |\mathbf{F}| \sin\theta$, and the force vector is $\mathbf{F} = \langle |\mathbf{F}| \cos\theta, |\mathbf{F}| \sin\theta \rangle$.



(a) (b)

Figure 13.21

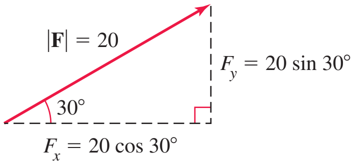


Figure 13.22

EXAMPLE 7 Finding force vectors A child pulls a wagon (Figure 13.21) with a force of $|\mathbf{F}| = 20$ lb at an angle of $\theta = 30°$ to the horizontal. Find the force vector $\mathbf{F}$.

SOLUTION The force vector (Figure 13.22) is

$$\mathbf{F} = \langle |\mathbf{F}| \cos \theta, |\mathbf{F}| \sin \theta \rangle = \langle 20 \cos 30°, 20 \sin 30° \rangle = \langle 10\sqrt{3}, 10 \rangle.$$

Related Exercise 60 ◀

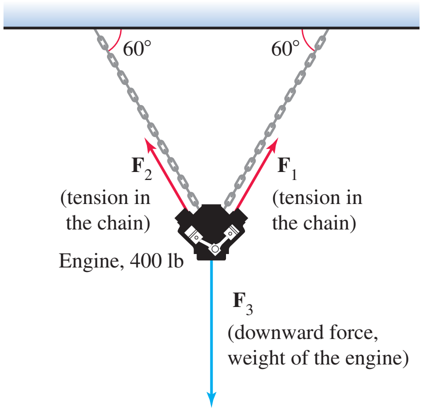


Figure 13.23

EXAMPLE 8 Balancing forces A 400-lb engine is suspended from two chains that form 60° angles with a horizontal ceiling (Figure 13.23). How much weight does each chain support?

SOLUTION Let $\mathbf{F}_1$ and $\mathbf{F}_2$ denote the forces exerted by the chains on the engine, and let $\mathbf{F}_3$ be the downward force due to the weight of the engine (Figure 13.23). Placing the vectors in a standard coordinate system (Figure 13.24), we find that $\mathbf{F}_1 = \langle |\mathbf{F}_1| \cos 60°, |\mathbf{F}_1| \sin 60° \rangle$, $\mathbf{F}_2 = \langle -|\mathbf{F}_2| \cos 60°, |\mathbf{F}_2| \sin 60° \rangle$, and $\mathbf{F}_3 = \langle 0, -400 \rangle$.

➤ The components of $\mathbf{F}_2$ in Example 8 can also be computed using an angle of 120°. That is, $\mathbf{F}_2 = \langle |\mathbf{F}_2| \cos 120°, |\mathbf{F}_2| \sin 120° \rangle$.

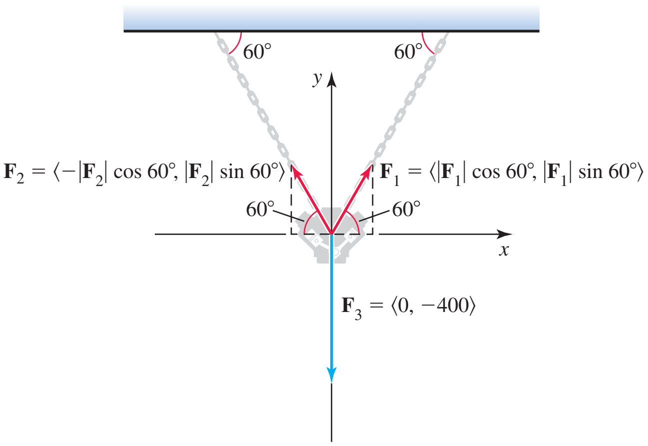


Figure 13.24

Because the engine is in equilibrium (the chains and engine are stationary), the sum of the forces is zero; that is, $\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = \mathbf{0}$ or $\mathbf{F}_1 + \mathbf{F}_2 = -\mathbf{F}_3$. Therefore,

$$\langle |\mathbf{F}_1| \cos 60° - |\mathbf{F}_2| \cos 60°, |\mathbf{F}_1| \sin 60° + |\mathbf{F}_2| \sin 60° \rangle = \langle 0, 400 \rangle.$$

Equating corresponding components, we obtain two equations to be solved for $|\mathbf{F}_1|$ and $|\mathbf{F}_2|$:

$$|\mathbf{F}_1| \cos 60° - |\mathbf{F}_2| \cos 60° = 0 \text{ and}$$
$$|\mathbf{F}_1| \sin 60° + |\mathbf{F}_2| \sin 60° = 400.$$

Factoring the first equation, we find that $(|\mathbf{F}_1| - |\mathbf{F}_2|) \cos 60° = 0$, which implies that $|\mathbf{F}_1| = |\mathbf{F}_2|$. Replacing $|\mathbf{F}_2|$ with $|\mathbf{F}_1|$ in the second equation gives $2|\mathbf{F}_1| \sin 60° = 400$. Noting that $\sin 60° = \sqrt{3}/2$ and solving for $|\mathbf{F}_1|$, we find that $|\mathbf{F}_1| = 400/\sqrt{3} \approx 231$. Each chain must be able to support a weight of approximately 231 lb.

Related Exercise 63 ◀

SECTION 13.1 EXERCISES

Getting Started

1. Interpret the following statement: Points have a location, but no size or direction; nonzero vectors have a size and direction, but no location.

2. What is a position vector?

3. Given a position vector $\mathbf{v}$, why are there infinitely many vectors equal to $\mathbf{v}$?

4. Use the points $P(3, 1)$ and $Q(7, 1)$ to find position vectors equal to $\overrightarrow{PQ}$ and $\overrightarrow{QP}$.

5. If $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$, how do you find $\mathbf{u} + \mathbf{v}$?

6. Find two unit vectors parallel to $\langle 2, 3 \rangle$.

7. Is $\langle 1, 1 \rangle$ a unit vector? Explain.

8. Evaluate $\langle 3, 1 \rangle + \langle 2, 4 \rangle$ and illustrate the sum geometrically using the Parallelogram Rule.

9. How do you compute the magnitude of $\mathbf{v} = \langle v_1, v_2 \rangle$?

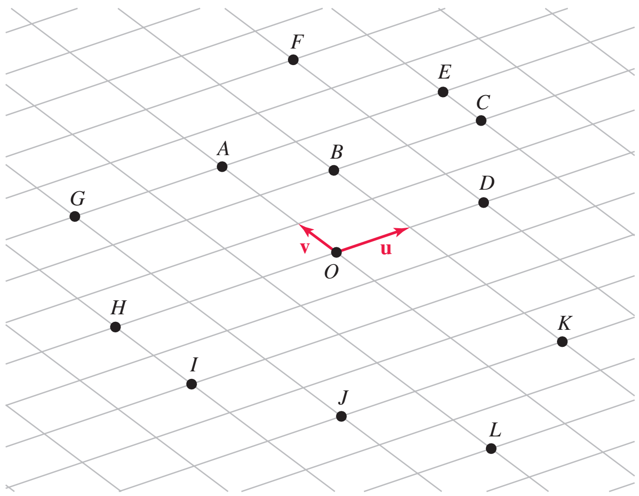
10. Write the vector $\mathbf{v} = \langle v_1, v_2 \rangle$ in terms of the unit vectors **i** and **j**.

11. How do you compute $|\overrightarrow{PQ}|$ from the coordinates of the points P and Q?

12. The velocity of a kayak on a lake is $\mathbf{v} = \langle 2\sqrt{2}, 2\sqrt{2} \rangle$. Find the speed and heading of the kayak. Assume the positive x-axis points east and the positive y-axis points north. Assume the coordinates of **v** are in feet per second.

Practice Exercises

13–18. Vector operations *Refer to the figure and carry out the following vector operations.*



13. Scalar multiples Which of the following vectors equal $\overrightarrow{CE}$? (There may be more than one correct answer.)

a. $\mathbf{v}$ **b.** $\frac{1}{2}\overrightarrow{HI}$ **c.** $\frac{1}{3}\overrightarrow{OA}$ **d.** $\mathbf{u}$ **e.** $\frac{1}{2}\overrightarrow{IH}$

14. Scalar multiples Which of the following vectors equal $\overrightarrow{BK}$? (There may be more than one correct answer.)

a. $6\mathbf{v}$ **b.** $-6\mathbf{v}$ **c.** $3\,\overrightarrow{HI}$ **d.** $3\,\overrightarrow{IH}$ **e.** $3\,\overrightarrow{AO}$

15. Scalar multiples Write the following vectors as scalar multiples of **u** or **v**.

a. $\overrightarrow{OA}$ **b.** $\overrightarrow{OD}$ **c.** $\overrightarrow{OH}$ **d.** $\overrightarrow{AG}$ **e.** $\overrightarrow{CE}$

16. Scalar multiples Write the following vectors as scalar multiples of **u** or **v**.

a. $\overrightarrow{IH}$ **b.** $\overrightarrow{HI}$ **c.** $\overrightarrow{JK}$ **d.** $\overrightarrow{FD}$ **e.** $\overrightarrow{EA}$

17. Vector addition Write the following vectors as sums of scalar multiples of **u** and **v**.

a. $\overrightarrow{OE}$ **b.** $\overrightarrow{OB}$ **c.** $\overrightarrow{OF}$ **d.** $\overrightarrow{OG}$ **e.** $\overrightarrow{OC}$
f. $\overrightarrow{OI}$ **g.** $\overrightarrow{OJ}$ **h.** $\overrightarrow{OK}$ **i.** $\overrightarrow{OL}$

18. Vector addition Write the following vectors as sums of scalar multiples of **u** and **v**.

a. $\overrightarrow{BF}$ **b.** $\overrightarrow{DE}$ **c.** $\overrightarrow{AF}$ **d.** $\overrightarrow{AD}$ **e.** $\overrightarrow{CD}$
f. $\overrightarrow{JD}$ **g.** $\overrightarrow{JI}$ **h.** $\overrightarrow{DB}$ **i.** $\overrightarrow{IL}$

19. Components and magnitudes Define the points $O(0, 0)$, $P(3, 2)$, $Q(4, 2)$, and $R(-6, -1)$. For each vector, do the following.

(i) Sketch the vector in an xy-coordinate system.
(ii) Compute the magnitude of the vector.

a. $\overrightarrow{OP}$ **b.** $\overrightarrow{QP}$ **c.** $\overrightarrow{RQ}$

20. Finding vectors from two points Given the points $A(-2, 0)$, $B(6, 16)$, $C(1, 4)$, $D(5, 4)$, $E(\sqrt{2}, \sqrt{2})$, and $F(3\sqrt{2}, -4\sqrt{2})$, find the position vector equal to the following vectors.

a. $\overrightarrow{AB}$ **b.** $\overrightarrow{AC}$ **c.** $\overrightarrow{EF}$ **d.** $\overrightarrow{CD}$

21–23. Components and equality *Define the points* $P(-3, -1)$, $Q(-1, 2)$, $R(1, 2)$, $S(3, 5)$, $T(4, 2)$, *and* $U(6, 4)$.

21. Sketch $\overrightarrow{QU}$, $\overrightarrow{PT}$, and $\overrightarrow{RS}$ and their corresponding position vectors.

22. Find the equal vectors among $\overrightarrow{PQ}$, $\overrightarrow{RS}$, and $\overrightarrow{TU}$.

23. Consider the vectors $\overrightarrow{QT}$ and $\overrightarrow{SU}$: Which vector is equal to $\langle 5, 0 \rangle$?

24–27. Vector operations *Let* $\mathbf{u} = \langle 4, -2 \rangle$, $\mathbf{v} = \langle -4, 6 \rangle$, *and* $\mathbf{w} = \langle 0, 8 \rangle$. *Express the following vectors in the form* $\langle a, b \rangle$.

24. $\mathbf{u} + \mathbf{v}$

25. $\mathbf{w} - \mathbf{u}$

26. $2\mathbf{u} + 3\mathbf{v}$

27. $10\mathbf{u} - 3\mathbf{v} + \mathbf{w}$

28–31. Vector operations *Let* $\mathbf{u} = \langle 3, -4 \rangle$, $\mathbf{v} = \langle 1, 1 \rangle$, *and* $\mathbf{w} = \langle -1, 0 \rangle$.

28. Find $|\mathbf{u} + \mathbf{v} + \mathbf{w}|$.

29. Find $|-2\mathbf{v}|$.

30. Find two vectors parallel to **u** with four times the magnitude of **u**.

31. Which has the greater magnitude, $\mathbf{u} - \mathbf{v}$ or $\mathbf{w} - \mathbf{u}$?

32. Find a unit vector in the direction of $\mathbf{v} = \langle -6, 8 \rangle$.

33. Write $\mathbf{v} = \langle -5, 12 \rangle$ as a product of its magnitude and a unit vector in the direction of **v**.

34. Consider the points $P(2, 7)$ and $Q(6, 4)$. Write $\overrightarrow{PQ}$ as a product of its magnitude and a unit vector in the direction of $\overrightarrow{PQ}$.

35. Find the vector **v** of length 6 that has the same direction as the unit vector $\langle 1/2, \sqrt{3}/2 \rangle$.

36. Find the vector **v** that has a magnitude of 10 and a direction opposite that of the unit vector $\langle 3/5, -4/5 \rangle$.

37. Find the vector in the direction of $\langle 5, -12 \rangle$ with length 3.

38. Find the vector pointing in the direction opposite that of $\langle 6, -8 \rangle$ with length 20.

39. Find a vector in the same direction as $\langle 3, -2 \rangle$ with length 10.

40. Let $\mathbf{v} = \langle 8, 15 \rangle$.

a. Find a vector in the direction of **v** that is three times as long as **v**.
b. Find a vector in the direction of **v** that has length 3.

41–46. Unit vectors *Define the points* $P(-4, 1)$, $Q(3, -4)$, *and* $R(2, 6)$.

41. Express $\overrightarrow{QR}$ in the form $a\mathbf{i} + b\mathbf{j}$.

42. Express $\overrightarrow{PQ}$ in the form $a\mathbf{i} + b\mathbf{j}$.

43. Find two unit vectors parallel to $\overrightarrow{PR}$.

44. Find the unit vector with the same direction as $\overrightarrow{QR}$.

45. Find two vectors parallel to $\overrightarrow{QP}$ with length 4.

46. Find two vectors parallel to $\overrightarrow{RP}$ with length 4.

47. Unit vectors

a. Find two unit vectors parallel to $\mathbf{v} = 6\mathbf{i} - 8\mathbf{j}$.
b. Find b if $\mathbf{v} = \langle 1/3, b \rangle$ is a unit vector.
c. Find all values of a such that $\mathbf{w} = a\mathbf{i} - \frac{a}{3}\mathbf{j}$ is a unit vector.

48. Vectors from polar coordinates Suppose O is the origin and P has polar coordinates (r, θ). Show that $\overrightarrow{OP} = \langle r\cos\theta, r\sin\theta \rangle$.

49. Vectors from polar coordinates Find the position vector $\overrightarrow{OP}$ if O is the origin and P has polar coordinates $(8, 5\pi/6)$.

50. Find the velocity $\mathbf{v}$ of an ocean freighter that is traveling northeast (45° east of north) at 40 km/hr.

51. Find the velocity $\mathbf{v}$ of an ocean freighter that is traveling 30° south of east at 30 km/hr.

52. Find a force vector of magnitude 100 that is directed 45° south of east.

T 53–55. Airplanes and crosswinds *Assume each plane flies horizontally in a crosswind that blows horizontally.*

53. An airplane flies east to west at 320 mi/hr relative to the air in a crosswind that blows at 40 mi/hr toward the southwest (45° south of west).

a. Find the velocity of the plane relative to the air $\mathbf{v}_a$, the velocity of the crosswind $\mathbf{w}$, and the velocity of the plane relative to the ground $\mathbf{v}_g$.
b. Find the ground speed and heading of the plane relative to the ground.

T 54. A commercial jet flies west to east at 400 mi/hr relative to the air, and it flies at 420 mi/hr at a heading of 5° north of east relative to the ground.

a. Find the velocity of the plane relative to the air $\mathbf{v}_a$, the velocity of the plane relative to the ground $\mathbf{v}_g$, and the crosswind $\mathbf{w}$.
b. Find the speed and heading of the wind.

55. Determine the necessary air speed and heading that a pilot must maintain in order to fly her commercial jet north at a speed of 480 mi/hr relative to the ground in a crosswind that is blowing 60° south of east at 20 mi/hr.

56. A boat in a current The water in a river moves south at 10 mi/hr. A motorboat travels due east at a speed of 20 mi/hr relative to the shore. Determine the speed and direction of the boat relative to the moving water.

57. Another boat in a current The water in a river moves south at 5 km/hr. A motorboat travels due east at a speed of 40 km/hr relative to the water. Determine the speed of the boat relative to the shore.

58. Parachute in the wind In still air, a parachute with a payload falls vertically at a terminal speed of 4 m/s. Find the direction and magnitude of its terminal velocity relative to the ground if it falls in a steady wind blowing horizontally from west to east at 10 m/s.

59. Boat in a wind A sailboat floats in a current that flows due east at 1 m/s. Because of a wind, the boat's actual speed relative to the shore is $\sqrt{3}$ m/s in a direction 30° north of east. Find the speed and direction of the wind.

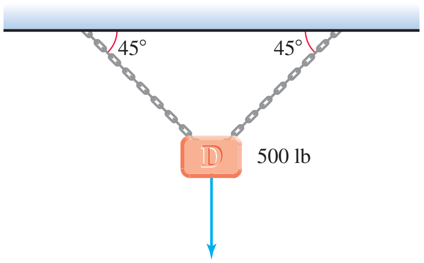
60. Towing a boat A boat is towed with a force of 150 lb with a rope that makes an angle of 30° to the horizontal. Find the horizontal and vertical components of the force.

61. Pulling a suitcase Suppose you pull a suitcase with a strap that makes a 60° angle with the horizontal. The magnitude of the force you exert on the suitcase is 40 lb.

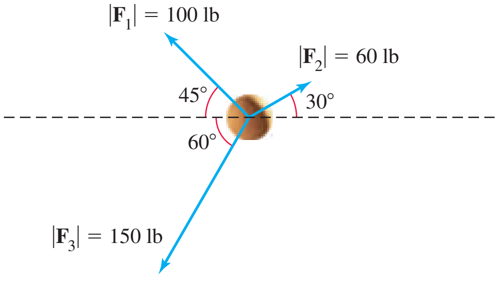
a. Find the horizontal and vertical components of the force.
b. Is the horizontal component of the force greater if the angle of the strap is 45° instead of 60°?
c. Is the vertical component of the force greater if the angle of the strap is 45° instead of 60°?

62. Which is greater? Which has a greater horizontal component, a 100-N force directed at an angle of 60° above the horizontal or a 60-N force directed at an angle of 30° above the horizontal?

63. Suspended load If a 500-lb load is suspended by two chains (see figure), what is the magnitude of the force each chain must be able to support?



64. Net force Three forces are applied to an object, as shown in the figure. Find the magnitude and direction of the sum of the forces.



65. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. José travels from point A to point B in the plane by following vector $\mathbf{u}$, then vector $\mathbf{v}$, and then vector $\mathbf{w}$. If he starts at A and follows $\mathbf{w}$, then $\mathbf{v}$, and then $\mathbf{u}$, he still arrives at B.
b. Maria travels from A to B in the plane by following the vector $\mathbf{u}$. By following $-\mathbf{u}$, she returns from B to A.
c. $|\mathbf{u} + \mathbf{v}| \geq |\mathbf{u}|$, for all vectors $\mathbf{u}$ and $\mathbf{v}$.
d. $|\mathbf{u} + \mathbf{v}| \geq |\mathbf{u}| + |\mathbf{v}|$, for all vectors $\mathbf{u}$ and $\mathbf{v}$.
e. Parallel vectors have the same length.
f. If $\overrightarrow{AB} = \overrightarrow{CD}$, then $A = C$ and $B = D$.
g. If $\mathbf{u}$ and $\mathbf{v}$ are perpendicular, then $|\mathbf{u} + \mathbf{v}| = |\mathbf{u}| + |\mathbf{v}|$.
h. If $\mathbf{u}$ and $\mathbf{v}$ are parallel and have the same direction, then $|\mathbf{u} + \mathbf{v}| = |\mathbf{u}| + |\mathbf{v}|$.

66. Equal vectors For the points $A(3, 4)$, $B(6, 10)$, $C(a + 2, b + 5)$, and $D(b + 4, a - 2)$, find the values of a and b such that $\overrightarrow{AB} = \overrightarrow{CD}$.

67–69. Vector equations *Use the properties of vectors to solve the following equations for the unknown vector* $\mathbf{x} = \langle a, b \rangle$. *Let* $\mathbf{u} = \langle 2, -3 \rangle$ *and* $\mathbf{v} = \langle -4, 1 \rangle$.

67. $10\mathbf{x} = \mathbf{u}$ **68.** $2\mathbf{x} + \mathbf{u} = \mathbf{v}$ **69.** $3\mathbf{x} - 4\mathbf{u} = \mathbf{v}$

70. Solve the pair of equations $2\mathbf{u} + 3\mathbf{v} = \mathbf{i}$, $\mathbf{u} - \mathbf{v} = \mathbf{j}$ for the vectors $\mathbf{u}$ and $\mathbf{v}$. Assume $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$.

Explorations and Challenges

71–73. Linear combinations *A sum of scalar multiples of two or more vectors (such as $c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w}$, where c_i are scalars) is called a **linear combination** of the vectors. Let $\mathbf{i} = \langle 1, 0 \rangle$, $\mathbf{j} = \langle 0, 1 \rangle$, $\mathbf{u} = \langle 1, 1 \rangle$, and $\mathbf{v} = \langle -1, 1 \rangle$.*

71. Express $\langle 4, -8 \rangle$ as a linear combination of **i** and **j** (that is, find scalars c_1 and c_2 such that $\langle 4, -8 \rangle = c_1\mathbf{i} + c_2\mathbf{j}$).

72. Express $\langle 4, -8 \rangle$ as a linear combination of **u** and **v**.

73. For arbitrary real numbers a and b, express $\langle a, b \rangle$ as a linear combination of **u** and **v**.

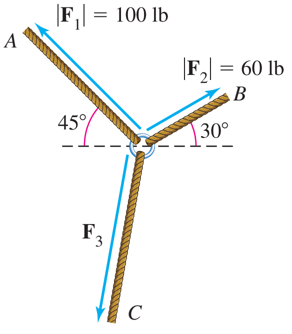
74. Ant on a page An ant walks due east at a constant speed of 2 mi/hr on a sheet of paper that rests on a table. Suddenly, the sheet of paper starts moving southeast at $\sqrt{2}$ mi/hr. Describe the motion of the ant relative to the table.

75. Clock vectors Consider the 12 vectors that have their tails at the center of a (circular) clock and their heads at the numbers on the edge of the clock.

- **a.** What is the sum of these 12 vectors?
- **b.** If the 12:00 vector is removed, what is the sum of the remaining 11 vectors?
- **c.** By removing one or more of these 12 clock vectors, explain how to make the sum of the remaining vectors as large as possible in magnitude.
- **d.** Consider the 11 vectors that originate at the number 12 at the top of the clock and point to the other 11 numbers. What is the sum of these vectors?

(*Source: Calculus*, by Gilbert Strang, Wellesley-Cambridge Press, 1991)

76. Three-way tug-of-war Three people located at A, B, and C pull on ropes tied to a ring. Find the magnitude and direction of the force with which the person at C must pull so that no one moves (the system is at equilibrium).



77–81. *Prove the following vector properties using components. Then make a sketch to illustrate the property geometrically. Suppose **u**, **v**, and **w** are vectors in the xy-plane and a and c are scalars.*

77. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ — Commutative property

78. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ — Associative property

79. $a(c\mathbf{v}) = (ac)\mathbf{v}$ — Associative property

80. $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$ — Distributive property 1

81. $(a + c)\mathbf{v} = a\mathbf{v} + c\mathbf{v}$ — Distributive property 2

82. Midpoint of a line segment Use vectors to show that the midpoint of the line segment joining $P(x_1, y_1)$ and $Q(x_2, y_2)$ is the point $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$. (*Hint:* Let O be the origin and let M be the midpoint of PQ. Draw a picture and show that
$$\overrightarrow{OM} = \overrightarrow{OP} + \frac{1}{2}\overrightarrow{PQ} = \overrightarrow{OP} + \frac{1}{2}(\overrightarrow{OQ} - \overrightarrow{OP}).)$$

83. Magnitude of scalar multiple Prove that $|c\mathbf{v}| = |c||\mathbf{v}|$, where c is a scalar and **v** is a vector.

84. Equality of vectors Assume $\overrightarrow{PQ}$ equals $\overrightarrow{RS}$. Does it follow that $\overrightarrow{PR}$ is equal to $\overrightarrow{QS}$? Prove your conclusion.

85. Linear independence A pair of nonzero vectors in the plane is *linearly dependent* if one vector is a scalar multiple of the other. Otherwise, the pair is *linearly independent*.

- **a.** Which pairs of the following vectors are linearly dependent and which are linearly independent: $\mathbf{u} = \langle 2, -3 \rangle$, $\mathbf{v} = \langle -12, 18 \rangle$, and $\mathbf{w} = \langle 4, 6 \rangle$?
- **b.** Geometrically, what does it mean for a pair of nonzero vectors in the plane to be linearly dependent? Linearly independent?
- **c.** Prove that if a pair of vectors **u** and **v** is linearly independent, then given any vector **w**, there are constants c_1 and c_2 such that $\mathbf{w} = c_1\mathbf{u} + c_2\mathbf{v}$.

86. Perpendicular vectors Show that two nonzero vectors $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ are perpendicular to each other if $u_1v_1 + u_2v_2 = 0$.

87. Parallel and perpendicular vectors Let $\mathbf{u} = \langle a, 5 \rangle$ and $\mathbf{v} = \langle 2, 6 \rangle$.

- **a.** Find the value of a such that **u** is parallel to **v**.
- **b.** Find the value of a such that **u** is perpendicular to **v**.

88. The Triangle Inequality Suppose **u** and **v** are vectors in the plane.

- **a.** Use the Triangle Rule for adding vectors to explain why $|\mathbf{u} + \mathbf{v}| \le |\mathbf{u}| + |\mathbf{v}|$. This result is known as the *Triangle Inequality*.
- **b.** Under what conditions is $|\mathbf{u} + \mathbf{v}| = |\mathbf{u}| + |\mathbf{v}|$?

QUICK CHECK ANSWERS

1. The vector $-5\mathbf{v}$ is five times as long as **v** and points in the opposite direction. **2.** $\mathbf{v}_a + \mathbf{w}$ points in a northeasterly direction. **3.** Constructing $\mathbf{u} + \mathbf{v}$ and $\mathbf{v} + \mathbf{u}$ using the Triangle Rule produces vectors having the same length and direction. **4.** $\overrightarrow{PQ} = \langle -6, -2 \rangle$ **5.** $10\mathbf{u} = \langle 6, 8 \rangle$ and $-10\mathbf{u} = \langle -6, -8 \rangle$

6. $\left|\left\langle \frac{5}{13}, \frac{12}{13} \right\rangle\right| = \sqrt{\frac{25 + 144}{169}} = \sqrt{\frac{169}{169}} = 1$

7. $\mathbf{u} = -\frac{4}{3}\mathbf{v} + 4\mathbf{w}$ ◄

13.2 Vectors in Three Dimensions

Up to this point, our study of calculus has been limited to functions, curves, and vectors that can be plotted in the two-dimensional xy-plane. However, a two-dimensional coordinate system is insufficient for modeling many physical phenomena. For example, to describe the trajectory of a jet gaining altitude, we need two coordinates, say x and y, to measure east–west and north–south distances. In addition, another coordinate, say z, is needed to measure the altitude of the jet. By adding a third coordinate and creating an ordered triple (x, y, z), the location of the jet can be described. The set of all points described by the triples (x, y, z) is called *three-dimensional space*, xyz-space, or $\mathbb{R}^3$. Many of the properties of xyz-space are extensions of familiar ideas you have seen in the xy-plane.

The xyz-Coordinate System

➤ The notation $\mathbb{R}^3$ (pronounced *R-three*) stands for the set of all ordered triples of real numbers.

A three-dimensional coordinate system is created by adding a new axis, called the ***z*-axis**, to the familiar xy-coordinate system. The new z-axis is inserted through the origin perpendicular to the x- and y-axes (Figure 13.25). The result is a new coordinate system called the **three-dimensional rectangular coordinate system** or the ***xyz*-coordinate system**.

We use a conventional **right-handed coordinate system**: If the curled fingers of the right hand are rotated from the positive x-axis to the positive y-axis, the thumb points in the direction of the positive z-axis (Figure 13.25).

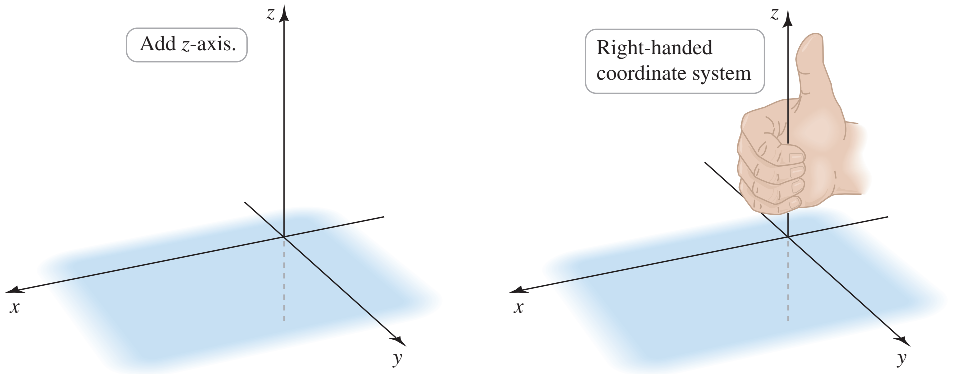


Figure 13.25

The coordinate plane containing the x-axis and y-axis is still called the xy-plane. We now have two new coordinate planes: the ***xz*-plane** containing the x-axis and the z-axis, and the ***yz*-plane** containing the y-axis and the z-axis. Taken together, these three coordinate planes divide xyz-space into eight regions called **octants** (Figure 13.26).

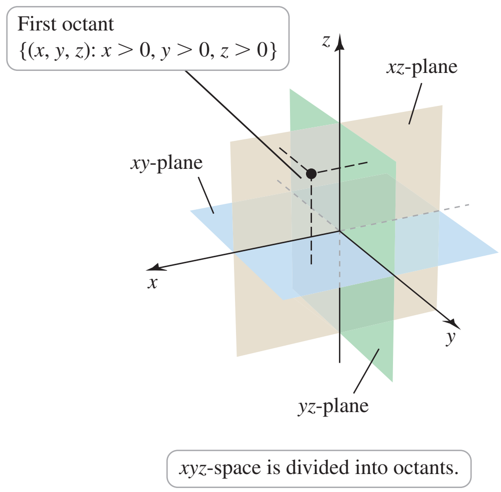


Figure 13.26

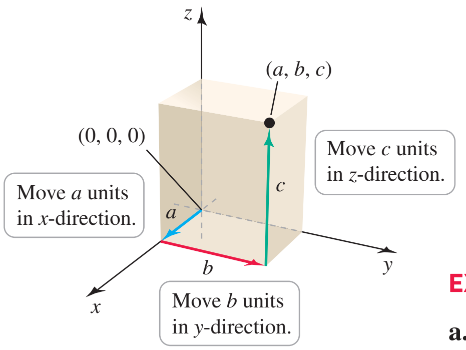


Figure 13.27

The point where all three axes intersect is the **origin**, which has coordinates $(0, 0, 0)$. An ordered triple (a, b, c) refers to the point in xyz-space that is found by starting at the origin, moving a units in the x-direction, b units in the y-direction, and c units in the z-direction. With a negative coordinate, you move in the negative direction along the corresponding coordinate axis. To visualize this point, it's helpful to construct a rectangular box with one vertex at the origin and the opposite vertex at the point (a, b, c) (Figure 13.27).

EXAMPLE 1 **Plotting points in *xyz*-space** Plot the following points.

a. $(3, 4, 5)$ **b.** $(-2, -3, 5)$

SOLUTION

a. Starting at $(0, 0, 0)$, we move 3 units in the x-direction to the point $(3, 0, 0)$, then 4 units in the y-direction to the point $(3, 4, 0)$, and finally 5 units in the z-direction to reach the point $(3, 4, 5)$ (Figure 13.28).

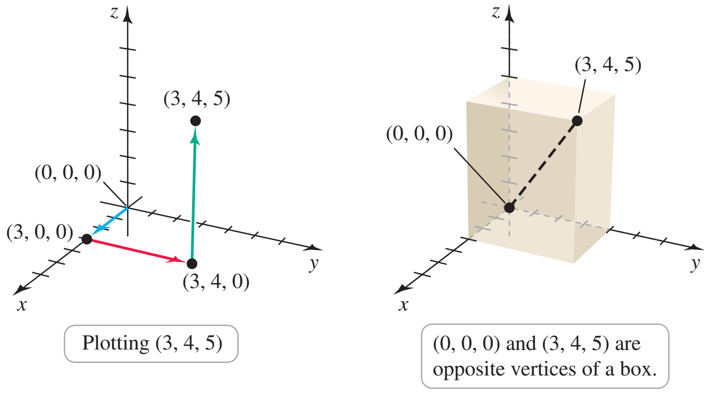


Figure 13.28

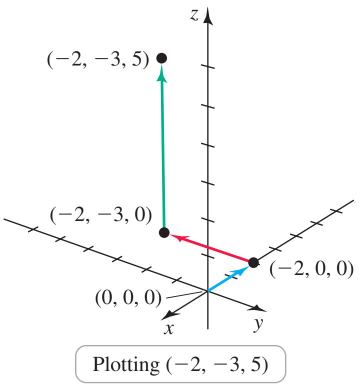


Figure 13.29

b. We move -2 units in the x-direction to $(-2, 0, 0)$, -3 units in the y-direction to $(-2, -3, 0)$, and 5 units in the z-direction to reach $(-2, -3, 5)$ (Figure 13.29).

Related Exercises 13–14 ◀

QUICK CHECK 1 Suppose the positive x-, y-, and z-axes point east, north, and upward, respectively. Describe the location of the points $(-1, -1, 0)$, $(1, 0, 1)$, and $(-1, -1, -1)$ relative to the origin. ◀

Equations of Simple Planes

The xy-plane consists of all points in xyz-space that have a z-coordinate of 0. Therefore, the xy-plane is the set $\{(x, y, z): z = 0\}$; it is represented by the equation $z = 0$. Similarly, the xz-plane has the equation $y = 0$, and the yz-plane has the equation $x = 0$.

Planes parallel to one of the coordinate planes are easy to describe. For example, the equation $x = 2$ describes the set of all points whose x-coordinate is 2 and whose y- and z-coordinates are arbitrary; this plane is parallel to and 2 units from the yz-plane. Similarly, the equation $y = a$ describes a plane that is everywhere $|a|$ units from the xz-plane, and $z = a$ is the equation of a horizontal plane $|a|$ units from the xy-plane (Figure 13.30).

➤ Planes that are not parallel to the coordinate planes are discussed in Section 13.5.

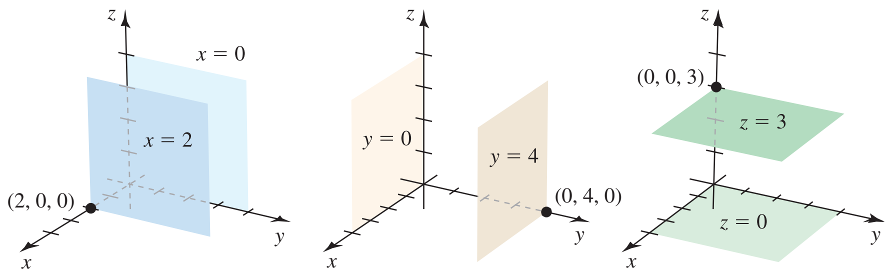


Figure 13.30

QUICK CHECK 2 To which coordinate planes are the planes $x = -2$ and $z = 16$ parallel? ◀

EXAMPLE 2 **Parallel planes** Determine the equation of the plane parallel to the xz-plane passing through the point $(2, -3, 7)$.

SOLUTION Points on a plane parallel to the xz-plane have the same y-coordinate. Therefore, the plane passing through the point $(2, -3, 7)$ with a y-coordinate of -3 has the equation $y = -3$ (Figure 13.31).

Related Exercises 20–22 ◄

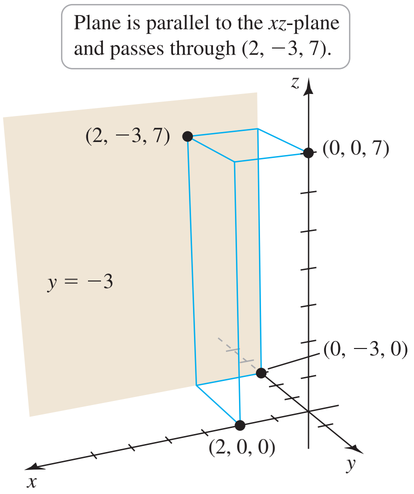


Figure 13.31

Distances in xyz-Space

Recall that the distance between two points (x_1, y_1) and (x_2, y_2) in the xy-plane is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. This distance formula is useful in deriving a similar formula for the distance between two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ in xyz-space.

Figure 13.32 shows the points P and Q, together with the auxiliary point $R(x_2, y_2, z_1)$, which has the same z-coordinate as P and the same x- and y-coordinates as Q. The line segment PR has length $|PR| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ and is one leg of the right triangle $\triangle PRQ$. The length of the hypotenuse of that triangle is the distance between P and Q:

$$|PQ| = \sqrt{|PR|^2 + |RQ|^2} = \sqrt{\underbrace{(x_2 - x_1)^2 + (y_2 - y_1)^2}_{|PR|^2} + \underbrace{(z_2 - z_1)^2}_{|RQ|^2}}.$$

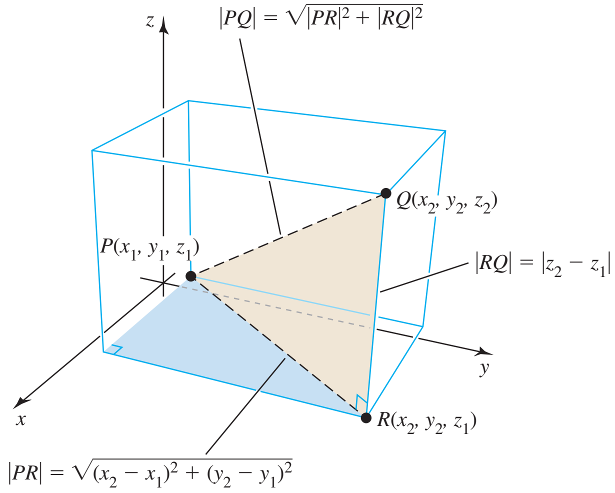


Figure 13.32

> **Distance Formula in xyz-Space**
>
> The distance between the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is
>
> $$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

By using the distance formula, we can derive the formula (Exercise 81) for the **midpoint** of the line segment joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$, which is found by averaging the x-, y-, and z-coordinates (Figure 13.33):

$$\text{Midpoint} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$

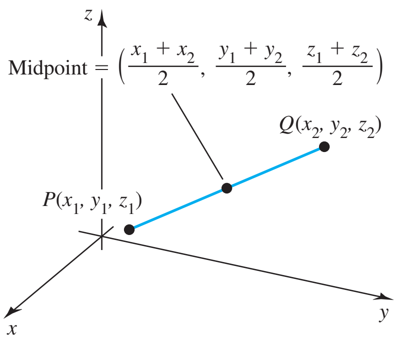
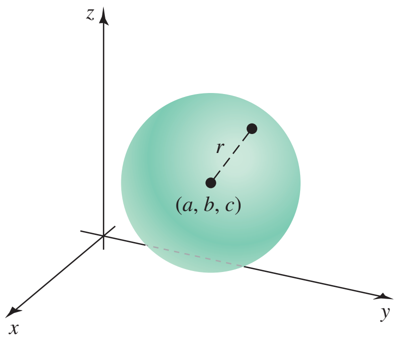


Figure 13.33

Equation of a Sphere

A *sphere* is the set of all points that are a constant distance r from a point (a, b, c); r is the *radius* of the sphere, and (a, b, c) is the *center* of the sphere. A *ball* centered at (a, b, c) with radius r consists of all the points inside and on the sphere centered at (a, b, c) with radius r (Figure 13.34). We now use the distance formula to translate these statements.



Sphere: $(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$
Ball: $(x-a)^2 + (y-b)^2 + (z-c)^2 \le r^2$

Figure 13.34

➤ Just as a circle is the boundary of a disk in two dimensions, a *sphere* is the boundary of a *ball* in three dimensions. We have defined a *closed ball*, which includes its boundary. An *open ball* does not contain its boundary.

DEFINITION Spheres and Balls

A **sphere** centered at (a, b, c) with radius r is the set of points satisfying the equation

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2.$$

A **ball** centered at (a, b, c) with radius r is the set of points satisfying the inequality

$$(x-a)^2 + (y-b)^2 + (z-c)^2 \le r^2.$$

EXAMPLE 3 Equation of a sphere Consider the points $P(1, -2, 5)$ and $Q(3, 4, -6)$. Find an equation of the sphere for which the line segment PQ is a diameter.

SOLUTION The center of the sphere is the midpoint of PQ:

$$\left(\frac{1+3}{2}, \frac{-2+4}{2}, \frac{5-6}{2}\right) = \left(2, 1, -\frac{1}{2}\right).$$

The diameter of the sphere is the distance $|PQ|$, which is

$$\sqrt{(3-1)^2 + (4+2)^2 + (-6-5)^2} = \sqrt{161}.$$

Therefore, the sphere's radius is $\frac{1}{2}\sqrt{161}$, its center is $\left(2, 1, -\frac{1}{2}\right)$, and it is described by the equation

$$(x-2)^2 + (y-1)^2 + \left(z + \frac{1}{2}\right)^2 = \left(\frac{1}{2}\sqrt{161}\right)^2 = \frac{161}{4}.$$

Related Exercises 27–28 ◄

EXAMPLE 4 Identifying equations Describe the set of points that satisfy the equation $x^2 + y^2 + z^2 - 2x + 6y - 8z = -1$.

QUICK CHECK 3 Describe the solution set of the equation $(x-1)^2 + y^2 + (z+1)^2 + 4 = 0$. ◄

SOLUTION We simplify the equation by completing the square and factoring:

$$\begin{aligned} (x^2 - 2x) + (y^2 + 6y) + (z^2 - 8z) &= -1 && \text{Group terms.} \\ (x^2 - 2x + 1) + (y^2 + 6y + 9) + (z^2 - 8z + 16) &= 25 && \text{Complete the square.} \\ (x-1)^2 + (y+3)^2 + (z-4)^2 &= 25. && \text{Factor.} \end{aligned}$$

The equation describes a sphere of radius 5 with center $(1, -3, 4)$.

Related Exercises 31–32 ◄

Vectors in $\mathbb{R}^3$

Vectors in $\mathbb{R}^3$ are straightforward extensions of vectors in the xy-plane; we simply include a third component. The position vector $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ has its tail at the origin and its head at the point (v_1, v_2, v_3). Vectors having the same length and direction are equal. Therefore, the vector from $P(x_1, y_1, z_1)$ to $Q(x_2, y_2, z_2)$ is denoted $\overrightarrow{PQ}$ and is equal to the position vector $\langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$. It is also equal to all vectors such as $\overrightarrow{RS}$ (**Figure 13.35**) that have the same length and direction as $\mathbf{v}$.

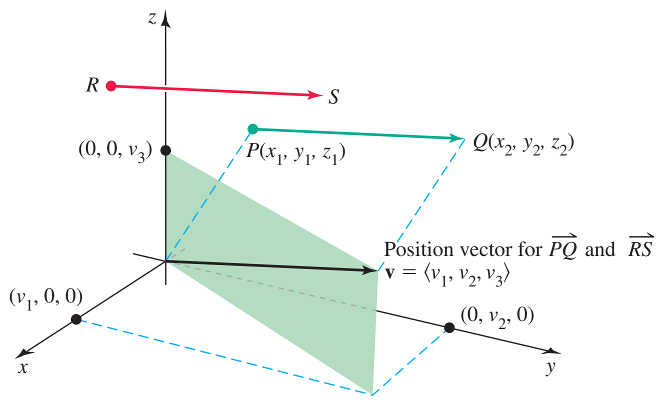


Figure 13.35

QUICK CHECK 4 Which of the following vectors are parallel to each other?

a. $\mathbf{u} = \langle -2, 4, -6 \rangle$

b. $\mathbf{v} = \langle 4, -8, 12 \rangle$

c. $\mathbf{w} = \langle -1, 2, 3 \rangle$ ◄

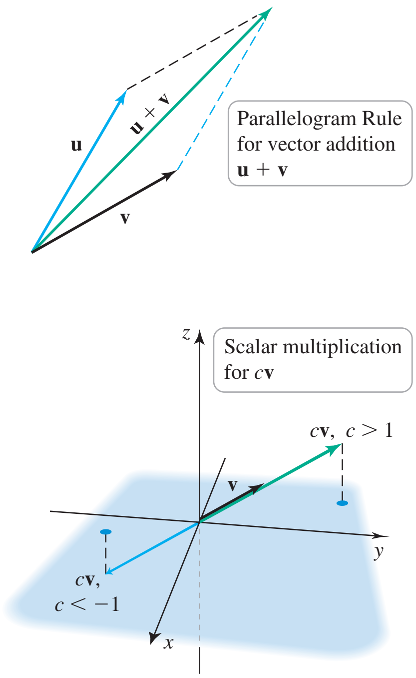


Figure 13.36

The operations of vector addition and scalar multiplication in $\mathbb{R}^2$ generalize in a natural way to three dimensions. For example, the sum of two vectors is found geometrically using the Triangle Rule or the Parallelogram Rule (Section 13.1). The sum is found analytically by adding the respective components of the two vectors. As with two-dimensional vectors, scalar multiplication corresponds to stretching or compressing a vector, possibly with a reversal of direction. Two nonzero vectors are parallel if one is a scalar multiple of the other (Figure 13.36).

DEFINITION Vector Operations in $\mathbb{R}^3$

Let c be a scalar, $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$.

$$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle \quad \text{Vector addition}$$

$$\mathbf{u} - \mathbf{v} = \langle u_1 - v_1, u_2 - v_2, u_3 - v_3 \rangle \quad \text{Vector subtraction}$$

$$c\mathbf{u} = \langle cu_1, cu_2, cu_3 \rangle \quad \text{Scalar multiplication}$$

EXAMPLE 5 Vectors in $\mathbb{R}^3$ Let $\mathbf{u} = \langle 2, -4, 1 \rangle$ and $\mathbf{v} = \langle 3, 0, -1 \rangle$. Find the components of the following vectors and draw them in $\mathbb{R}^3$.

a. $\frac{1}{2}\mathbf{u}$ **b.** $-2\mathbf{v}$ **c.** $\mathbf{u} + 2\mathbf{v}$

SOLUTION

a. Using the definition of scalar multiplication, $\frac{1}{2}\mathbf{u} = \frac{1}{2}\langle 2, -4, 1 \rangle = \left\langle 1, -2, \frac{1}{2} \right\rangle$. The vector $\frac{1}{2}\mathbf{u}$ has the same direction as $\mathbf{u}$ with half the length of $\mathbf{u}$ (Figure 13.37).

b. Using scalar multiplication, $-2\mathbf{v} = -2\langle 3, 0, -1 \rangle = \langle -6, 0, 2 \rangle$. The vector $-2\mathbf{v}$ has the direction opposite that of $\mathbf{v}$ and twice the length of $\mathbf{v}$ (Figure 13.38).

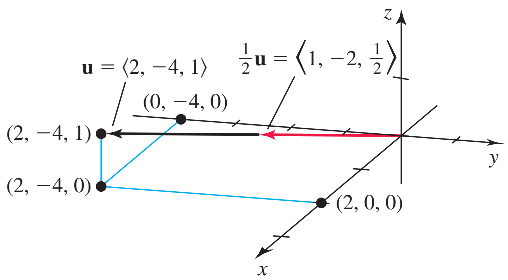


Figure 13.37

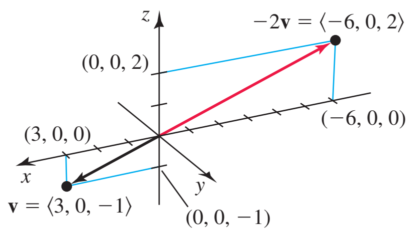


Figure 13.38

c. Using vector addition and scalar multiplication,

$$\mathbf{u} + 2\mathbf{v} = \langle 2, -4, 1 \rangle + 2\langle 3, 0, -1 \rangle = \langle 8, -4, -1 \rangle.$$

The vector $\mathbf{u} + 2\mathbf{v}$ is drawn by applying the Parallelogram Rule to $\mathbf{u}$ and $2\mathbf{v}$ (Figure 13.39).

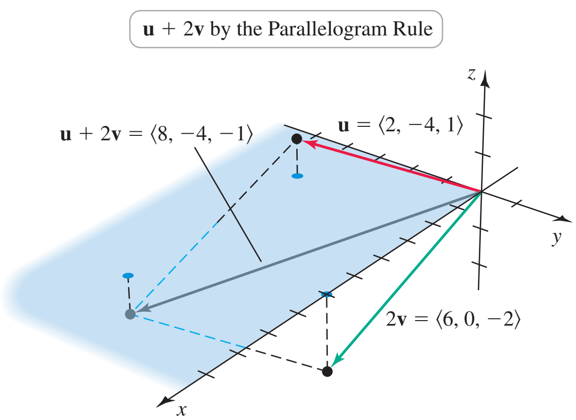


Figure 13.39

Related Exercise 39 ◄

Magnitude and Unit Vectors

The magnitude of the vector $\overrightarrow{PQ}$ from $P(x_1, y_1, z_1)$ to $Q(x_2, y_2, z_2)$ is denoted $|\overrightarrow{PQ}|$; it is the distance between P and Q and is given by the distance formula (**Figure 13.40**).

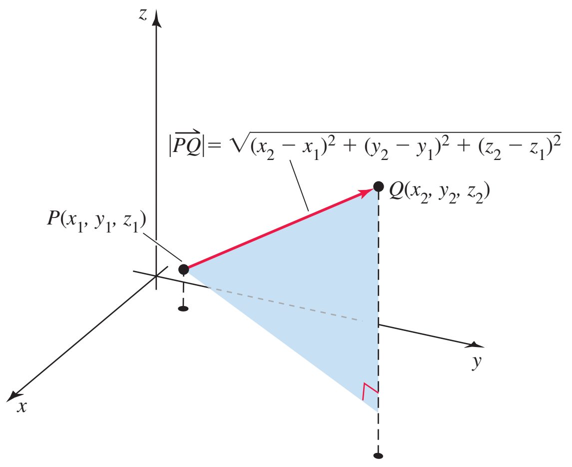


Figure 13.40

> **DEFINITION Magnitude of a Vector**
>
> The **magnitude** (or **length**) of the vector $\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$ is the distance from $P(x_1, y_1, z_1)$ to $Q(x_2, y_2, z_2)$:
>
> $$|\overrightarrow{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

The coordinate unit vectors introduced in Section 13.1 extend naturally to three dimensions. The three coordinate unit vectors in $\mathbb{R}^3$ (**Figure 13.41**) are

$$\mathbf{i} = \langle 1, 0, 0 \rangle, \quad \mathbf{j} = \langle 0, 1, 0 \rangle, \quad \text{and} \quad \mathbf{k} = \langle 0, 0, 1 \rangle.$$

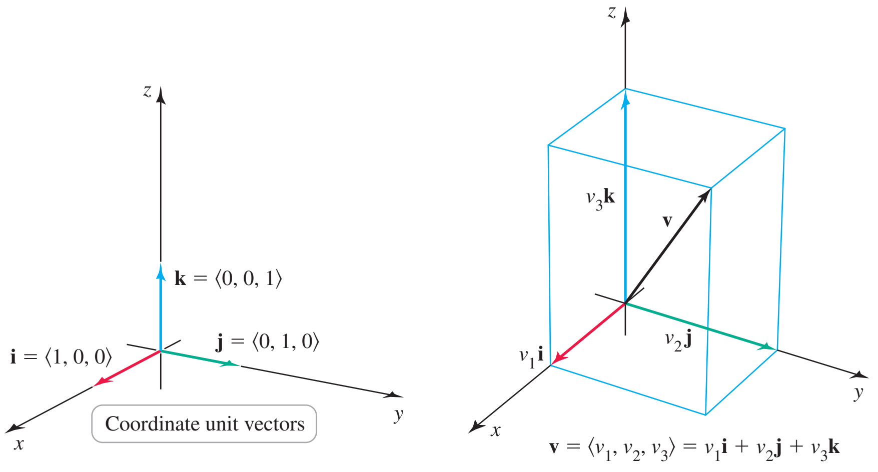


Figure 13.41

These unit vectors give an alternative way of expressing position vectors. If $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, then we have

$$\mathbf{v} = v_1\langle 1, 0, 0 \rangle + v_2\langle 0, 1, 0 \rangle + v_3\langle 0, 0, 1 \rangle = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}.$$

EXAMPLE 6 **Magnitudes and unit vectors** Consider the points $P(5, 3, 1)$ and $Q(-7, 8, 1)$.

a. Express $\overrightarrow{PQ}$ in terms of the unit vectors **i**, **j**, and **k**.

b. Find the magnitude of $\overrightarrow{PQ}$.

c. Find the position vector of magnitude 10 in the direction of $\overrightarrow{PQ}$.

SOLUTION

a. $\overrightarrow{PQ}$ is equal to the position vector $\langle -7 - 5, 8 - 3, 1 - 1 \rangle = \langle -12, 5, 0 \rangle$. Therefore, $\overrightarrow{PQ} = -12\mathbf{i} + 5\mathbf{j}$.

b. $|\overrightarrow{PQ}| = |-12\mathbf{i} + 5\mathbf{j}| = \sqrt{12^2 + 5^2} = \sqrt{169} = 13$

c. The unit vector in the direction of $\overrightarrow{PQ}$ is $\mathbf{u} = \dfrac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|} = \dfrac{1}{13}\langle -12, 5, 0 \rangle$. Therefore, the vector in the direction of **u** with a magnitude of 10 is $10\mathbf{u} = \dfrac{10}{13}\langle -12, 5, 0 \rangle$.

Related Exercises 45, 68 ◄

QUICK CHECK 5 Which vector has the smaller magnitude: $\mathbf{u} = 3\mathbf{i} - \mathbf{j} - \mathbf{k}$ or $\mathbf{v} = 2(\mathbf{i} + \mathbf{j} + \mathbf{k})$? ◄

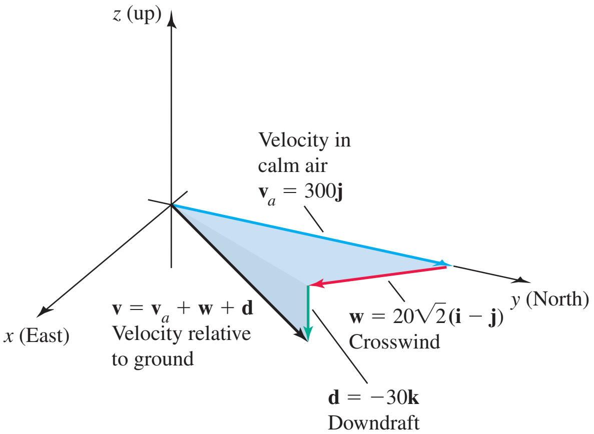


Figure 13.42

EXAMPLE 7 **Flight in crosswinds** A plane is flying horizontally due north in calm air at 300 mi/hr when it encounters a horizontal crosswind blowing southeast at 40 mi/hr and a downdraft blowing vertically downward at 30 mi/hr. What are the resulting speed and direction of the plane relative to the ground?

SOLUTION Let the unit vectors **i**, **j**, and **k** point east, north, and upward, respectively (Figure 13.42). The velocity of the plane relative to the air (300 mi/hr due north) is $\mathbf{v}_a = 300\mathbf{j}$. The crosswind blows 45° south of east, so its component to the east is $40 \cos 45° = 20\sqrt{2}$ (in the **i**-direction) and its component to the south is $40 \cos 45° = 20\sqrt{2}$ (in the negative **j**-direction). Therefore, the crosswind may be expressed as $\mathbf{w} = 20\sqrt{2}\mathbf{i} - 20\sqrt{2}\mathbf{j}$. Finally, the downdraft in the negative **k**-direction is $\mathbf{d} = -30\mathbf{k}$. The velocity of the plane relative to the ground is the sum of $\mathbf{v}_a$, **w**, and **d**:

$$\begin{aligned}\mathbf{v} &= \mathbf{v}_a + \mathbf{w} + \mathbf{d}\\ &= 300\mathbf{j} + (20\sqrt{2}\mathbf{i} - 20\sqrt{2}\mathbf{j}) - 30\mathbf{k}\\ &= 20\sqrt{2}\mathbf{i} + (300 - 20\sqrt{2})\mathbf{j} - 30\mathbf{k}.\end{aligned}$$

Figure 13.42 shows the velocity vector of the plane. A quick calculation shows that the speed is $|\mathbf{v}| \approx 275$ mi/hr. The direction of the plane is slightly east of north and downward. In the next section, we present methods for precisely determining the direction of a vector.

Related Exercises 51–52 ◄

SECTION 13.2 EXERCISES

Getting Started

1. Explain how to plot the point $(3, -2, 1)$ in $\mathbb{R}^3$.

2. What is the y-coordinate of all points in the xz-plane?

3. Describe the plane $x = 4$.

4. What position vector is equal to the vector from $(3, 5, -2)$ to $(0, -6, 3)$?

5. Let $\mathbf{u} = \langle 3, 5, -7 \rangle$ and $\mathbf{v} = \langle 6, -5, 1 \rangle$. Evaluate $\mathbf{u} + \mathbf{v}$ and $3\mathbf{u} - \mathbf{v}$.

6. What is the magnitude of a vector joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$?

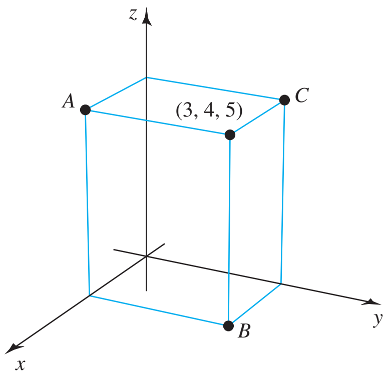
7. Which point is farther from the origin, $(3, -1, 2)$ or $(0, 0, -4)$?

8. Express the vector from $P(-1, -4, 6)$ to $Q(1, 3, -6)$ as a position vector in terms of **i**, **j**, and **k**.

Practice Exercises

9–12. Points in $\mathbb{R}^3$ *Find the coordinates of the vertices A, B, and C of the following rectangular boxes.*

9.



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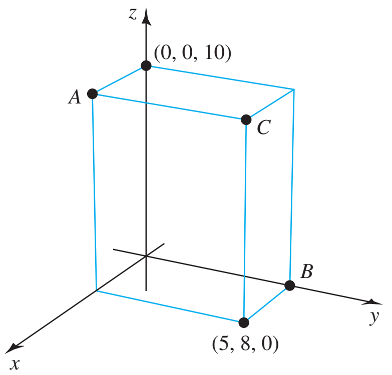

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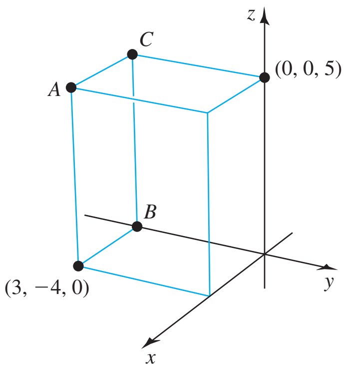
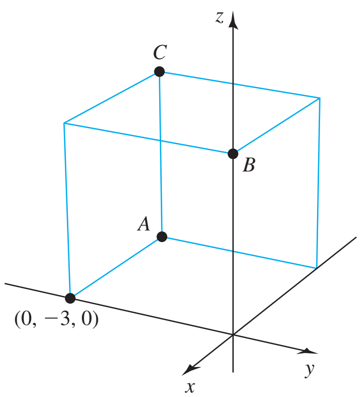


12. Assume all the edges have the same length.



13–14. Plotting points in $\mathbb{R}^3$ *For each point $P(x, y, z)$ given below, let $A(x, y, 0)$, $B(x, 0, z)$, and $C(0, y, z)$ be points in the xy-, xz-, and yz-planes, respectively. Plot and label the points A, B, C, and P in $\mathbb{R}^3$.*

13. **a.** $P(2, 2, 4)$ **b.** $P(1, 2, 5)$ **c.** $P(-2, 0, 5)$

14. **a.** $P(-3, 2, 4)$ **b.** $P(4, -2, -3)$ **c.** $P(-2, -4, -3)$

15–20. Sketching planes *Sketch the following planes in the window* $[0, 5] \times [0, 5] \times [0, 5]$.

15. $x = 2$ **16.** $z = 3$ **17.** $y = 2$ **18.** $z = y$

19. The plane that passes through $(2, 0, 0)$, $(0, 3, 0)$, and $(0, 0, 4)$

20. The plane parallel to the xz-plane containing the point $(1, 2, 3)$

21. Planes Sketch the plane parallel to the xy-plane through $(2, 4, 2)$ and find its equation.

22. Planes Sketch the plane parallel to the yz-plane through $(2, 4, 2)$ and find its equation.

23–26. Spheres and balls *Find an equation or inequality that describes the following objects.*

23. A sphere with center $(1, 2, 3)$ and radius 4

24. A sphere with center $(1, 2, 0)$ passing through the point $(3, 4, 5)$

25. A ball with center $(-2, 0, 4)$ and radius 1

26. A ball with center $(0, -2, 6)$ with the point $(1, 4, 8)$ on its boundary

27. Midpoints and spheres Find an equation of the sphere passing through $P(1, 0, 5)$ and $Q(2, 3, 9)$ with its center at the midpoint of PQ.

28. Midpoints and spheres Find an equation of the sphere passing through $P(-4, 2, 3)$ and $Q(0, 2, 7)$ with its center at the midpoint of PQ.

29–38. Identifying sets *Give a geometric description of the following sets of points.*

29. $(x - 1)^2 + y^2 + z^2 - 9 = 0$

30. $(x + 1)^2 + y^2 + z^2 - 2y - 24 = 0$

31. $x^2 + y^2 + z^2 - 2y - 4z - 4 = 0$

32. $x^2 + y^2 + z^2 - 6x + 6y - 8z - 2 = 0$

33. $x^2 + y^2 - 14y + z^2 \geq -13$

34. $x^2 + y^2 - 14y + z^2 \leq -13$

35. $x^2 + y^2 + z^2 - 8x - 14y - 18z \leq 79$

36. $x^2 + y^2 + z^2 - 8x + 14y - 18z \geq 65$

37. $x^2 - 2x + y^2 + 6y + z^2 + 10 = 0$

38. $x^2 - 4x + y^2 + 6y + z^2 + 14 = 0$

39–44. Vector operations *For the given vectors* **u** *and* **v**, *evaluate the following expressions.*

a. $3\mathbf{u} + 2\mathbf{v}$ *b.* $4\mathbf{u} - \mathbf{v}$ *c.* $|\mathbf{u} + 3\mathbf{v}|$

39. $\mathbf{u} = \langle 4, -3, 0 \rangle$, $\mathbf{v} = \langle 0, 1, 1 \rangle$

40. $\mathbf{u} = \langle -2, -3, 0 \rangle$, $\mathbf{v} = \langle 1, 2, 1 \rangle$

41. $\mathbf{u} = \langle -2, 1, -2 \rangle$, $\mathbf{v} = \langle 1, 1, 1 \rangle$

42. $\mathbf{u} = \langle -5, 0, 2 \rangle$, $\mathbf{v} = \langle 3, 1, 1 \rangle$

43. $\mathbf{u} = \langle -7, 11, 8 \rangle$, $\mathbf{v} = \langle 3, -5, -1 \rangle$

44. $\mathbf{u} = \langle -4, -8\sqrt{3}, 2\sqrt{2} \rangle$, $\mathbf{v} = \langle 2, 3\sqrt{3}, -\sqrt{2} \rangle$

45–50. Unit vectors and magnitude *Consider the following points P and Q.*

a. *Find $\overrightarrow{PQ}$ and state your answer in two forms: $\langle a, b, c \rangle$ and $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$.*

b. *Find the magnitude of $\overrightarrow{PQ}$.*

c. *Find two unit vectors parallel to $\overrightarrow{PQ}$.*

45. $P(1, 5, 0)$, $Q(3, 11, 2)$

46. $P(5, 11, 12)$, $Q(1, 14, 13)$

47. $P(-3, 1, 0)$, $Q(-3, -4, 1)$

48. $P(3, 8, 12)$, $Q(3, 9, 11)$

49. $P(0, 0, 2)$, $Q(-2, 4, 0)$

50. $P(a, b, c)$, $Q(1, 1, -1)$ (a, b, and c are real numbers)

51. **Flight in crosswinds** A model airplane is flying horizontally due north at 20 mi/hr when it encounters a horizontal crosswind blowing east at 20 mi/hr and a downdraft blowing vertically downward at 10 mi/hr.
   - **a.** Find the position vector that represents the velocity of the plane relative to the ground.
   - **b.** Find the speed of the plane relative to the ground.

52. **Another crosswind flight** A model airplane is flying horizontally due east at 10 mi/hr when it encounters a horizontal crosswind blowing south at 5 mi/hr and an updraft blowing vertically upward at 5 mi/hr.
   - **a.** Find the position vector that represents the velocity of the plane relative to the ground.
   - **b.** Find the speed of the plane relative to the ground.

53. **Crosswinds** A small plane is flying horizontally due east in calm air at 250 mi/hr when it encounters a horizontal crosswind blowing southwest at 50 mi/hr and a 30-mi/hr updraft. Find the resulting speed of the plane, and describe with a sketch the approximate direction of the velocity relative to the ground.

54. **Combined force** An object at the origin is acted on by the forces $\mathbf{F}_1 = 20\mathbf{i} - 10\mathbf{j}$, $\mathbf{F}_2 = 30\mathbf{j} + 10\mathbf{k}$, and $\mathbf{F}_3 = 40\mathbf{j} + 20\mathbf{k}$. Find the magnitude of the combined force, and describe the approximate direction of the force.

55. **Submarine course** A submarine climbs at an angle of 30° above the horizontal with a heading to the northeast. If its speed is 20 knots, find the components of the velocity in the east, north, and vertical directions.

56. **Maintaining equilibrium** An object is acted on by the forces $\mathbf{F}_1 = \langle 10, 6, 3 \rangle$ and $\mathbf{F}_2 = \langle 0, 4, 9 \rangle$. Find the force $\mathbf{F}_3$ that must act on the object so that the sum of the forces is zero.

57. **Explain why or why not** Determine whether the following statements are true and give an explanation or counterexample.
   - **a.** Suppose $\mathbf{u}$ and $\mathbf{v}$ are distinct vectors that both make a 45° angle with $\mathbf{w}$ in $\mathbb{R}^3$. Then $\mathbf{u} + \mathbf{v}$ makes a 45° angle with $\mathbf{w}$.
   - **b.** Suppose $\mathbf{u}$ and $\mathbf{v}$ are distinct vectors that both make a 90° angle with $\mathbf{w}$ in $\mathbb{R}^3$. Then $\mathbf{u} + \mathbf{v}$ can never make a 90° angle with $\mathbf{w}$.
   - **c.** $\mathbf{i} + \mathbf{j} + \mathbf{k} = \mathbf{0}$
   - **d.** The intersection of the planes $x = 1$, $y = 1$, and $z = 1$ is a point.

58–60. Sets of points *Describe with a sketch the sets of points (x, y, z) satisfying the following equations.*

58. $(x + 1)(y - 3) = 0$

59. $x^2y^2z^2 > 0$

60. $y - z = 0$

61–64. Sets of points

61. Give a geometric description of the set of points (x, y, z) satisfying the pair of equations $z = 0$ and $x^2 + y^2 = 1$. Sketch a figure of this set of points.

62. Give a geometric description of the set of points (x, y, z) satisfying the pair of equations $z = x^2$ and $y = 0$. Sketch a figure of this set of points.

63. Give a geometric description of the set of points (x, y, z) that lie on the intersection of the sphere $x^2 + y^2 + z^2 = 5$ and the plane $z = 1$.

64. Give a geometric description of the set of points (x, y, z) that lie on the intersection of the sphere $x^2 + y^2 + z^2 = 36$ and the plane $z = 6$.

65. **Describing a circle** Find a pair of equations describing a circle of radius 3 centered at $(2, 4, 1)$ that lies in a plane parallel to the xz-plane.

66. **Describing a line** Find a pair of equations describing a line passing through the point $(-2, -5, 1)$ that is parallel to the x-axis.

67. Write the vector $\mathbf{v} = \langle 2, -4, 4 \rangle$ as a product of its magnitude and a unit vector with the same direction as $\mathbf{v}$.

68. Find the vector of length 10 with the same direction as $\mathbf{w} = \langle 2, \sqrt{2}, \sqrt{3} \rangle$.

69. Find a vector of length 5 in the direction opposite that of $\langle 3, -2, \sqrt{3} \rangle$.

70–73. Parallel vectors of varying lengths *Find vectors parallel to* $\mathbf{v}$ *of the given length.*

70. $\mathbf{v} = \langle 3, -2, 6 \rangle$; length $= 10$

71. $\mathbf{v} = \langle 6, -8, 0 \rangle$; length $= 20$

72. $\mathbf{v} = \overrightarrow{PQ}$ with $P(1, 0, 1)$ and $Q(2, -1, 1)$; length $= 3$

73. $\mathbf{v} = \overrightarrow{PQ}$ with $P(3, 4, 0)$ and $Q(2, 3, 1)$; length $= 3$

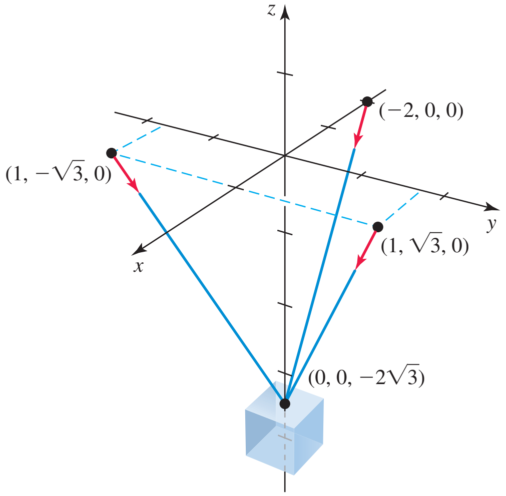
74. **Collinear points** Determine the values of x and y such that the points $(1, 2, 3)$, $(4, 7, 1)$, and $(x, y, 2)$ are collinear (lie on a line).

75. **Collinear points** Determine whether the points P, Q, and R are collinear (lie on a line) by comparing $\overrightarrow{PQ}$ and $\overrightarrow{PR}$. If the points are collinear, determine which point lies between the other two points.
   - **a.** $P(1, 6, -5)$, $Q(2, 5, -3)$, $R(4, 3, 1)$
   - **b.** $P(1, 5, 7)$, $Q(5, 13, -1)$, $R(0, 3, 9)$
   - **c.** $P(1, 2, 3)$, $Q(2, -3, 6)$, $R(3, -1, 9)$
   - **d.** $P(9, 5, 1)$, $Q(11, 18, 4)$, $R(6, 3, 0)$

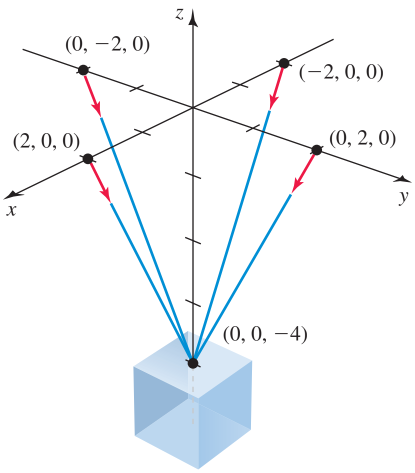
76. **Lengths of the diagonals of a box** What is the longest diagonal of a rectangular 2 ft × 3 ft × 4 ft box?

Explorations and Challenges

77. Three-cable load A 500-lb load hangs from three cables of equal length that are anchored at the points $(-2, 0, 0)$, $(1, \sqrt{3}, 0)$, and $(1, -\sqrt{3}, 0)$. The load is located at $(0, 0, -2\sqrt{3})$. Find the vectors describing the forces on the cables due to the load.



78. Four-cable load A 500-lb load hangs from four cables of equal length that are anchored at the points $(\pm 2, 0, 0)$ and $(0, \pm 2, 0)$. The load is located at $(0, 0, -4)$. Find the vectors describing the forces on the cables due to the load.



79. Possible parallelograms The points $O(0, 0, 0)$, $P(1, 4, 6)$, and $Q(2, 4, 3)$ lie at three vertices of a parallelogram. Find all possible locations of the fourth vertex.

80. Diagonals of parallelograms Two sides of a parallelogram are formed by the vectors $\mathbf{u}$ and $\mathbf{v}$. Prove that the diagonals of the parallelogram are $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$.

81. Midpoint formula Prove that the midpoint of the line segment joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is

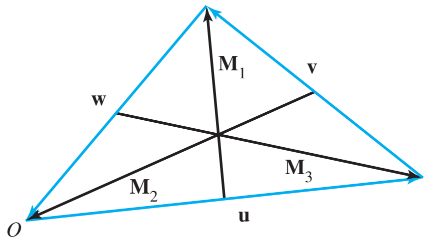
$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$

82. Equation of a sphere For constants a, b, c, and d, show that the equation

$$x^2 + y^2 + z^2 - 2ax - 2by - 2cz = d$$

describes a sphere centered at (a, b, c) with radius r, where $r^2 = d + a^2 + b^2 + c^2$, provided $d + a^2 + b^2 + c^2 > 0$.

83. Medians of a triangle—without coordinates Assume $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in $\mathbb{R}^3$ that form the sides of a triangle (see figure). Use the following steps to prove that the medians intersect at a point that divides each median in a 2:1 ratio. The proof does not use a coordinate system.



a. Show that $\mathbf{u} + \mathbf{v} + \mathbf{w} = \mathbf{0}$.

b. Let $\mathbf{M}_1$ be the median vector from the midpoint of $\mathbf{u}$ to the opposite vertex. Define $\mathbf{M}_2$ and $\mathbf{M}_3$ similarly. Using the geometry of vector addition, show that $\mathbf{M}_1 = \dfrac{\mathbf{u}}{2} + \mathbf{v}$. Find analogous expressions for $\mathbf{M}_2$ and $\mathbf{M}_3$.

c. Let $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ be the vectors from O to the points one-third of the way along $\mathbf{M}_1$, $\mathbf{M}_2$, and $\mathbf{M}_3$, respectively. Show that $\mathbf{a} = \mathbf{b} = \mathbf{c} = \dfrac{\mathbf{u} - \mathbf{w}}{3}$.

d. Conclude that the medians intersect at a point that divides each median in a 2:1 ratio.

84. Medians of a triangle—with coordinates In contrast to the proof in Exercise 83, we now use coordinates and position vectors to prove the same result. Without loss of generality, let $P(x_1, y_1, 0)$ and $Q(x_2, y_2, 0)$ be two points in the xy-plane, and let $R(x_3, y_3, z_3)$ be a third point such that P, Q, and R do not lie on a line. Consider ΔPQR.

a. Let M_1 be the midpoint of the side PQ. Find the coordinates of M_1 and the components of the vector $\overrightarrow{RM_1}$.

b. Find the vector $\overrightarrow{OZ_1}$ from the origin to the point Z_1 two-thirds of the way along $\overrightarrow{RM_1}$.

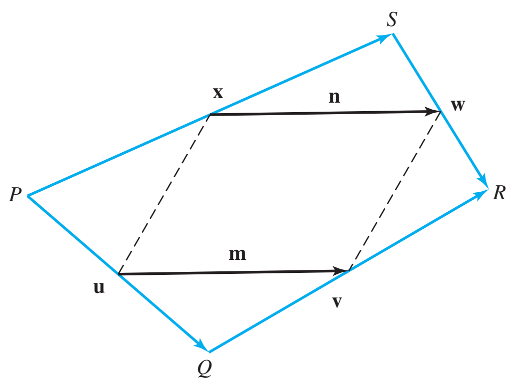
c. Repeat the calculation of part (b) with the midpoint M_2 of RQ and the vector $\overrightarrow{PM_2}$ to obtain the vector $\overrightarrow{OZ_2}$.

d. Repeat the calculation of part (b) with the midpoint M_3 of PR and the vector $\overrightarrow{QM_3}$ to obtain the vector $\overrightarrow{OZ_3}$.

e. Conclude that the medians of ΔPQR intersect at a point. Give the coordinates of the point.

f. With $P(2, 4, 0)$, $Q(4, 1, 0)$, and $R(6, 3, 4)$, find the point at which the medians of ΔPQR intersect.

85. The amazing quadrilateral property—without coordinates The points P, Q, R, and S, joined by the vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{x}$, are the vertices of a quadrilateral in $\mathbb{R}^3$. *The four points need not lie in a plane* (see figure). Use the following steps to prove that the line segments joining the midpoints of the sides of the quadrilateral form a parallelogram. The proof does not use a coordinate system.



a. Use vector addition to show that $\mathbf{u} + \mathbf{v} = \mathbf{w} + \mathbf{x}$.

b. Let $\mathbf{m}$ be the vector that joins the midpoints of PQ and QR. Show that $\mathbf{m} = \dfrac{\mathbf{u} + \mathbf{v}}{2}$.

c. Let $\mathbf{n}$ be the vector that joins the midpoints of PS and SR. Show that $\mathbf{n} = \dfrac{\mathbf{x} + \mathbf{w}}{2}$.

d. Combine parts (a), (b), and (c) to conclude that $\mathbf{m} = \mathbf{n}$.

e. Explain why part (d) implies that the line segments joining the midpoints of the sides of the quadrilateral form a parallelogram.

86. The amazing quadrilateral property—with coordinates Prove the quadrilateral property in Exercise 85, assuming the coordinates of P, Q, R, and S are $P(x_1, y_1, 0)$, $Q(x_2, y_2, 0)$, $R(x_3, y_3, 0)$, and $S(x_4, y_4, z_4)$, where we assume P, Q, and R lie in the xy-plane without loss of generality.

QUICK CHECK ANSWERS

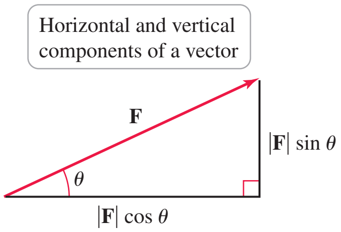
1. Southwest; due east and upward; southwest and downward **2.** yz-plane; xy-plane **3.** No solution **4.** $\mathbf{u}$ and $\mathbf{v}$ are parallel. **5.** $|\mathbf{u}| = \sqrt{11}$ and $|\mathbf{v}| = \sqrt{12} = 2\sqrt{3}$; $\mathbf{u}$ has the smaller magnitude. ◄

13.3 Dot Products

➤ The dot product is also called the *scalar product*, a term we do not use in order to avoid confusion with *scalar multiplication.*

The *dot product* is used to determine the angle between two vectors. It is also a tool for calculating *projections*—the measure of how much of a given vector lies in the direction of another vector.

To see the usefulness of the dot product, consider an example. Recall that the work done by a constant force F in moving an object a distance d is $W = Fd$ (Section 6.7). This rule is valid provided the force acts in the direction of motion (Figure 13.43a). Now assume the force is a vector $\mathbf{F}$ applied at an angle θ to the direction of motion; the resulting displacement of the object is a vector $\mathbf{d}$. In this case, the work done by the force is the component of the force in the direction of motion multiplied by the distance moved by the object, which is $W = (|\mathbf{F}| \cos \theta)|\mathbf{d}|$ (Figure 13.43b). We call this product of the magnitudes of two vectors and the cosine of the angle between them the dot product.



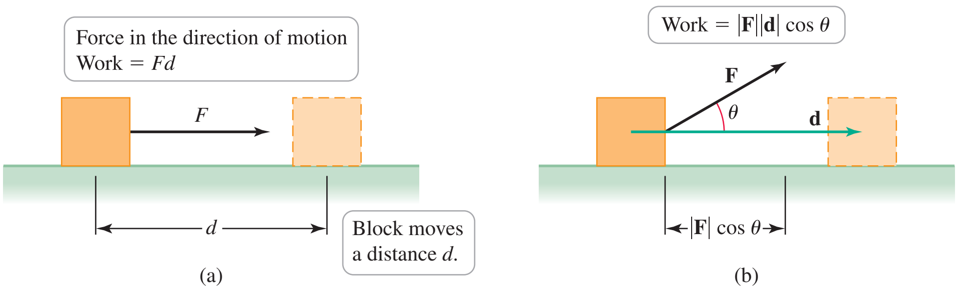


Figure 13.43

Two Forms of the Dot Product

The example of work done by a force leads to our first definition of the dot product. We then give an equivalent formula that is often better suited for computation.

> **DEFINITION Dot Product**
>
> Given two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in two or three dimensions, their **dot product** is
>
> $$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos \theta,$$
>
> where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$ with $0 \le \theta \le \pi$ (Figure 13.44). If $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$, then $\mathbf{u} \cdot \mathbf{v} = 0$, and θ is undefined.

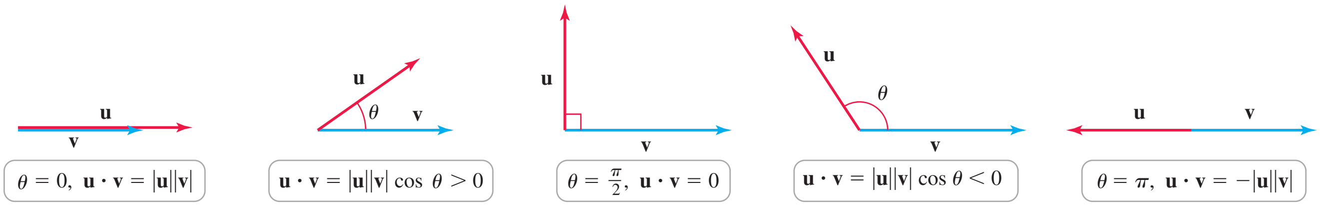


Figure 13.44

The dot product of two vectors is itself a scalar. Two special cases immediately arise:

- **u** and **v** are parallel ($\theta = 0$ or $\theta = \pi$) if and only if $\mathbf{u} \cdot \mathbf{v} = \pm|\mathbf{u}||\mathbf{v}|$.
- **u** and **v** are perpendicular ($\theta = \pi/2$) if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.

The second case gives rise to the important property of *orthogonality*.

➤ In two and three dimensions, the terms *orthogonal* and *perpendicular* are used interchangeably. *Orthogonal* is a more general term that also applies in more than three dimensions.

DEFINITION Orthogonal Vectors

Two vectors **u** and **v** are **orthogonal** if and only if $\mathbf{u} \cdot \mathbf{v} = 0$. The zero vector is orthogonal to all vectors. In two or three dimensions, two nonzero orthogonal vectors are perpendicular to each other.

QUICK CHECK 1 Sketch two nonzero vectors **u** and **v** with $\theta = 0$. Sketch two nonzero vectors **u** and **v** with $\theta = \pi$. ◄

EXAMPLE 1 **Dot products** Compute the dot products of the following vectors.

a. $\mathbf{u} = 2\mathbf{i} - 6\mathbf{j}$ and $\mathbf{v} = 12\mathbf{k}$

b. $\mathbf{u} = \langle \sqrt{3}, 1 \rangle$ and $\mathbf{v} = \langle 0, 1 \rangle$

SOLUTION

a. The vector **u** lies in the xy-plane and the vector **v** is perpendicular to the xy-plane. Therefore, $\theta = \dfrac{\pi}{2}$, **u** and **v** are orthogonal, and $\mathbf{u} \cdot \mathbf{v} = 0$ (Figure 13.45a).

b. As shown in Figure 13.45b, **u** and **v** form two sides of a 30–60–90 triangle in the xy-plane, with an angle of $\pi/3$ between them. Because $|\mathbf{u}| = 2$, $|\mathbf{v}| = 1$, and $\cos \pi/3 = 1/2$, the dot product is

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos \theta = 2 \cdot 1 \cdot \frac{1}{2} = 1.$$

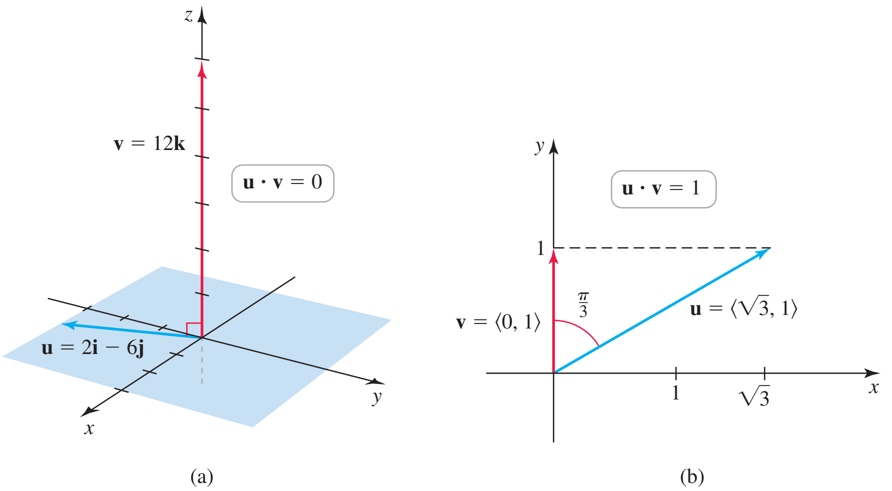


Figure 13.45

Related Exercises 16–17 ◄

Computing a dot product in this manner requires knowing the angle θ between the vectors. Often the angle is not known; in fact, it may be exactly what we seek. For this reason, we present another method for computing the dot product that does not require knowing θ.

➤ In $\mathbb{R}^2$ with $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$, $\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2$.

THEOREM 13.1 Dot Product

Given two vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$,

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3.$$

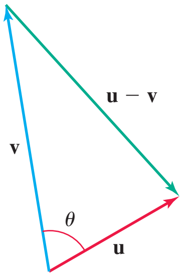
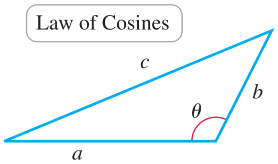


Figure 13.46



$c^2 = a^2 + b^2 - 2ab\cos\theta$

Proof: Consider two position vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, and suppose θ is the angle between them. The vector $\mathbf{u} - \mathbf{v}$ forms the third side of a triangle (**Figure 13.46**). By the Law of Cosines,

$$|\mathbf{u} - \mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2\underbrace{|\mathbf{u}||\mathbf{v}|\cos\theta}_{\mathbf{u}\cdot\mathbf{v}}.$$

The definition of the dot product, $\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta$, allows us to write

$$\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta = \frac{1}{2}\left(|\mathbf{u}|^2 + |\mathbf{v}|^2 - |\mathbf{u} - \mathbf{v}|^2\right). \quad (1)$$

Using the definition of magnitude, we find that

$$|\mathbf{u}|^2 = u_1^2 + u_2^2 + u_3^2, \qquad |\mathbf{v}|^2 = v_1^2 + v_2^2 + v_3^2,$$

and

$$|\mathbf{u} - \mathbf{v}|^2 = (u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2.$$

Expanding the terms in $|\mathbf{u} - \mathbf{v}|^2$ and simplifying yields

$$|\mathbf{u}|^2 + |\mathbf{v}|^2 - |\mathbf{u} - \mathbf{v}|^2 = 2(u_1v_1 + u_2v_2 + u_3v_3).$$

Substituting into expression (1) gives a compact expression for the dot product:

$$\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3.$$

➤

This new representation of $\mathbf{u}\cdot\mathbf{v}$ has two immediate consequences.

1. Combining it with the definition of dot product gives

$$\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3 = |\mathbf{u}||\mathbf{v}|\cos\theta.$$

If $\mathbf{u}$ and $\mathbf{v}$ are both nonzero, then

$$\cos\theta = \frac{u_1v_1 + u_2v_2 + u_3v_3}{|\mathbf{u}||\mathbf{v}|} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|},$$

and we have a way to compute θ.

2. Notice that $\mathbf{u}\cdot\mathbf{u} = u_1^2 + u_2^2 + u_3^2 = |\mathbf{u}|^2$. Therefore, we have a relationship between the dot product and the magnitude of a vector: $|\mathbf{u}| = \sqrt{\mathbf{u}\cdot\mathbf{u}}$ or $|\mathbf{u}|^2 = \mathbf{u}\cdot\mathbf{u}$.

QUICK CHECK 2 Use Theorem 13.1 to compute the dot products $\mathbf{i}\cdot\mathbf{j}$, $\mathbf{i}\cdot\mathbf{k}$, and $\mathbf{j}\cdot\mathbf{k}$ for the unit coordinate vectors. What do you conclude about the angles between these vectors? ➤

EXAMPLE 2 **Dot products and angles** Let $\mathbf{u} = \langle \sqrt{3}, 1, 0 \rangle$, $\mathbf{v} = \langle 1, \sqrt{3}, 0 \rangle$, and $\mathbf{w} = \langle 1, \sqrt{3}, 2\sqrt{3} \rangle$.

a. Compute $\mathbf{u}\cdot\mathbf{v}$.

b. Find the angle between $\mathbf{u}$ and $\mathbf{v}$.

c. Find the angle between $\mathbf{u}$ and $\mathbf{w}$.

SOLUTION

a. $\mathbf{u}\cdot\mathbf{v} = \langle \sqrt{3}, 1, 0 \rangle \cdot \langle 1, \sqrt{3}, 0 \rangle = \sqrt{3} + \sqrt{3} + 0 = 2\sqrt{3}$

b. Note that $|\mathbf{u}| = \sqrt{\mathbf{u}\cdot\mathbf{u}} = \sqrt{\langle \sqrt{3}, 1, 0 \rangle \cdot \langle \sqrt{3}, 1, 0 \rangle} = 2$, and similarly, $|\mathbf{v}| = 2$. Therefore,

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \frac{2\sqrt{3}}{2\cdot 2} = \frac{\sqrt{3}}{2}.$$

Because $0 \le \theta \le \pi$, it follows that $\theta = \cos^{-1}\dfrac{\sqrt{3}}{2} = \dfrac{\pi}{6}$.

c. $$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{w}}{|\mathbf{u}||\mathbf{w}|} = \frac{\langle \sqrt{3}, 1, 0 \rangle \cdot \langle 1, \sqrt{3}, 2\sqrt{3} \rangle}{|\langle \sqrt{3}, 1, 0 \rangle||\langle 1, \sqrt{3}, 2\sqrt{3} \rangle|} = \frac{2\sqrt{3}}{2\cdot 4} = \frac{\sqrt{3}}{4}$$

It follows that

$$\theta = \cos^{-1}\frac{\sqrt{3}}{4} \approx 1.12 \text{ rad} \approx 64.3°.$$

Related Exercises 20, 26 ➤

Properties of Dot Products The properties of the dot product in the following theorem are easily proved using vector components (Exercises 79–81).

➤ Theorem 13.1 extends to vectors with any number of components. If $\mathbf{u} = \langle u_1, \ldots, u_n \rangle$ and $\mathbf{v} = \langle v_1, \ldots, v_n \rangle$, then

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + \cdots + u_n v_n.$$

The properties in Theorem 13.2 also apply in two or more dimensions.

THEOREM 13.2 Properties of the Dot Product
Suppose $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors and let c be a scalar.

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ — Commutative property
2. $c(\mathbf{u} \cdot \mathbf{v}) = (c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v})$ — Associative property
3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ — Distributive property

Orthogonal Projections

Given vectors $\mathbf{u}$ and $\mathbf{v}$, how closely aligned are they? That is, how much of $\mathbf{u}$ points in the direction of $\mathbf{v}$? This question is answered using *projections*. As shown in Figure 13.47a, the projection of the vector $\mathbf{u}$ onto a nonzero vector $\mathbf{v}$, denoted $\text{proj}_{\mathbf{v}}\mathbf{u}$, is the "shadow" cast by $\mathbf{u}$ onto the line through $\mathbf{v}$. The projection of $\mathbf{u}$ onto $\mathbf{v}$ is itself a vector; it points in the same direction as $\mathbf{v}$ if the angle between $\mathbf{u}$ and $\mathbf{v}$ lies in the interval $0 \le \theta < \pi/2$ (Figure 13.47b); it points in the direction opposite that of $\mathbf{v}$ if the angle between $\mathbf{u}$ and $\mathbf{v}$ lies in the interval $\pi/2 < \theta \le \pi$ (Figure 13.47c). If $\theta = \dfrac{\pi}{2}$, $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, and there is no shadow.

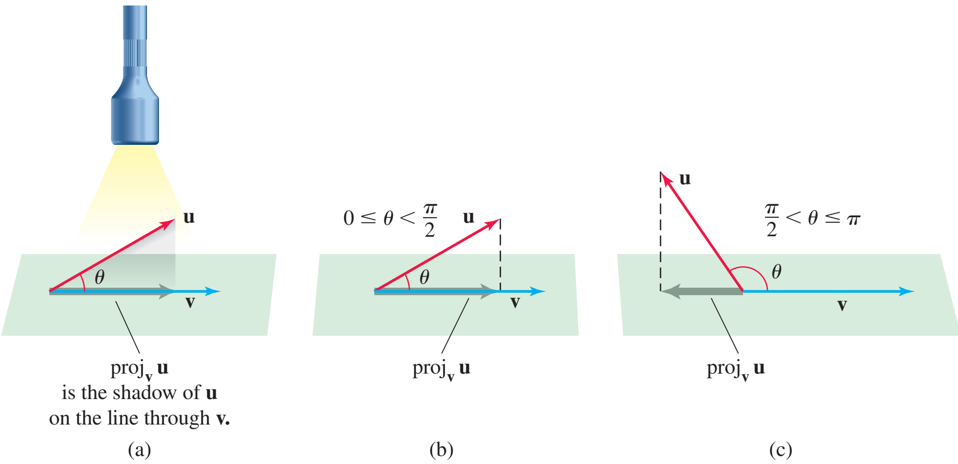


Figure 13.47

To find the projection of $\mathbf{u}$ onto $\mathbf{v}$, we proceed as follows: With the tails of $\mathbf{u}$ and $\mathbf{v}$ together, we drop a perpendicular line segment from the head of $\mathbf{u}$ to the point P on the line through $\mathbf{v}$ (Figure 13.48). The vector $\overrightarrow{OP}$ is the *orthogonal projection of* $\mathbf{u}$ *onto* $\mathbf{v}$. An expression for $\text{proj}_{\mathbf{v}}\mathbf{u}$ is found using two observations.

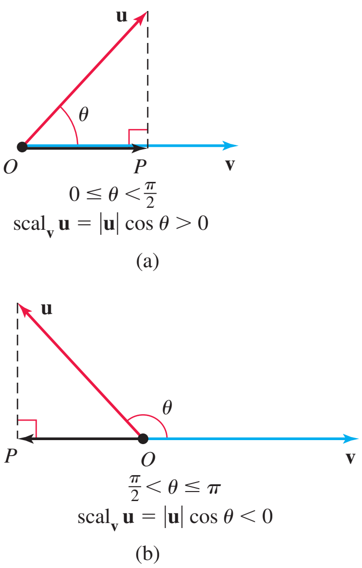


Figure 13.48

- If $0 \le \theta < \pi/2$, then $\text{proj}_{\mathbf{v}}\mathbf{u}$ has length $|\mathbf{u}| \cos\theta$ and points in the direction of the unit vector $\mathbf{v}/|\mathbf{v}|$ (Figure 13.48a). Therefore,

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \underbrace{|\mathbf{u}| \cos\theta}_{\text{length}} \underbrace{\left(\frac{\mathbf{v}}{|\mathbf{v}|}\right)}_{\text{direction}}.$$

  We define the *scalar component of* $\mathbf{u}$ *in the direction of* $\mathbf{v}$ to be $\text{scal}_{\mathbf{v}}\mathbf{u} = |\mathbf{u}| \cos\theta$. In this case, $\text{scal}_{\mathbf{v}}\mathbf{u}$ is the length of $\text{proj}_{\mathbf{v}}\mathbf{u}$.

- If $\pi/2 < \theta \le \pi$, then $\text{proj}_{\mathbf{v}}\mathbf{u}$ has length $-|\mathbf{u}| \cos\theta$ (which is positive) and points in the direction of $-\mathbf{v}/|\mathbf{v}|$ (Figure 13.48b). Therefore,

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \underbrace{-|\mathbf{u}| \cos\theta}_{\text{length}} \underbrace{\left(-\frac{\mathbf{v}}{|\mathbf{v}|}\right)}_{\text{direction}} = |\mathbf{u}| \cos\theta \left(\frac{\mathbf{v}}{|\mathbf{v}|}\right).$$

  In this case, $\text{scal}_{\mathbf{v}}\mathbf{u} = |\mathbf{u}| \cos\theta < 0$.

➤ Notice that $\text{scal}_\mathbf{v}\mathbf{u}$ may be positive, negative, or zero. However, $|\text{scal}_\mathbf{v}\mathbf{u}|$ is the length of $\text{proj}_\mathbf{v}\mathbf{u}$. The projection $\text{proj}_\mathbf{v}\mathbf{u}$ is defined for all vectors $\mathbf{u}$, but only for nonzero vectors $\mathbf{v}$.

We see that in both cases, the expression for $\text{proj}_\mathbf{v}\mathbf{u}$ is the same:

$$\text{proj}_\mathbf{v}\mathbf{u} = \underbrace{|\mathbf{u}|\cos\theta}_{\text{scal}_\mathbf{v}\mathbf{u}}\left(\frac{\mathbf{v}}{|\mathbf{v}|}\right) = \text{scal}_\mathbf{v}\mathbf{u}\left(\frac{\mathbf{v}}{|\mathbf{v}|}\right).$$

Note that if $\theta = \frac{\pi}{2}$, then $\text{proj}_\mathbf{v}\mathbf{u} = \mathbf{0}$ and $\text{scal}_\mathbf{v}\mathbf{u} = 0$.

Using properties of the dot product, $\text{proj}_\mathbf{v}\mathbf{u}$ may be written in different ways:

$$\begin{aligned}\text{proj}_\mathbf{v}\mathbf{u} &= |\mathbf{u}|\cos\theta\left(\frac{\mathbf{v}}{|\mathbf{v}|}\right)\\ &= \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}\left(\frac{\mathbf{v}}{|\mathbf{v}|}\right) && |\mathbf{u}|\cos\theta = \frac{|\mathbf{u}||\mathbf{v}|\cos\theta}{|\mathbf{v}|} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}\\ &= \underbrace{\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)}_{\text{scalar}}\mathbf{v}. && \text{Regroup terms; } |\mathbf{v}|^2 = \mathbf{v}\cdot\mathbf{v}\end{aligned}$$

QUICK CHECK 3 Let $\mathbf{u} = 4\mathbf{i} - 3\mathbf{j}$. By inspection (not calculations), find the orthogonal projection of $\mathbf{u}$ onto $\mathbf{i}$ and onto $\mathbf{j}$. Find the scalar component of $\mathbf{u}$ in the direction of $\mathbf{i}$ and in the direction of $\mathbf{j}$. ◄

The first two expressions show that $\text{proj}_\mathbf{v}\mathbf{u}$ is a scalar multiple of the unit vector $\frac{\mathbf{v}}{|\mathbf{v}|}$, whereas the last expression shows that $\text{proj}_\mathbf{v}\mathbf{u}$ is a scalar multiple of $\mathbf{v}$.

DEFINITION (Orthogonal) Projection of u onto v

The **orthogonal projection of u onto v**, denoted $\text{proj}_\mathbf{v}\mathbf{u}$, where $\mathbf{v} \neq \mathbf{0}$, is

$$\text{proj}_\mathbf{v}\mathbf{u} = |\mathbf{u}|\cos\theta\left(\frac{\mathbf{v}}{|\mathbf{v}|}\right).$$

The orthogonal projection may also be computed with the formulas

$$\text{proj}_\mathbf{v}\mathbf{u} = \text{scal}_\mathbf{v}\mathbf{u}\left(\frac{\mathbf{v}}{|\mathbf{v}|}\right) = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v},$$

where the **scalar component of u in the direction of v** is

$$\text{scal}_\mathbf{v}\mathbf{u} = |\mathbf{u}|\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}.$$

EXAMPLE 3 **Orthogonal projections** Find $\text{proj}_\mathbf{v}\mathbf{u}$ and $\text{scal}_\mathbf{v}\mathbf{u}$ for the following vectors and illustrate each result.

a. $\mathbf{u} = \langle 4, 1\rangle$, $\mathbf{v} = \langle 3, 4\rangle$

b. $\mathbf{u} = \langle -4, -3\rangle$, $\mathbf{v} = \langle 1, -1\rangle$

SOLUTION

a. The scalar component of $\mathbf{u}$ in the direction of $\mathbf{v}$ (Figure 13.49) is

$$\text{scal}_\mathbf{v}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|} = \frac{\langle 4, 1\rangle\cdot\langle 3, 4\rangle}{|\langle 3, 4\rangle|} = \frac{16}{5}.$$

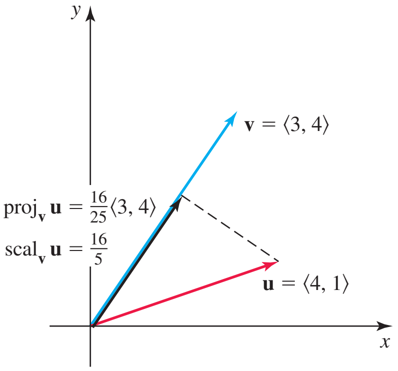


Figure 13.49

Because $\frac{\mathbf{v}}{|\mathbf{v}|} = \left\langle \frac{3}{5}, \frac{4}{5}\right\rangle$, we have

$$\text{proj}_\mathbf{v}\mathbf{u} = \text{scal}_\mathbf{v}\mathbf{u}\left(\frac{\mathbf{v}}{|\mathbf{v}|}\right) = \frac{16}{5}\left\langle\frac{3}{5}, \frac{4}{5}\right\rangle = \frac{16}{25}\langle 3, 4\rangle.$$

b. Using another formula for $\text{proj}_\mathbf{v}\mathbf{u}$, we have

$$\text{proj}_\mathbf{v}\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = \left(\frac{\langle -4, -3\rangle\cdot\langle 1, -1\rangle}{\langle 1, -1\rangle\cdot\langle 1, -1\rangle}\right)\langle 1, -1\rangle = -\frac{1}{2}\langle 1, -1\rangle.$$

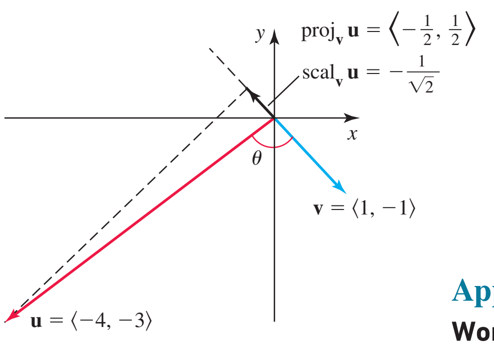


Figure 13.50

The vectors $\mathbf{v}$ and $\text{proj}_{\mathbf{v}}\mathbf{u}$ point in opposite directions because $\pi/2 < \theta \leq \pi$ (Figure 13.50). This fact is reflected in the scalar component of $\mathbf{u}$ in the direction of $\mathbf{v}$, which is negative:

$$\text{scal}_{\mathbf{v}}\mathbf{u} = \frac{\langle -4, -3 \rangle \cdot \langle 1, -1 \rangle}{|\langle 1, -1 \rangle|} = -\frac{1}{\sqrt{2}}.$$

Related Exercises 35–36 ◄

Applications of Dot Products

Work and Force In the opening of this section, we observed that if a constant force $\mathbf{F}$ acts at an angle θ to the direction of motion of an object (Figure 13.51), the work done by the force is

$$W = |\mathbf{F}| \cos\theta\, |\mathbf{d}| = \mathbf{F} \cdot \mathbf{d}.$$

Notice that the work is a scalar, and if the force acts in a direction orthogonal to the motion, then $\theta = \pi/2$, $\mathbf{F} \cdot \mathbf{d} = 0$, and no work is done by the force.

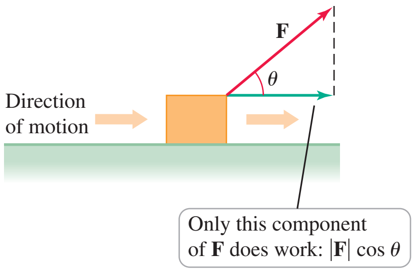


Figure 13.51

➤ If the unit of force is newtons (N) and the distance is measured in meters, then the unit of work is joules (J), where $1\text{ J} = 1\text{ N-m}$. If force is measured in pounds and distance is measured in feet, then work has units of ft-lb.

> **DEFINITION Work**
>
> Let a constant force $\mathbf{F}$ be applied to an object, producing a displacement $\mathbf{d}$. If the angle between $\mathbf{F}$ and $\mathbf{d}$ is θ, then the **work** done by the force is
>
> $$W = |\mathbf{F}||\mathbf{d}| \cos\theta = \mathbf{F} \cdot \mathbf{d}.$$

EXAMPLE 4 Calculating work A force $\mathbf{F} = \langle 3, 3, 2 \rangle$ (in newtons) moves an object along a line segment from $P(1, 1, 0)$ to $Q(6, 6, 0)$ (in meters). What is the work done by the force? Interpret the result.

SOLUTION The displacement of the object is $\mathbf{d} = \overrightarrow{PQ} = \langle 6 - 1, 6 - 1, 0 - 0 \rangle = \langle 5, 5, 0 \rangle$. Therefore, the work done by the force is

$$W = \mathbf{F} \cdot \mathbf{d} = \langle 3, 3, 2 \rangle \cdot \langle 5, 5, 0 \rangle = 30\text{ J}.$$

To interpret this result, notice that the angle between the force and the displacement vector satisfies

$$\cos\theta = \frac{\mathbf{F} \cdot \mathbf{d}}{|\mathbf{F}||\mathbf{d}|} = \frac{\langle 3, 3, 2 \rangle \cdot \langle 5, 5, 0 \rangle}{|\langle 3, 3, 2 \rangle||\langle 5, 5, 0 \rangle|} = \frac{30}{\sqrt{22}\sqrt{50}} \approx 0.905.$$

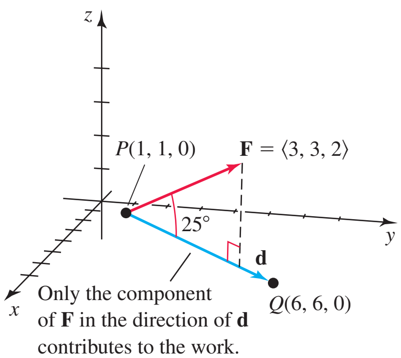


Figure 13.52

Therefore, $\theta \approx 0.44\text{ rad} \approx 25°$. The magnitude of the force is $|\mathbf{F}| = \sqrt{22} \approx 4.7\text{ N}$, but only the component of that force in the direction of motion, $|\mathbf{F}|\cos\theta \approx \sqrt{22}\cos 0.44 \approx 4.2\text{ N}$, contributes to the work (Figure 13.52).

Related Exercises 44, 46 ◄

Parallel and Normal Forces Projections find frequent use in expressing a force in terms of orthogonal components. A common situation arises when an object rests on an inclined plane (Figure 13.53). The gravitational force on the object equals its weight, which is directed vertically downward. The projections of the gravitational force in the directions **parallel** to and **normal** (or perpendicular) to the plane are of interest. Specifically, the projection of the force parallel to the plane determines the tendency of the object to slide down the plane, while the projection of the force normal to the plane determines its tendency to "stick" to the plane.

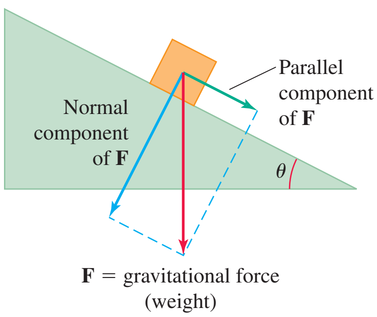


Figure 13.53

EXAMPLE 5 Components of a force A 10-lb block rests on a plane that is inclined at 30° above the horizontal. Find the components of the gravitational force parallel to and normal (perpendicular) to the plane.

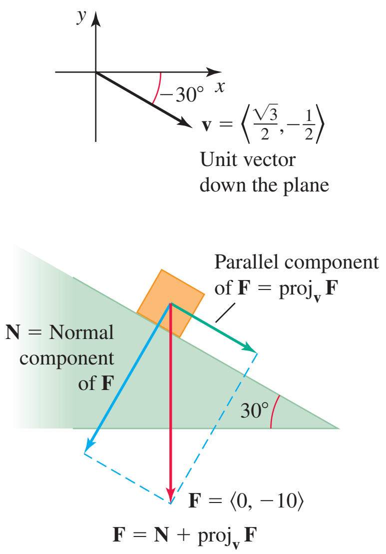


Figure 13.54

SOLUTION The gravitational force $\mathbf{F}$ acting on the block equals the weight of the block (10 lb); we regard the block as a point mass. Using the coordinate system shown in Figure 13.54, the force acts in the negative y-direction; therefore, $\mathbf{F} = \langle 0, -10 \rangle$. The direction *down* the plane is given by the unit vector $\mathbf{v} = \langle \cos(-30°), \sin(-30°) \rangle = \left\langle \frac{\sqrt{3}}{2}, -\frac{1}{2} \right\rangle$ (check that $|\mathbf{v}| = 1$). The component of the gravitational force parallel to the plane is

$$\text{proj}_{\mathbf{v}}\mathbf{F} = \underbrace{\left(\frac{\mathbf{F}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)}_{\mathbf{v}\cdot\mathbf{v}=1}\mathbf{v} = \left(\underbrace{\langle 0, -10\rangle}_{\mathbf{F}} \cdot \underbrace{\left\langle \frac{\sqrt{3}}{2}, -\frac{1}{2}\right\rangle}_{\mathbf{v}}\right)\underbrace{\left\langle \frac{\sqrt{3}}{2}, -\frac{1}{2}\right\rangle}_{\mathbf{v}} = 5\left\langle \frac{\sqrt{3}}{2}, -\frac{1}{2}\right\rangle.$$

Let the component of $\mathbf{F}$ normal to the plane be $\mathbf{N}$. Note that $\mathbf{F} = \text{proj}_{\mathbf{v}}\mathbf{F} + \mathbf{N}$, so

$$\mathbf{N} = \mathbf{F} - \text{proj}_{\mathbf{v}}\mathbf{F} = \langle 0, -10\rangle - 5\left\langle \frac{\sqrt{3}}{2}, -\frac{1}{2}\right\rangle = -5\left\langle \frac{\sqrt{3}}{2}, \frac{3}{2}\right\rangle.$$

Figure 13.54 shows how the components of $\mathbf{F}$ parallel to and normal to the plane combine to form the total force $\mathbf{F}$.

Related Exercises 47, 49 ◄

SECTION 13.3 EXERCISES

Getting Started

1. Express the dot product of $\mathbf{u}$ and $\mathbf{v}$ in terms of their magnitudes and the angle between them.
2. Express the dot product of $\mathbf{u}$ and $\mathbf{v}$ in terms of the components of the vectors.
3. Compute $\langle 2, 3, -6\rangle \cdot \langle 1, -8, 3\rangle$.
4. Use the definition of the dot product to explain why $\mathbf{v}\cdot\mathbf{v} = |\mathbf{v}|^2$.
5. Explain how to find the angle between two nonzero vectors.
6. Find the angle θ between $\mathbf{u}$ and $\mathbf{v}$ if $\text{scal}_{\mathbf{v}}\mathbf{u} = -2$ and $|\mathbf{u}| = 4$. Assume $0 \le \theta \le \pi$.
7. Find $\text{proj}_{\mathbf{v}}\mathbf{u}$ if $\text{scal}_{\mathbf{v}}\mathbf{u} = -2$ and $\mathbf{v} = \langle 2, -1, -2\rangle$.
8. Use a dot product to determine whether the vectors $\mathbf{u} = \langle 1, 2, 3\rangle$ and $\mathbf{v} = \langle 4, 1, -2\rangle$ are orthogonal.
9. Find $\mathbf{u}\cdot\mathbf{v}$ if $\mathbf{u}$ and $\mathbf{v}$ are unit vectors and the angle between $\mathbf{u}$ and $\mathbf{v}$ is π.
10. Explain how the work done by a force in moving an object is computed using dot products.
11. Suppose $\mathbf{v}$ is a nonzero position vector in the xy-plane. How many position vectors with length 2 in the xy-plane are orthogonal to $\mathbf{v}$?
12. Suppose $\mathbf{v}$ is a nonzero position vector in xyz-space. How many position vectors with length 2 in xyz-space are orthogonal to $\mathbf{v}$?

Practice Exercises

13–16. Dot product from the definition *Consider the following vectors* $\mathbf{u}$ *and* $\mathbf{v}$. *Sketch the vectors, find the angle between the vectors, and compute the dot product using the definition* $\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta$.

13. $\mathbf{u} = 4\mathbf{i}$ and $\mathbf{v} = 6\mathbf{j}$
14. $\mathbf{u} = \langle -3, 2, 0\rangle$ and $\mathbf{v} = \langle 0, 0, 6\rangle$
15. $\mathbf{u} = \langle 10, 0\rangle$ and $\mathbf{v} = \langle 10, 10\rangle$
16. $\mathbf{u} = \langle -\sqrt{3}, 1\rangle$ and $\mathbf{v} = \langle \sqrt{3}, 1\rangle$
17. **Dot product from the definition** Compute $\mathbf{u}\cdot\mathbf{v}$ if $\mathbf{u}$ and $\mathbf{v}$ are unit vectors and the angle between them is $\pi/3$.
18. **Dot product from the definition** Compute $\mathbf{u}\cdot\mathbf{v}$ if $\mathbf{u}$ is a unit vector, $|\mathbf{v}| = 2$, and the angle between them is $3\pi/4$.

T 19–28. Dot products and angles *Compute the dot product of the vectors* $\mathbf{u}$ *and* $\mathbf{v}$, *and find the angle between the vectors.*

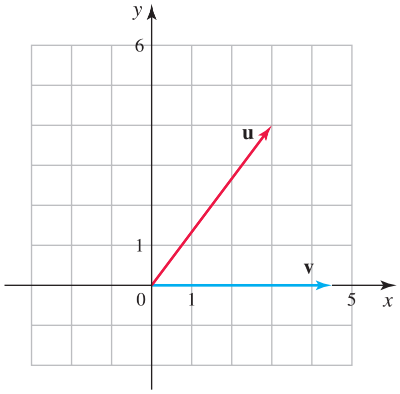
19. $\mathbf{u} = \mathbf{i} + \mathbf{j}$ and $\mathbf{v} = \mathbf{i} - \mathbf{j}$
20. $\mathbf{u} = \langle 10, 0\rangle$ and $\mathbf{v} = \langle -5, 5\rangle$
21. $\mathbf{u} = \mathbf{i}$ and $\mathbf{v} = \mathbf{i} + \sqrt{3}\,\mathbf{j}$
22. $\mathbf{u} = \sqrt{2}\,\mathbf{i} + \sqrt{2}\,\mathbf{j}$ and $\mathbf{v} = -\sqrt{2}\,\mathbf{i} - \sqrt{2}\,\mathbf{j}$
23. $\mathbf{u} = 4\mathbf{i} + 3\mathbf{j}$ and $\mathbf{v} = 4\mathbf{i} - 6\mathbf{j}$
24. $\mathbf{u} = \langle 3, 4, 0\rangle$ and $\mathbf{v} = \langle 0, 4, 5\rangle$
25. $\mathbf{u} = \langle -10, 0, 4\rangle$ and $\mathbf{v} = \langle 1, 2, 3\rangle$
26. $\mathbf{u} = \langle 3, -5, 2\rangle$ and $\mathbf{v} = \langle -9, 5, 1\rangle$
27. $\mathbf{u} = 2\mathbf{i} - 3\mathbf{k}$ and $\mathbf{v} = \mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$
28. $\mathbf{u} = \mathbf{i} - 4\mathbf{j} - 6\mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$

T 29–30. Angles of a triangle *For the given points P, Q, and R, find the approximate measurements of the angles of* ΔPQR.

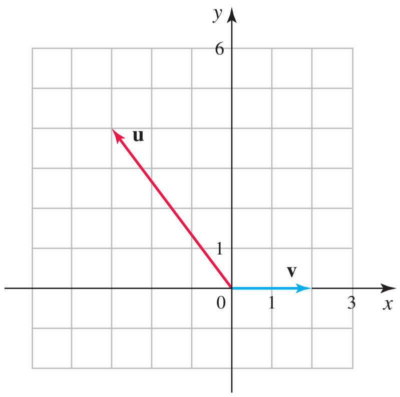
29. $P(0, -1, 3)$, $Q(2, 2, 1)$, $R(-2, 2, 4)$
30. $P(1, -4)$, $Q(2, 7)$, $R(-2, 2)$

31–34. Sketching orthogonal projections *Find* $\text{proj}_{\mathbf{v}}\mathbf{u}$ *and* $\text{scal}_{\mathbf{v}}\mathbf{u}$ *by inspection without using formulas.*

31.



32.



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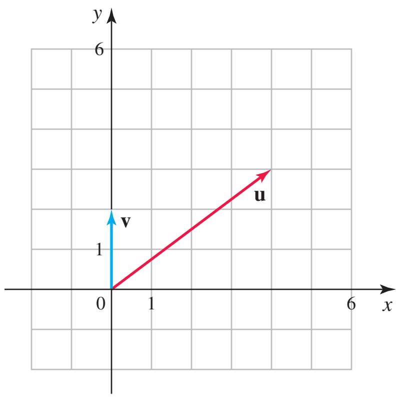

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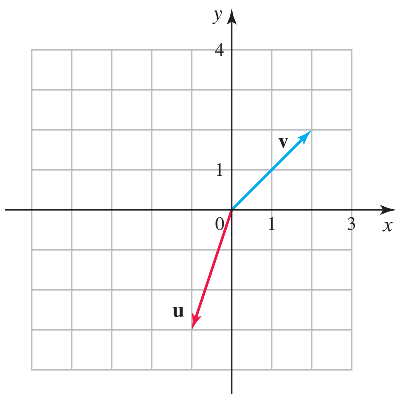


35–40. Calculating orthogonal projections *For the given vectors* $\mathbf{u}$ *and* $\mathbf{v}$*, calculate* $\text{proj}_{\mathbf{v}}\mathbf{u}$ *and* $\text{scal}_{\mathbf{v}}\mathbf{u}$.

35. $\mathbf{u} = \langle -1, 4 \rangle$ and $\mathbf{v} = \langle -4, 2 \rangle$

36. $\mathbf{u} = \langle 10, 5 \rangle$ and $\mathbf{v} = \langle 2, 6 \rangle$

37. $\mathbf{u} = \langle -8, 0, 2 \rangle$ and $\mathbf{v} = \langle 1, 3, -3 \rangle$

38. $\mathbf{u} = \langle 5, 0, 15 \rangle$ and $\mathbf{v} = \langle 0, 4, -2 \rangle$

39. $\mathbf{u} = 5\mathbf{i} + \mathbf{j} - 5\mathbf{k}$ and $\mathbf{v} = -\mathbf{i} + \mathbf{j} - 2\mathbf{k}$

40. $\mathbf{u} = \mathbf{i} + 4\mathbf{j} + 7\mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$

41–46. Computing work *Calculate the work done in the following situations.*

41. A suitcase is pulled 50 ft along a horizontal sidewalk with a constant force of 30 lb at an angle of 30° above the horizontal.

T **42.** A stroller is pushed 20 m along a horizontal sidewalk with a constant force of 10 N at an angle of 15° below the horizontal.

43. A sled is pulled 10 m along horizontal ground with a constant force of 5 N at an angle of 45° above the horizontal.

44. A constant force $\mathbf{F} = \langle 4, 3, 2 \rangle$ (in newtons) moves an object from $(0, 0, 0)$ to $(8, 6, 0)$. (Distance is measured in meters.)

45. A constant force $\mathbf{F} = \langle 40, 30 \rangle$ (in newtons) is used to move a sled horizontally 10 m.

46. A constant force $\mathbf{F} = \langle 2, 4, 1 \rangle$ (in newtons) moves an object from $(0, 0, 0)$ to $(2, 4, 6)$. (Distance is measured in meters.)

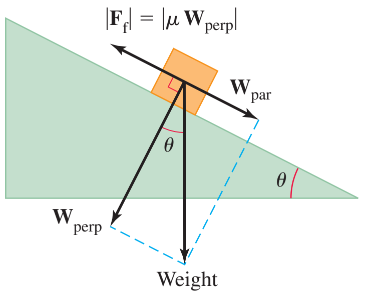
47–48. Parallel and normal forces *Find the components of the vertical force* $\mathbf{F} = \langle 0, -10 \rangle$ *in the directions parallel to and normal to the following planes. Show that the total force is the sum of the two component forces.*

47. A plane that makes an angle of $\pi/3$ with the positive x-axis

48. A plane that makes an angle of $\theta = \tan^{-1}\frac{4}{5}$ with the positive x-axis

49. Mass on a plane A 100-kg object rests on an inclined plane at an angle of 45° to the floor. Find the components of the force parallel to and perpendicular to the plane.

T **50. Forces on an inclined plane** An object on an inclined plane does not slide, provided the component of the object's weight parallel to the plane $|\mathbf{W}_{\text{par}}|$ is less than or equal to the magnitude of the opposing frictional force $|\mathbf{F}_{\text{f}}|$. The magnitude of the frictional force, in turn, is proportional to the component of the object's weight perpendicular to the plane $|\mathbf{W}_{\text{perp}}|$ (see figure). The constant of proportionality is the coefficient of static friction $\mu > 0$. Suppose a 100-lb block rests on a plane that is tilted at an angle of $\theta = 20°$ to the horizontal.



a. Find $|\mathbf{W}_{\text{par}}|$ and $|\mathbf{W}_{\text{perp}}|$. (*Hint:* It is not necessary to find $\mathbf{W}_{\text{par}}$ and $\mathbf{W}_{\text{perp}}$ first.)
b. The condition for the block not sliding is $|\mathbf{W}_{\text{par}}| \le \mu|\mathbf{W}_{\text{perp}}|$. If $\mu = 0.65$, does the block slide?
c. What is the critical angle above which the block slides?

51. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. $\text{proj}_{\mathbf{v}}\mathbf{u} = \text{proj}_{\mathbf{u}}\mathbf{v}$.
b. If nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ have the same magnitude, they make equal angles with $\mathbf{u} + \mathbf{v}$.
c. $(\mathbf{u}\cdot\mathbf{i})^2 + (\mathbf{u}\cdot\mathbf{j})^2 + (\mathbf{u}\cdot\mathbf{k})^2 = |\mathbf{u}|^2$.
d. If $\mathbf{u}$ is orthogonal to $\mathbf{v}$ and $\mathbf{v}$ is orthogonal to $\mathbf{w}$, then $\mathbf{u}$ is orthogonal to $\mathbf{w}$.
e. The vectors orthogonal to $\langle 1, 1, 1\rangle$ lie on the same line.
f. If $\text{proj}_{\mathbf{v}}\mathbf{u} = \mathbf{0}$, then vectors $\mathbf{u}$ and $\mathbf{v}$ (both nonzero) are orthogonal.

52. For what value of a is the vector $\mathbf{v} = \langle 4, -3, 7\rangle$ orthogonal to $\mathbf{w} = \langle a, 8, 3\rangle$?

53. For what value of c is the vector $\mathbf{v} = \langle 2, -5, c\rangle$ orthogonal to $\mathbf{w} = \langle 3, 2, 9\rangle$?

54. Find two vectors that are orthogonal to $\langle 0, 1, 1\rangle$ and to each other.

55. Let a and b be real numbers. Find all vectors $\langle 1, a, b\rangle$ orthogonal to $\langle 4, -8, 2\rangle$.

56. Find three mutually orthogonal unit vectors in $\mathbb{R}^3$ besides $\pm\mathbf{i}$, $\pm\mathbf{j}$, and $\pm\mathbf{k}$.

57. Equal angles Consider the set of all unit position vectors $\mathbf{u}$ in $\mathbb{R}^3$ that make a $60°$ angle with the unit vector $\mathbf{k}$ in $\mathbb{R}^3$.

a. Prove that $\text{proj}_{\mathbf{k}}\mathbf{u}$ is the same for all vectors in this set.
b. Is $\text{scal}_{\mathbf{k}}\mathbf{u}$ the same for all vectors in this set?

58–61. Vectors with equal projections *Given a fixed vector* $\mathbf{v}$, *there is an infinite set of vectors* $\mathbf{u}$ *with the same value of* $\text{proj}_{\mathbf{v}}\mathbf{u}$.

58. Find another vector that has the same projection onto $\mathbf{v} = \langle 1, 1\rangle$ as $\mathbf{u} = \langle 1, 2\rangle$. Draw a picture.

59. Let $\mathbf{v} = \langle 1, 1\rangle$. Give a description of the position vectors $\mathbf{u}$ such that $\text{proj}_{\mathbf{v}}\mathbf{u} = \text{proj}_{\mathbf{v}}\langle 1, 2\rangle$.

60. Find another vector that has the same projection onto $\mathbf{v} = \langle 1, 1, 1\rangle$ as $\mathbf{u} = \langle 1, 2, 3\rangle$.

61. Let $\mathbf{v} = \langle 0, 0, 1\rangle$. Give a description of all position vectors $\mathbf{u}$ such that $\text{proj}_{\mathbf{v}}\mathbf{u} = \text{proj}_{\mathbf{v}}\langle 1, 2, 3\rangle$.

62–65. Decomposing vectors *For the following vectors* $\mathbf{u}$ *and* $\mathbf{v}$, *express* $\mathbf{u}$ *as the sum* $\mathbf{u} = \mathbf{p} + \mathbf{n}$, *where* $\mathbf{p}$ *is parallel to* $\mathbf{v}$ *and* $\mathbf{n}$ *is orthogonal to* $\mathbf{v}$.

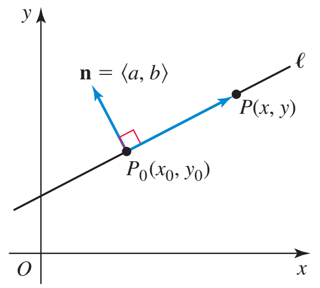
62. $\mathbf{u} = \langle 4, 3\rangle$, $\mathbf{v} = \langle 1, 1\rangle$

63. $\mathbf{u} = \langle -2, 2\rangle$, $\mathbf{v} = \langle 2, 1\rangle$

64. $\mathbf{u} = \langle 4, 3, 0\rangle$, $\mathbf{v} = \langle 1, 1, 1\rangle$

65. $\mathbf{u} = \langle -1, 2, 3\rangle$, $\mathbf{v} = \langle 2, 1, 1\rangle$

66–69. An alternative line definition *Given a fixed point* $P_0(x_0, y_0)$ *and a nonzero vector* $\mathbf{n} = \langle a, b\rangle$, *the set of points* $P(x, y)$ *for which* $\overrightarrow{P_0P}$ *is orthogonal to* $\mathbf{n}$ *is a line* ℓ *(see figure). The vector* $\mathbf{n}$ *is called a normal vector or a vector normal to* ℓ.



66. Show that the equation of the line passing through $P_0(x_0, y_0)$ with a normal vector $\mathbf{n} = \langle a, b\rangle$ is $a(x - x_0) + b(y - y_0) = 0$. (*Hint:* For a point $P(x, y)$ on ℓ, examine $\mathbf{n}\cdot\overrightarrow{P_0P}$.)

67. Use the result of Exercise 66 to find an equation of the line passing through the point $P_0(2, 6)$ with a normal vector $\mathbf{n} = \langle 3, -7\rangle$. Write the final answer in the form $ax + by = c$.

68. Use the result of Exercise 66 to find an equation of the line passing through the point $P_0(1, -3)$ with a normal vector $\mathbf{n} = \langle 4, 0\rangle$.

69. Suppose a line is normal to $\mathbf{n} = \langle 5, 3\rangle$. What is the slope of the line?

Explorations and Challenges

70–72. Orthogonal unit vectors in $\mathbb{R}^2$ *Consider the vectors* $\mathbf{I} = \langle 1/\sqrt{2}, 1/\sqrt{2}\rangle$ *and* $\mathbf{J} = \langle -1/\sqrt{2}, 1/\sqrt{2}\rangle$.

70. Show that $\mathbf{I}$ and $\mathbf{J}$ are orthogonal unit vectors.

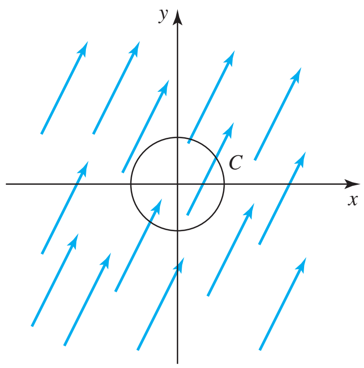
71. Express $\mathbf{I}$ and $\mathbf{J}$ in terms of the usual unit coordinate vectors $\mathbf{i}$ and $\mathbf{j}$. Then write $\mathbf{i}$ and $\mathbf{j}$ in terms of $\mathbf{I}$ and $\mathbf{J}$.

72. Write the vector $\langle 2, -6\rangle$ in terms of $\mathbf{I}$ and $\mathbf{J}$.

73. Orthogonal unit vectors in $\mathbb{R}^3$ Consider the vectors $\mathbf{I} = \langle 1/2, 1/2, 1/\sqrt{2}\rangle$, $\mathbf{J} = \langle -1/\sqrt{2}, 1/\sqrt{2}, 0\rangle$, and $\mathbf{K} = \langle 1/2, 1/2, -1/\sqrt{2}\rangle$.

a. Sketch $\mathbf{I}$, $\mathbf{J}$, and $\mathbf{K}$ and show that they are unit vectors.
b. Show that $\mathbf{I}$, $\mathbf{J}$, and $\mathbf{K}$ are pairwise orthogonal.
c. Express the vector $\langle 1, 0, 0\rangle$ in terms of $\mathbf{I}$, $\mathbf{J}$, and $\mathbf{K}$.

74. Flow through a circle Suppose water flows in a thin sheet over the xy-plane with a uniform velocity given by the vector $\mathbf{v} = \langle 1, 2\rangle$; this means that at all points of the plane, the velocity of the water has components 1 m/s in the x-direction and 2 m/s in the y-direction (see figure). Let C be an imaginary unit circle (that does not interfere with the flow).

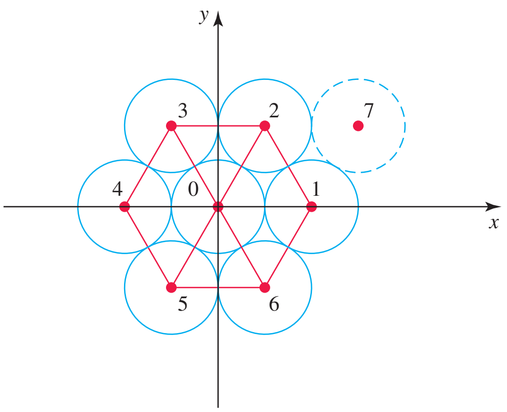


a. Show that at the point (x, y) on the circle C, the outward-pointing unit vector normal to C is $\mathbf{n} = \langle x, y\rangle$.
b. Show that at the point $(\cos\theta, \sin\theta)$ on the circle C, the outward-pointing unit vector normal to C is also $\mathbf{n} = \langle \cos\theta, \sin\theta\rangle$.

c. Find all points on C at which the velocity is normal to C.
d. Find all points on C at which the velocity is tangential to C.
e. At each point on C, find the component of $\mathbf{v}$ normal to C. Express the answer as a function of (x, y) and as a function of θ.
f. What is the net flow through the circle? Does water accumulate inside the circle?

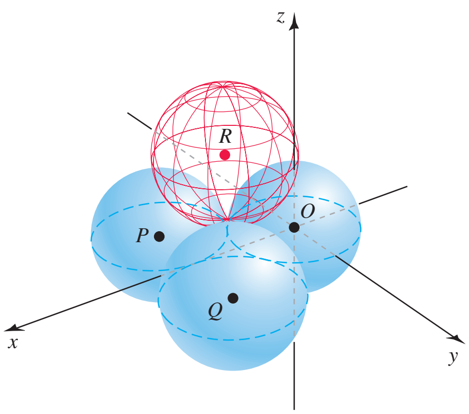
75. Heat flux Let D be a solid heat-conducting cube formed by the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$. The heat flow at every point of D is given by the constant vector $\mathbf{Q} = \langle 0, 2, 1 \rangle$.
a. Through which faces of D does $\mathbf{Q}$ point into D?
b. Through which faces of D does $\mathbf{Q}$ point out of D?
c. On which faces of D is $\mathbf{Q}$ tangential to D (pointing neither in nor out of D)?
d. Find the scalar component of $\mathbf{Q}$ normal to the face $x = 0$.
e. Find the scalar component of $\mathbf{Q}$ normal to the face $z = 1$.
f. Find the scalar component of $\mathbf{Q}$ normal to the face $y = 0$.

76. Hexagonal circle packing The German mathematician Carl Friedrich Gauss proved that the densest way to pack circles with the same radius in the plane is to place the centers of the circles on a hexagonal grid (see figure). Some molecular structures use this packing or its three-dimensional analog. Assume all circles have a radius of 1, and let $\mathbf{r}_{ij}$ be the vector that extends from the center of circle i to the center of circle j, for $i, j = 0, 1, \ldots, 6$.



a. Find $\mathbf{r}_{0j}$ for $j = 1, 2, \ldots, 6$.
b. Find $\mathbf{r}_{12}$, $\mathbf{r}_{34}$, and $\mathbf{r}_{61}$.
c. Imagine that circle 7 is added to the arrangement as shown in the figure. Find $\mathbf{r}_{07}$, $\mathbf{r}_{17}$, $\mathbf{r}_{47}$, and $\mathbf{r}_{75}$.

77. Hexagonal sphere packing Imagine three unit spheres (radius equal to 1) with centers at $O(0, 0, 0)$, $P(\sqrt{3}, -1, 0)$, and $Q(\sqrt{3}, 1, 0)$. Now place another unit sphere symmetrically on top of these spheres with its center at R (see figure).



a. Find the coordinates of R. (*Hint:* The distance between the centers of any two spheres is 2.)
b. Let $\mathbf{r}_{IJ}$ be the vector from point I to point J. Find $\mathbf{r}_{OP}$, $\mathbf{r}_{OQ}$, $\mathbf{r}_{PQ}$, $\mathbf{r}_{OR}$, and $\mathbf{r}_{PR}$.

78–81. Properties of dot products *Let* $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, *and* $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$. *Prove the following vector properties, where c is a scalar.*

78. $|\mathbf{u} \cdot \mathbf{v}| \le |\mathbf{u}||\mathbf{v}|$

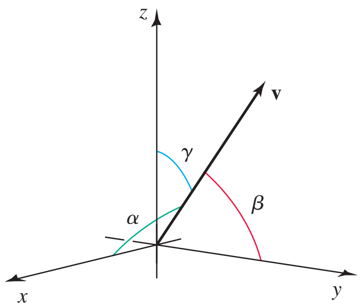
79. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ Commutative property

80. $c(\mathbf{u} \cdot \mathbf{v}) = (c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v})$ Associative property

81. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ Distributive property

82. Distributive properties
a. Show that $(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = |\mathbf{u}|^2 + 2\mathbf{u} \cdot \mathbf{v} + |\mathbf{v}|^2$.
b. Show that $(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = |\mathbf{u}|^2 + |\mathbf{v}|^2$ if $\mathbf{u}$ is orthogonal to $\mathbf{v}$.
c. Show that $(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = |\mathbf{u}|^2 - |\mathbf{v}|^2$.

83. Direction angles and cosines Let $\mathbf{v} = \langle a, b, c \rangle$ and let α, β, and γ be the angles between $\mathbf{v}$ and the positive x-axis, the positive y-axis, and the positive z-axis, respectively (see figure).



a. Prove that $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$.
b. Find a vector that makes a $45°$ angle with $\mathbf{i}$ and $\mathbf{j}$. What angle does it make with $\mathbf{k}$?
c. Find a vector that makes a $60°$ angle with $\mathbf{i}$ and $\mathbf{j}$. What angle does it make with $\mathbf{k}$?
d. Is there a vector that makes a $30°$ angle with $\mathbf{i}$ and $\mathbf{j}$? Explain.
e. Find a vector $\mathbf{v}$ such that $\alpha = \beta = \gamma$. What is the angle?

84–88. Cauchy-Schwarz Inequality *The definition* $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta$ *implies that* $|\mathbf{u} \cdot \mathbf{v}| \le |\mathbf{u}||\mathbf{v}|$ *(because* $|\cos\theta| \le 1$*). This inequality, known as the Cauchy-Schwarz Inequality, holds in any number of dimensions and has many consequences.*

84. What conditions on $\mathbf{u}$ and $\mathbf{v}$ lead to equality in the Cauchy-Schwarz Inequality?

85. Verify that the Cauchy-Schwarz Inequality holds for $\mathbf{u} = \langle 3, -5, 6 \rangle$ and $\mathbf{v} = \langle -8, 3, 1 \rangle$.

86. Geometric-arithmetic mean Use the vectors $\mathbf{u} = \langle \sqrt{a}, \sqrt{b} \rangle$ and $\mathbf{v} = \langle \sqrt{b}, \sqrt{a} \rangle$ to show that $\sqrt{ab} \le \dfrac{a + b}{2}$, where $a \ge 0$ and $b \ge 0$.

87. Triangle Inequality Consider the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} + \mathbf{v}$ (in any number of dimensions). Use the following steps to prove that $|\mathbf{u} + \mathbf{v}| \le |\mathbf{u}| + |\mathbf{v}|$.
a. Show that $|\mathbf{u} + \mathbf{v}|^2 = (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = |\mathbf{u}|^2 + 2\mathbf{u} \cdot \mathbf{v} + |\mathbf{v}|^2$.

b. Use the Cauchy-Schwarz Inequality to show that $|\mathbf{u} + \mathbf{v}|^2 \le (|\mathbf{u}| + |\mathbf{v}|)^2$.

c. Conclude that $|\mathbf{u} + \mathbf{v}| \le |\mathbf{u}| + |\mathbf{v}|$.

d. Interpret the Triangle Inequality geometrically in $\mathbb{R}^2$ or $\mathbb{R}^3$.

88. Algebra inequality Show that

$$(u_1 + u_2 + u_3)^2 \le 3(u_1^2 + u_2^2 + u_3^2),$$

for any real numbers u_1, u_2, and u_3. (*Hint:* Use the Cauchy-Schwarz Inequality in three dimensions with $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and choose $\mathbf{v}$ in the right way.)

89. Diagonals of a parallelogram Consider the parallelogram with adjacent sides $\mathbf{u}$ and $\mathbf{v}$.

a. Show that the diagonals of the parallelogram are $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$.

b. Prove that the diagonals have the same length if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.

c. Show that the sum of the squares of the lengths of the diagonals equals the sum of the squares of the lengths of the sides.

QUICK CHECK ANSWERS

1. If $\theta = 0$, then $\mathbf{u}$ and $\mathbf{v}$ are parallel and point in the same direction. If $\theta = \pi$, then $\mathbf{u}$ and $\mathbf{v}$ are parallel and point in opposite directions. **2.** All these dot products are zero, and the unit vectors are mutually orthogonal. The angle between two different unit vectors is $\pi/2$. **3.** $\text{proj}_{\mathbf{i}}\mathbf{u} = 4\mathbf{i}$, $\text{proj}_{\mathbf{j}}\mathbf{u} = -3\mathbf{j}$, $\text{scal}_{\mathbf{i}}\mathbf{u} = 4$, $\text{scal}_{\mathbf{j}}\mathbf{u} = -3$ ➤

13.4 Cross Products

The dot product combines two vectors to produce a *scalar* result. There is an equally fundamental way to combine two vectors in $\mathbb{R}^3$ and obtain a *vector* result. This operation, known as the *cross product* (or *vector product*), may be motivated by a physical application.

Suppose you want to loosen a bolt with a wrench. As you apply force to the end of the wrench in the plane perpendicular to the bolt, the "twisting power" you generate depends on three variables:

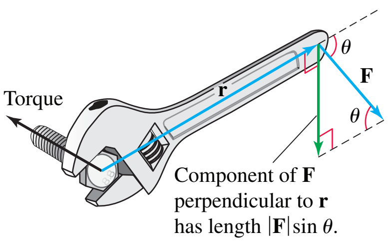


Figure 13.55

- the magnitude of the force $\mathbf{F}$ applied to the wrench;
- the length $|\mathbf{r}|$ of the wrench;
- the angle at which the force is applied to the wrench.

The twisting generated by a force acting at a distance from a pivot point is called **torque** (from the Latin *to twist*). The torque is a vector whose magnitude is proportional to $|\mathbf{F}|$, $|\mathbf{r}|$, and $\sin\theta$, where θ is the angle between $\mathbf{F}$ and $\mathbf{r}$ (Figure 13.55). If the force is applied parallel to the wrench—for example, if you pull the wrench ($\theta = 0$) or push the wrench ($\theta = \pi$)—there is no twisting effect; if the force is applied perpendicular to the wrench ($\theta = \pi/2$), the twisting effect is maximized. The direction of the torque vector is defined to be orthogonal to both $\mathbf{F}$ and $\mathbf{r}$. As we will see shortly, the torque is expressed in terms of the cross product of $\mathbf{F}$ and $\mathbf{r}$.

The Cross Product

The preceding physical example leads to the following definition of the cross product.

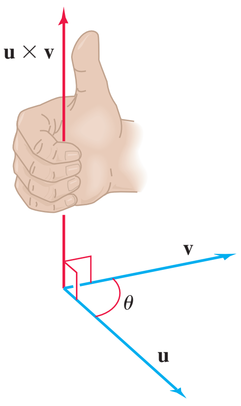


Figure 13.56

> **DEFINITION Cross Product**
>
> Given two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$, the **cross product** $\mathbf{u} \times \mathbf{v}$ is a vector with magnitude
>
> $$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin\theta,$$
>
> where $0 \le \theta \le \pi$ is the angle between $\mathbf{u}$ and $\mathbf{v}$. The direction of $\mathbf{u} \times \mathbf{v}$ is given by the **right-hand rule**: When you put the vectors tail to tail and let the fingers of your right hand curl from $\mathbf{u}$ to $\mathbf{v}$, the direction of $\mathbf{u} \times \mathbf{v}$ is the direction of your thumb, orthogonal to both $\mathbf{u}$ and $\mathbf{v}$ (Figure 13.56). When $\mathbf{u} \times \mathbf{v} = \mathbf{0}$, the direction of $\mathbf{u} \times \mathbf{v}$ is undefined.

QUICK CHECK 1 Sketch the vectors $\mathbf{u} = \langle 1, 2, 0 \rangle$ and $\mathbf{v} = \langle -1, 2, 0 \rangle$. Which way does $\mathbf{u} \times \mathbf{v}$ point? Which way does $\mathbf{v} \times \mathbf{u}$ point? ◄

The following theorem is an immediate consequence of the definition of the cross product.

THEOREM 13.3 Geometry of the Cross Product

Let $\mathbf{u}$ and $\mathbf{v}$ be two nonzero vectors in $\mathbb{R}^3$.

1. The vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel ($\theta = 0$ or $\theta = \pi$) if and only if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.
2. If $\mathbf{u}$ and $\mathbf{v}$ are two sides of a parallelogram (Figure 13.57), then the area of the parallelogram is

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin \theta.$$

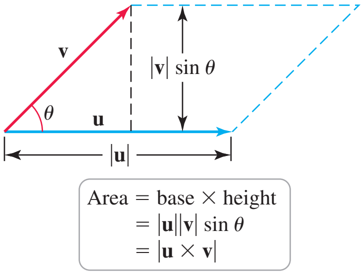


Figure 13.57

EXAMPLE 1 **A cross product** Find the magnitude and direction of $\mathbf{u} \times \mathbf{v}$, where $\mathbf{u} = \langle 1, 1, 0 \rangle$ and $\mathbf{v} = \langle 1, 1, \sqrt{2} \rangle$.

SOLUTION Because $\mathbf{u}$ is one side of a 45–45–90 triangle and $\mathbf{v}$ is the hypotenuse (Figure 13.58), we have $\theta = \pi/4$ and $\sin \theta = \frac{1}{\sqrt{2}}$. Also, $|\mathbf{u}| = \sqrt{2}$ and $|\mathbf{v}| = 2$, so the magnitude of $\mathbf{u} \times \mathbf{v}$ is

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin \theta = \sqrt{2} \cdot 2 \cdot \frac{1}{\sqrt{2}} = 2.$$

The direction of $\mathbf{u} \times \mathbf{v}$ is given by the right-hand rule: $\mathbf{u} \times \mathbf{v}$ is orthogonal to $\mathbf{u}$ and $\mathbf{v}$ (Figure 13.58).

Related Exercises 14–15 ◄

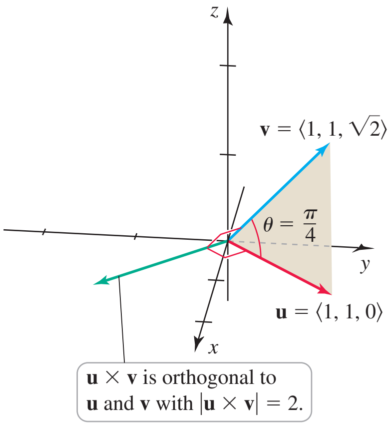


Figure 13.58

Properties of the Cross Product

The cross product has several algebraic properties that simplify calculations. For example, scalars factor out of a cross product; that is, if a and b are scalars, then (Exercise 69)

$$(a\mathbf{u}) \times (b\mathbf{v}) = ab(\mathbf{u} \times \mathbf{v}).$$

The order in which the cross product is performed is important. The magnitudes of $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$ are equal. However, applying the right-hand rule shows that $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$ point in opposite directions. Therefore, $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$. There are two distributive properties for the cross product, whose proofs are omitted.

THEOREM 13.4 Properties of the Cross Product

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be nonzero vectors in $\mathbb{R}^3$, and let a and b be scalars.

1. $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$ — Anticommutative property
2. $(a\mathbf{u}) \times (b\mathbf{v}) = ab(\mathbf{u} \times \mathbf{v})$ — Associative property
3. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$ — Distributive property
4. $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$ — Distributive property

QUICK CHECK 2 Explain why the vector $2\mathbf{u} \times 3\mathbf{v}$ points in the same direction as the vector $\mathbf{u} \times \mathbf{v}$. ◄

EXAMPLE 2 **Cross products of unit vectors** Evaluate all the cross products among the coordinate unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

SOLUTION These vectors are mutually orthogonal, which means the angle between any two distinct vectors is $\theta = \pi/2$ and $\sin \theta = 1$. Furthermore, $|\mathbf{i}| = |\mathbf{j}| = |\mathbf{k}| = 1$. Therefore, the cross product of any two distinct vectors has magnitude 1. By the right-hand rule, when the fingers of the right hand curl from $\mathbf{i}$ to $\mathbf{j}$, the thumb points in the direction of the positive z-axis (Figure 13.59). The unit vector in the positive z-direction is $\mathbf{k}$, so $\mathbf{i} \times \mathbf{j} = \mathbf{k}$. Similar calculations show that $\mathbf{j} \times \mathbf{k} = \mathbf{i}$ and $\mathbf{k} \times \mathbf{i} = \mathbf{j}$.

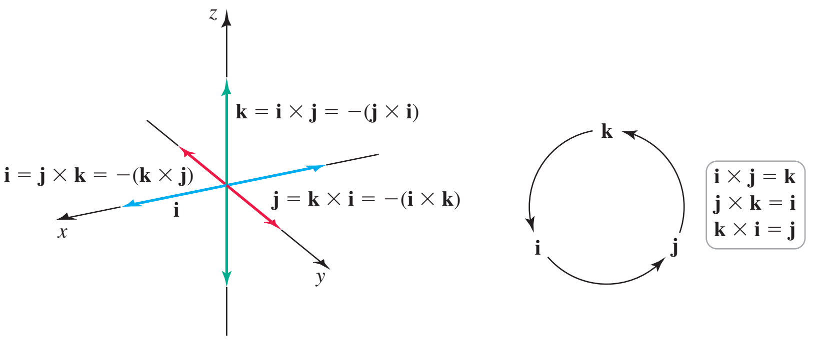


Figure 13.59

By property 1 of Theorem 13.4, $\mathbf{j} \times \mathbf{i} = -(\mathbf{i} \times \mathbf{j}) = -\mathbf{k}$, so $\mathbf{j} \times \mathbf{i}$ and $\mathbf{i} \times \mathbf{j}$ point in opposite directions. Similarly, $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$ and $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$. These relationships are easily remembered with the circle diagram in Figure 13.59. Finally, the angle between any unit vector and itself is $\theta = 0$. Therefore, $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$.

Related Exercises 17–19

THEOREM 13.5 Cross Products of Coordinate Unit Vectors

$$\mathbf{i} \times \mathbf{j} = -(\mathbf{j} \times \mathbf{i}) = \mathbf{k} \qquad \mathbf{j} \times \mathbf{k} = -(\mathbf{k} \times \mathbf{j}) = \mathbf{i}$$
$$\mathbf{k} \times \mathbf{i} = -(\mathbf{i} \times \mathbf{k}) = \mathbf{j} \qquad \mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$$

What is missing so far is an efficient method for finding the components of the cross product of two vectors in $\mathbb{R}^3$. Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. Using the distributive properties of the cross product (Theorem 13.4), we have

➤ The determinant of the matrix A is denoted both $|A|$ and det A. The formula for the determinant of a 3×3 matrix A is

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix},$$

where

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

$$\begin{aligned}
\mathbf{u} \times \mathbf{v} &= (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \\
&= u_1v_1\underbrace{(\mathbf{i} \times \mathbf{i})}_{\mathbf{0}} + u_1v_2\underbrace{(\mathbf{i} \times \mathbf{j})}_{\mathbf{k}} + u_1v_3\underbrace{(\mathbf{i} \times \mathbf{k})}_{-\mathbf{j}} \\
&\quad + u_2v_1\underbrace{(\mathbf{j} \times \mathbf{i})}_{-\mathbf{k}} + u_2v_2\underbrace{(\mathbf{j} \times \mathbf{j})}_{\mathbf{0}} + u_2v_3\underbrace{(\mathbf{j} \times \mathbf{k})}_{\mathbf{i}} \\
&\quad + u_3v_1\underbrace{(\mathbf{k} \times \mathbf{i})}_{\mathbf{j}} + u_3v_2\underbrace{(\mathbf{k} \times \mathbf{j})}_{-\mathbf{i}} + u_3v_3\underbrace{(\mathbf{k} \times \mathbf{k})}_{\mathbf{0}}.
\end{aligned}$$

This formula looks impossible to remember until we see that it fits the pattern used to evaluate 3×3 determinants. Specifically, if we compute the determinant of the matrix

$$\begin{array}{r c} \text{Unit vectors} \rightarrow \\ \text{Components of } \mathbf{u} \rightarrow \\ \text{Components of } \mathbf{v} \rightarrow \end{array} \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix}$$

(expanding about the first row), the following formula for the cross product emerges (see margin note).

THEOREM 13.6 Evaluating the Cross Product

Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. Then

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\mathbf{k}.$$

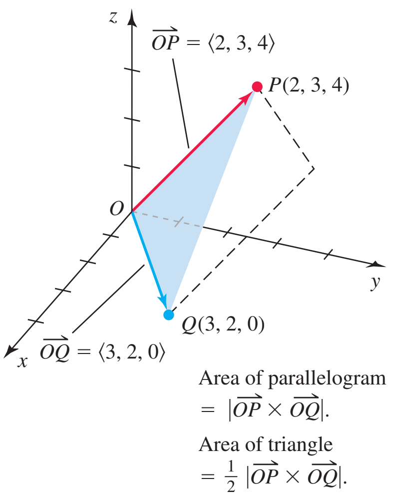


Figure 13.60

EXAMPLE 3 Area of a triangle Find the area of the triangle with vertices $O(0, 0, 0)$, $P(2, 3, 4)$, and $Q(3, 2, 0)$ (**Figure 13.60**).

SOLUTION First consider the parallelogram, two of whose sides are the vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$. By Theorem 13.3, the area of this parallelogram is $|\overrightarrow{OP} \times \overrightarrow{OQ}|$. Computing the cross product, we find that

$$\overrightarrow{OP} \times \overrightarrow{OQ} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 4 \\ 3 & 2 & 0 \end{vmatrix} = \begin{vmatrix} 3 & 4 \\ 2 & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 4 \\ 3 & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix}\mathbf{k}$$
$$= -8\mathbf{i} + 12\mathbf{j} - 5\mathbf{k}.$$

Therefore, the area of the parallelogram is

$$|\overrightarrow{OP} \times \overrightarrow{OQ}| = |-8\mathbf{i} + 12\mathbf{j} - 5\mathbf{k}| = \sqrt{233} \approx 15.26.$$

The triangle with vertices O, P, and Q forms half of the parallelogram, so its area is $\sqrt{233}/2 \approx 7.63$.

Related Exercises 34–36 ◄

EXAMPLE 4 **Vector orthogonal to two vectors** Find a vector orthogonal to the two vectors $\mathbf{u} = -\mathbf{i} + 6\mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} - 5\mathbf{j} - 3\mathbf{k}$.

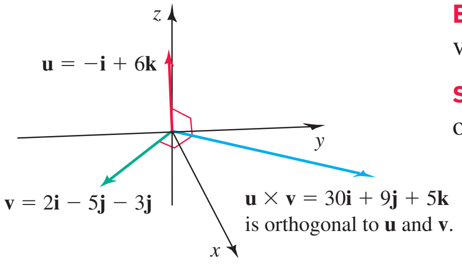


Figure 13.61

SOLUTION A vector orthogonal to $\mathbf{u}$ and $\mathbf{v}$ is parallel to $\mathbf{u} \times \mathbf{v}$ (Figure 13.61). One such orthogonal vector is

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 0 & 6 \\ 2 & -5 & -3 \end{vmatrix}$$
$$= (0 + 30)\mathbf{i} - (3 - 12)\mathbf{j} + (5 - 0)\mathbf{k}$$
$$= 30\mathbf{i} + 9\mathbf{j} + 5\mathbf{k}.$$

Any scalar multiple of this vector is also orthogonal to $\mathbf{u}$ and $\mathbf{v}$.

Related Exercises 42–44 ◄

QUICK CHECK 3 A good check on a cross product calculation is to verify that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal to the computed $\mathbf{u} \times \mathbf{v}$. In Example 4, verify that $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ and $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$. ◄

Applications of the Cross Product

We now investigate two physical applications of the cross product.

Torque Returning to the example of applying a force to a wrench, suppose a force $\mathbf{F}$ is applied to the point P at the head of a vector $\mathbf{r} = \overrightarrow{OP}$ (Figure 13.62). The **torque**, or twisting effect, produced by the force about the point O is given by $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$. The torque vector has a magnitude of

$$|\boldsymbol{\tau}| = |\mathbf{r} \times \mathbf{F}| = |\mathbf{r}||\mathbf{F}| \sin \theta,$$

where θ is the angle between $\mathbf{r}$ and $\mathbf{F}$. The direction of the torque is given by the right-hand rule; it is orthogonal to both $\mathbf{r}$ and $\mathbf{F}$. As noted earlier, if $\mathbf{r}$ and $\mathbf{F}$ are parallel, then $\sin \theta = 0$ and the torque is zero. For a given $\mathbf{r}$ and $\mathbf{F}$, the maximum torque occurs when $\mathbf{F}$ is applied in a direction orthogonal to $\mathbf{r}$ ($\theta = \pi/2$).

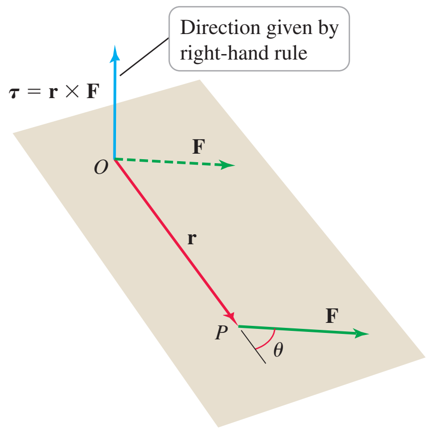
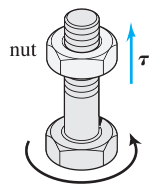


Figure 13.62

EXAMPLE 5 **Tightening a bolt** A force of 20 N is applied to a wrench attached to a bolt in a direction perpendicular to the bolt (Figure 13.63). Which produces more torque: applying the force at an angle of 60° on a wrench that is 0.15 m long or applying the force at an angle of 135° on a wrench that is 0.25 m long? In each case, what is the direction of the torque?

➤ When standard threads are added to the bolt in Figure 13.63, the forces used in Example 5 cause the bolt to move upward into a nut—in the direction of the torque.



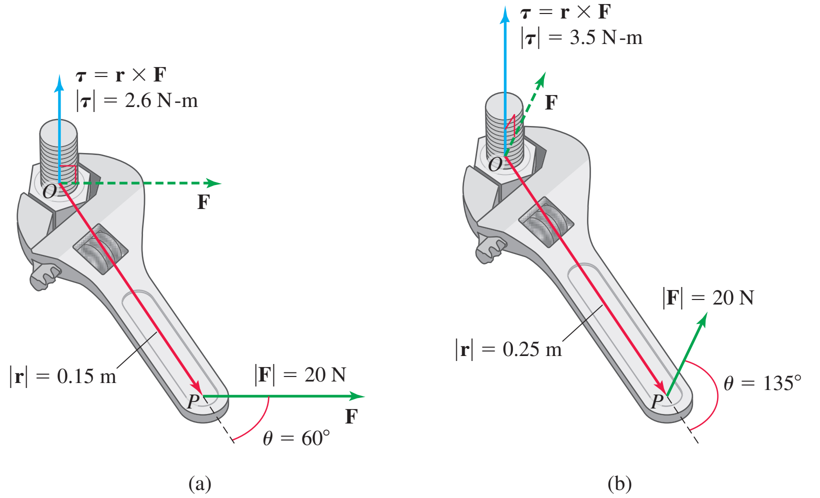


Figure 13.63

SOLUTION The magnitude of the torque in the first case is

$$|\boldsymbol{\tau}| = |\mathbf{r}||\mathbf{F}| \sin\theta = (0.15 \text{ m})(20 \text{ N}) \sin 60° \approx 2.6 \text{ N-m}.$$

In the second case, the magnitude of the torque is

$$|\boldsymbol{\tau}| = |\mathbf{r}||\mathbf{F}| \sin\theta = (0.25 \text{ m})(20 \text{ N}) \sin 135° \approx 3.5 \text{ N-m}.$$

The second instance gives the greater torque. In both cases, the torque is orthogonal to **r** and **F**, parallel to the shaft of the bolt (Figure 13.63).

Related Exercises 47, 49 ◄

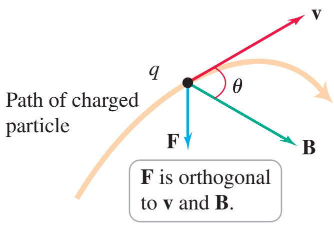


Figure 13.64

Magnetic Force on a Moving Charge Moving electric charges (either an isolated charge or a current in a wire) experience a force when they pass through a magnetic field. For an isolated charge q, the force is given by $\mathbf{F} = q(\mathbf{v} \times \mathbf{B})$, where $\mathbf{v}$ is the velocity of the charge and $\mathbf{B}$ is the magnetic field. The magnitude of the force is

$$|\mathbf{F}| = |q||\mathbf{v} \times \mathbf{B}| = |q||\mathbf{v}||\mathbf{B}| \sin\theta,$$

where θ is the angle between $\mathbf{v}$ and $\mathbf{B}$ (Figure 13.64). Note that the sign of the charge also determines the direction of the force. If the velocity vector is parallel to the magnetic field, the charge experiences no force. The maximum force occurs when the velocity is orthogonal to the magnetic field.

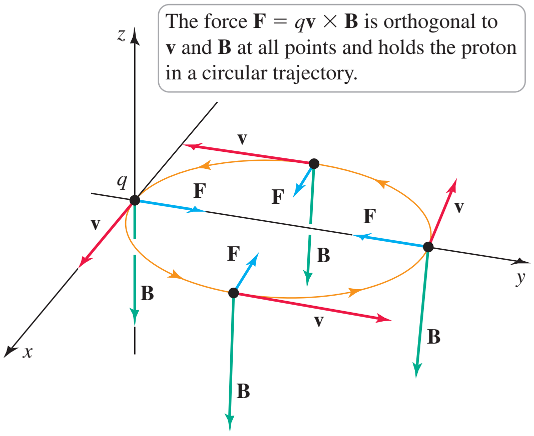


Figure 13.65

EXAMPLE 6 **Force on a proton** A proton with a mass of 1.7×10^{-27} kg and a charge of $q = +1.6 \times 10^{-19}$ coulombs (C) moves along the x-axis with a speed of $|\mathbf{v}| = 9 \times 10^5$ m/s. When it reaches $(0, 0, 0)$, a uniform magnetic field is turned on. The field has a constant strength of 1 tesla (1 T) and is directed along the negative z-axis (Figure 13.65).

a. Find the magnitude and direction of the force on the proton at the instant it enters the magnetic field.

b. Assume the proton loses no energy and the force in part (a) acts as a *centripetal force* with magnitude $|\mathbf{F}| = m|\mathbf{v}|^2/R$ that keeps the proton in a circular orbit of radius R. Find the radius of the orbit.

➤ The standard unit of magnetic field strength is the tesla (T, named after Nicola Tesla). A typical strong bar magnet has a strength of about 1 T. In terms of other units, $1\text{ T} = 1\text{ kg}/(\text{C-s})$, where C is the unit of charge called the *coulomb*. Therefore, the units of force in Example 6a are $\text{kg-m}/\text{s}^2$, or newtons.

SOLUTION

a. Expressed as vectors, we have $\mathbf{v} = 9 \times 10^5\, \mathbf{i}$ and $\mathbf{B} = -\mathbf{k}$. Therefore, the force on the proton in newtons is

$$\begin{aligned}\mathbf{F} = q(\mathbf{v} \times \mathbf{B}) &= 1.6 \times 10^{-19}\big((9 \times 10^5\, \mathbf{i}) \times (-\mathbf{k})\big)\\ &= 1.44 \times 10^{-13}\mathbf{j}.\end{aligned}$$

As shown in Figure 13.65, when the proton enters the magnetic field in the positive x-direction, the force acts in the positive y-direction, which changes the path of the proton.

b. The magnitude of the force acting on the proton remains 1.44×10^{-13} N at all times (from part (a)). Equating this force to the centripetal force $|\mathbf{F}| = m|\mathbf{v}|^2/R$, we find that

$$R = \frac{m|\mathbf{v}|^2}{|\mathbf{F}|} = \frac{(1.7 \times 10^{-27}\text{ kg})\,(9 \times 10^5\text{ m/s})^2}{1.44 \times 10^{-13}\text{ N}} \approx 0.01\text{ m}.$$

Assuming no energy loss, the proton moves in a circular orbit of radius 0.01 m.

Related Exercises 55, 67 ◀

SECTION 13.4 EXERCISES

Getting Started

1. What is the magnitude of the cross product of two parallel vectors?

2. If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, what is the magnitude of $\mathbf{u} \times \mathbf{v}$?

3. Suppose $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors. What is the geometric relationship between $\mathbf{u}$ and $\mathbf{v}$ under each of the following conditions?
   a. $\mathbf{u} \cdot \mathbf{v} = 0$ **b.** $\mathbf{u} \times \mathbf{v} = \mathbf{0}$

4. Use a geometric argument to explain why $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$.

5. Compute $|\mathbf{u} \times \mathbf{v}|$ if $\mathbf{u}$ and $\mathbf{v}$ are unit vectors and the angle between them is $\pi/4$.

6. Compute $|\mathbf{u} \times \mathbf{v}|$ if $|\mathbf{u}| = 3$ and $|\mathbf{v}| = 4$ and the angle between $\mathbf{u}$ and $\mathbf{v}$ is $2\pi/3$.

7. Find $\mathbf{v} \times \mathbf{u}$ if $\mathbf{u} \times \mathbf{v} = 3\mathbf{i} + 2\mathbf{j} - 7\mathbf{k}$.

8. For any vector $\mathbf{v}$ in $\mathbb{R}^3$, explain why $\mathbf{v} \times \mathbf{v} = \mathbf{0}$.

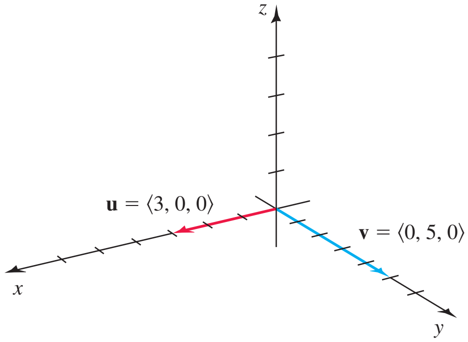
9. Explain how to use a determinant to compute $\mathbf{u} \times \mathbf{v}$.

10. Explain how to find the torque produced by a force using cross products.

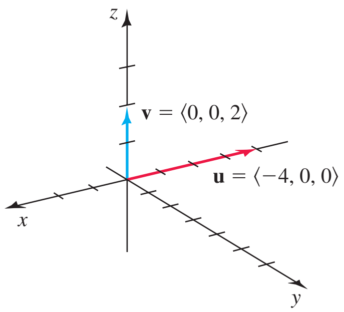
Practice Exercises

11–12. Cross products from the definition *Find the cross product* $\mathbf{u} \times \mathbf{v}$ *in each figure.*

11.



12.



13–16. Cross products from the definition *Sketch the following vectors* $\mathbf{u}$ *and* $\mathbf{v}$*. Then compute* $|\mathbf{u} \times \mathbf{v}|$ *and show the cross product on your sketch.*

13. $\mathbf{u} = \langle 0, -2, 0 \rangle$, $\mathbf{v} = \langle 0, 1, 0 \rangle$

14. $\mathbf{u} = \langle 0, 4, 0 \rangle$, $\mathbf{v} = \langle 0, 0, -8 \rangle$

15. $\mathbf{u} = \langle 3, 3, 0 \rangle$, $\mathbf{v} = \langle 3, 3, 3\sqrt{2} \rangle$

16. $\mathbf{u} = \langle 0, -2, -2 \rangle$, $\mathbf{v} = \langle 0, 2, -2 \rangle$

17–22. Coordinate unit vectors *Compute the following cross products. Then make a sketch showing the two vectors and their cross product.*

17. $\mathbf{j} \times \mathbf{k}$ 18. $\mathbf{i} \times \mathbf{k}$ 19. $-\mathbf{j} \times \mathbf{k}$

20. $3\mathbf{j} \times \mathbf{i}$ 21. $-2\mathbf{i} \times 3\mathbf{k}$ 22. $2\mathbf{j} \times (-5)\mathbf{i}$

23–28. Computing cross products *Find the cross products* $\mathbf{u} \times \mathbf{v}$ *and* $\mathbf{v} \times \mathbf{u}$ *for the following vectors* $\mathbf{u}$ *and* $\mathbf{v}$.

23. $\mathbf{u} = \langle 3, 5, 0 \rangle$, $\mathbf{v} = \langle 0, 3, -6 \rangle$

24. $\mathbf{u} = \langle -4, 1, 1 \rangle$, $\mathbf{v} = \langle 0, 1, -1 \rangle$

25. $\mathbf{u} = \langle 2, 3, -9 \rangle$, $\mathbf{v} = \langle -1, 1, -1 \rangle$

26. $\mathbf{u} = \langle 3, -4, 6 \rangle$, $\mathbf{v} = \langle 1, 2, -1 \rangle$

27. $\mathbf{u} = 3\mathbf{i} - \mathbf{j} - 2\mathbf{k}$, $\mathbf{v} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$

28. $\mathbf{u} = 2\mathbf{i} - 10\mathbf{j} + 15\mathbf{k}$, $\mathbf{v} = 0.5\mathbf{i} + \mathbf{j} - 0.6\mathbf{k}$

29–32. Area of a parallelogram *Find the area of the parallelogram that has two adjacent sides* $\mathbf{u}$ *and* $\mathbf{v}$.

29. $\mathbf{u} = 3\mathbf{i} - \mathbf{j}$, $\mathbf{v} = 3\mathbf{j} + 2\mathbf{k}$

30. $\mathbf{u} = -3\mathbf{i} + 2\mathbf{k}$, $\mathbf{v} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

31. $\mathbf{u} = 2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$, $\mathbf{v} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

32. $\mathbf{u} = 8\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$, $\mathbf{v} = 2\mathbf{i} + 4\mathbf{j} - 4\mathbf{k}$

33–38. Areas of triangles *Find the area of the following triangles* T.

33. The vertices of T are $A(0, 0, 0)$, $B(3, 0, 1)$, and $C(1, 1, 0)$.

34. The vertices of T are $O(0, 0, 0)$, $P(1, 2, 3)$, and $Q(6, 5, 4)$.

35. The vertices of T are $A(5, 6, 2)$, $B(7, 16, 4)$, and $C(6, 7, 3)$.

36. The vertices of T are $A(-1, -5, -3)$, $B(-3, -2, -1)$, and $C(0, -5, -1)$.

37. The sides of T are $\mathbf{u} = \langle 3, 3, 3 \rangle$, $\mathbf{v} = \langle 6, 0, 6 \rangle$, and $\mathbf{u} - \mathbf{v}$.

38. The sides of T are $\mathbf{u} = \langle 0, 6, 0 \rangle$, $\mathbf{v} = \langle 4, 4, 4 \rangle$, and $\mathbf{u} - \mathbf{v}$.

39. Collinear points and cross products Explain why the points A, B, and C in $\mathbb{R}^3$ are collinear if and only if $\overrightarrow{AB} \times \overrightarrow{AC} = \mathbf{0}$.

40–41. Collinear points *Use cross products to determine whether the points A, B, and C are collinear.*

40. $A(3, 2, 1)$, $B(5, 4, 7)$, and $C(9, 8, 19)$

41. $A(-3, -2, 1)$, $B(1, 4, 7)$, and $C(4, 10, 14)$

42–44. Orthogonal vectors *Find a vector orthogonal to the given vectors.*

42. $\langle 1, 2, 3 \rangle$ and $\langle -2, 4, -1 \rangle$

43. $\langle 0, 1, 2 \rangle$ and $\langle -2, 0, 3 \rangle$

44. $\langle 8, 0, 4 \rangle$ and $\langle -8, 2, 1 \rangle$

45–48. Computing torque *Answer the following questions about torque.*

45. Let $\mathbf{r} = \overrightarrow{OP} = \mathbf{i} + \mathbf{j} + \mathbf{k}$. A force $\mathbf{F} = \langle 20, 0, 0 \rangle$ is applied at P. Find the torque about O that is produced.

46. Let $\mathbf{r} = \overrightarrow{OP} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$. A force $\mathbf{F} = \langle 10, 10, 0 \rangle$ is applied at P. Find the torque about O that is produced.

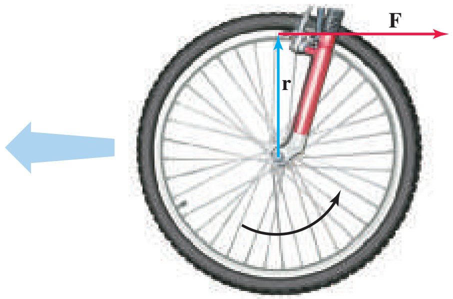
47. Let $\mathbf{r} = \overrightarrow{OP} = 10\mathbf{i}$. Which is greater (in magnitude): the torque about O when a force $\mathbf{F} = 5\mathbf{i} - 5\mathbf{k}$ is applied at P or the torque about O when a force $\mathbf{F} = 4\mathbf{i} - 3\mathbf{j}$ is applied at P?

48. A pump handle has a pivot at $(0, 0, 0)$ and extends to $P(5, 0, -5)$. A force $\mathbf{F} = \langle 1, 0, -10 \rangle$ is applied at P. Find the magnitude and direction of the torque about the pivot.

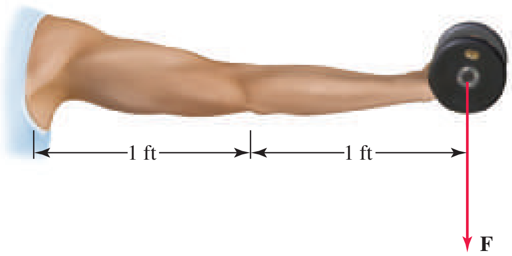
49. Tightening a bolt Suppose you apply a force of 20 N to a 0.25-meter-long wrench attached to a bolt in a direction perpendicular to the bolt. Determine the magnitude of the torque when the force is applied at an angle of 45° to the wrench.

50. Opening a laptop A force of 1.5 lb is applied in a direction perpendicular to the screen of a laptop at a distance of 10 in from the hinge of the screen. Find the magnitude of the torque (in ft-lb) that you apply.

51. Bicycle brakes A set of caliper brakes exerts a force on the rim of a bicycle wheel that creates a frictional force $\mathbf{F}$ of 40 N perpendicular to the radius of the wheel (see figure). Assuming the wheel has a radius of 33 cm, find the magnitude and direction of the torque about the axle of the wheel.



52. Arm torque A horizontally outstretched arm supports a weight of 20 lb in a hand (see figure). If the distance from the shoulder to the elbow is 1 ft and the distance from the elbow to the hand is 1 ft, find the magnitude and describe the direction of the torque about (a) the shoulder and (b) the elbow. (The units of torque in this case are ft-lb.)



53–56. Force on a moving charge *Answer the following questions about force on a moving charge.*

53. A particle with a unit positive charge $(q = 1)$ enters a constant magnetic field $\mathbf{B} = \mathbf{i} + \mathbf{j}$ with a velocity $\mathbf{v} = 20\mathbf{k}$. Find the magnitude and direction of the force on the particle. Make a sketch of the magnetic field, the velocity, and the force.

54. A particle with a unit negative charge $(q = -1)$ enters a constant magnetic field $\mathbf{B} = 5\mathbf{k}$ with a velocity $\mathbf{v} = \mathbf{i} + 2\mathbf{j}$. Find the magnitude and direction of the force on the particle. Make a sketch of the magnetic field, the velocity, and the force.

55. An electron $(q = -1.6 \times 10^{-19}\,\mathrm{C})$ enters a constant 2-T magnetic field at an angle of 45° to the field with a speed of $2 \times 10^5\,\mathrm{m/s}$. Find the magnitude of the force on the electron.

56. A proton $(q = 1.6 \times 10^{-19}\,\mathrm{C})$ with velocity $2 \times 10^6\,\mathbf{j}\,\mathrm{m/s}$ experiences a force in newtons of $\mathbf{F} = 5 \times 10^{-12}\,\mathbf{k}$ as it passes through the origin. Find the magnitude and direction of the magnetic field at that instant.

57. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. The cross product of two nonzero vectors is a nonzero vector.
b. $|\mathbf{u} \times \mathbf{v}|$ is less than both $|\mathbf{u}|$ and $|\mathbf{v}|$.
c. If $\mathbf{u}$ points east and $\mathbf{v}$ points south, then $\mathbf{u} \times \mathbf{v}$ points west.
d. If $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ and $\mathbf{u} \cdot \mathbf{v} = 0$, then either $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$.
e. Law of Cancellation? If $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w}$, then $\mathbf{v} = \mathbf{w}$.

58. Finding an unknown Find the value of a such that $\langle a, a, 2 \rangle \times \langle 1, a, 3 \rangle = \langle 2, -4, 2 \rangle$.

59. Vector equation Find all vectors **u** that satisfy the equation

$$\langle 1, 1, 1 \rangle \times \mathbf{u} = \langle -1, -1, 2 \rangle.$$

60. Vector equation Find all vectors **u** that satisfy the equation

$$\langle 1, 1, 1 \rangle \times \mathbf{u} = \langle 0, 0, 1 \rangle.$$

61. Area of a triangle Find the area of the triangle with vertices on the coordinate axes at the points $(a, 0, 0)$, $(0, b, 0)$, and $(0, 0, c)$, in terms of a, b, and c.

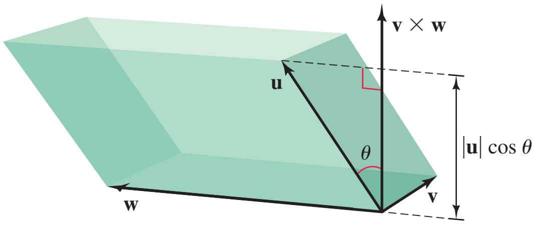
Explorations and Challenges

62–66. Scalar triple product *Another operation with vectors is the* ***scalar triple product****, defined to be* $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$, *for nonzero vectors* **u**, **v**, *and* **w** *in* $\mathbb{R}^3$.

62. Express **u**, **v**, and **w** in terms of their components, and show that $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ equals the determinant

$$\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

63. Consider the *parallelepiped* (slanted box) determined by the position vectors **u**, **v**, and **w** (see figure). Show that the volume of the parallelepiped is $|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$, the absolute value of the scalar triple product.

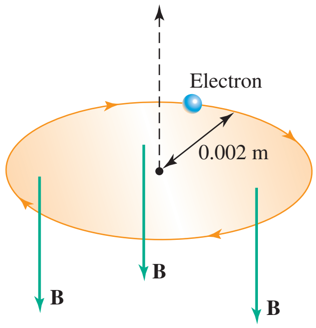


64. Find the volume of the parallelepiped determined by the position vectors $\mathbf{u} = \langle 3, 1, 0 \rangle$, $\mathbf{v} = \langle 2, 4, 1 \rangle$, and $\mathbf{w} = \langle 1, 1, 5 \rangle$ (see Exercise 63).

65. Explain why the position vectors **u**, **v**, and **w** are coplanar if and only if $|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = 0$. (*Hint:* See Exercise 63).

66. Prove that $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$.

67. Electron speed An electron with a mass of 9.1×10^{-31} kg and a charge of -1.6×10^{-19} C travels in a circular path with no loss of energy in a magnetic field of 0.05 T that is orthogonal to the path of the electron (see figure). If the radius of the path is 0.002 m, what is the speed of the electron?



68. Three proofs Prove that $\mathbf{u} \times \mathbf{u} = \mathbf{0}$ in three ways.

a. Use the definition of the cross product.
b. Use the determinant formulation of the cross product.
c. Use the property that $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$.

69. Associative property Prove in two ways that for scalars a and b, $(a\mathbf{u}) \times (b\mathbf{v}) = ab(\mathbf{u} \times \mathbf{v})$. Use the definition of the cross product and the determinant formula.

70–72. Possible identities *Determine whether the following statements are true using a proof or counterexample. Assume* **u**, **v**, *and* **w** *are nonzero vectors in* $\mathbb{R}^3$.

70. $\mathbf{u} \times (\mathbf{u} \times \mathbf{v}) = \mathbf{0}$

71. $(\mathbf{u} - \mathbf{v}) \times (\mathbf{u} + \mathbf{v}) = 2\mathbf{u} \times \mathbf{v}$

72. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})$

73–74. Identities *Prove the following identities. Assume* **u**, **v**, **w**, *and* **x** *are nonzero vectors in* $\mathbb{R}^3$.

73. $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$ Vector triple product

74. $(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{w} \times \mathbf{x}) = (\mathbf{u} \cdot \mathbf{w})(\mathbf{v} \cdot \mathbf{x}) - (\mathbf{u} \cdot \mathbf{x})(\mathbf{v} \cdot \mathbf{w})$

75. Cross product equations Suppose **u** and **v** are nonzero vectors in $\mathbb{R}^3$.

a. Prove that the equation $\mathbf{u} \times \mathbf{z} = \mathbf{v}$ has a nonzero solution **z** if and only if $\mathbf{u} \cdot \mathbf{v} = 0$. (*Hint:* Take the dot product of both sides with **v**.)
b. Explain this result geometrically.

QUICK CHECK ANSWERS

1. $\mathbf{u} \times \mathbf{v}$ points in the positive z-direction; $\mathbf{v} \times \mathbf{u}$ points in the negative z-direction. **2.** The vector $2\mathbf{u}$ points in the same direction as **u**, and the vector $3\mathbf{v}$ points in the same direction as **v**. So the right-hand rule gives the same direction for $2\mathbf{u} \times 3\mathbf{v}$ as it does for $\mathbf{u} \times \mathbf{v}$. **3.** $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = \langle -1, 0, 6 \rangle \cdot \langle 30, 9, 5 \rangle = -30 + 0 + 30 = 0$. A similar calculation shows that $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$. ◄

13.5 Lines and Planes in Space

In Chapter 1, we reviewed the catalog of standard functions and their associated graphs. For example, the graph of a linear equation in two variables $(y = mx + b)$ is a line, the graph of a quadratic equation $(y = ax^2 + bx + c)$ is a parabola, and both of these graphs lie in the xy-plane (two-dimensional space). Our immediate aim is to begin a similar journey through three-dimensional space. What are the basic geometrical objects in three dimensions, and how do we describe them with equations?

Certainly among the most fundamental objects in three dimensions are the line and plane. In this section, we develop equations for both lines and planes and explore their properties and uses.

Lines in Space

Two distinct points in $\mathbb{R}^3$ determine a unique line. Alternatively, one point and a direction also determine a unique line. We use both of these properties to derive two different descriptions of lines in space: one using parametric equations, and one using vector equations.

Let ℓ be the line passing through the point $P_0(x_0, y_0, z_0)$ parallel to the nonzero vector $\mathbf{v} = \langle a, b, c \rangle$, where P_0 and $\mathbf{v}$ are given. The fixed point P_0 is associated with the position vector $\mathbf{r}_0 = \overrightarrow{OP_0} = \langle x_0, y_0, z_0 \rangle$. We let $P(x, y, z)$ be a variable point on ℓ and let $\mathbf{r} = \overrightarrow{OP} = \langle x, y, z \rangle$ be the position vector associated with P (**Figure 13.66**). Because ℓ is parallel to $\mathbf{v}$, the vector $\overrightarrow{P_0P}$ is also parallel to $\mathbf{v}$; therefore, $\overrightarrow{P_0P} = t\mathbf{v}$, where t is a real number. By vector addition, we see that $\overrightarrow{OP} = \overrightarrow{OP_0} + \overrightarrow{P_0P}$, or $\overrightarrow{OP} = \overrightarrow{OP_0} + t\mathbf{v}$. Expressing these vectors in terms of their components, we obtain a *vector equation* for a line:

Component form $$\underbrace{\langle x, y, z \rangle}_{\mathbf{r} = \overrightarrow{OP}} = \underbrace{\langle x_0, y_0, z_0 \rangle}_{\mathbf{r}_0 = \overrightarrow{OP_0}} + t\underbrace{\langle a, b, c \rangle}_{\mathbf{v}} \quad \text{or}$$

Vector form $$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}.$$

Equating components, we obtain *parametric equations* for a line:

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct, \quad \text{for } -\infty < t < \infty.$$

QUICK CHECK 1 Describe the line $\mathbf{r} = t\mathbf{k}$, for $-\infty < t < \infty$. Describe the line $\mathbf{r} = t(\mathbf{i} + \mathbf{j} + 0\mathbf{k})$, for $-\infty < t < \infty$. ◄

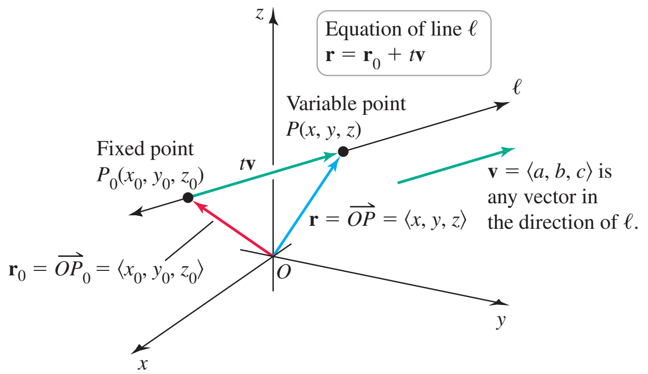


Figure 13.66

The parameter t determines the location of points on the line, where $t = 0$ corresponds to P_0. If t increases from 0, we move along the line in the direction of $\mathbf{v}$, and if t decreases from 0, we move along the line in the direction of $-\mathbf{v}$. As t varies over all real numbers $(-\infty < t < \infty)$, the entire line ℓ is generated. If, instead of knowing the direction $\mathbf{v}$ of the line, we are given two points $P_0(x_0, y_0, z_0)$ and $P_1(x_1, y_1, z_1)$, then the direction of the line is $\mathbf{v} = \overrightarrow{P_0P_1} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$.

➤ There are infinitely many equations for the same line. The direction vector is determined only up to a scalar multiple.

Equation of a Line

A **vector equation of the line** passing through the point $P_0(x_0, y_0, z_0)$ in the direction of the vector $\mathbf{v} = \langle a, b, c \rangle$ is $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$, or

$$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle, \quad \text{for} \quad -\infty < t < \infty.$$

Equivalently, the corresponding **parametric equations of the line** are

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct, \quad \text{for} \quad -\infty < t < \infty.$$

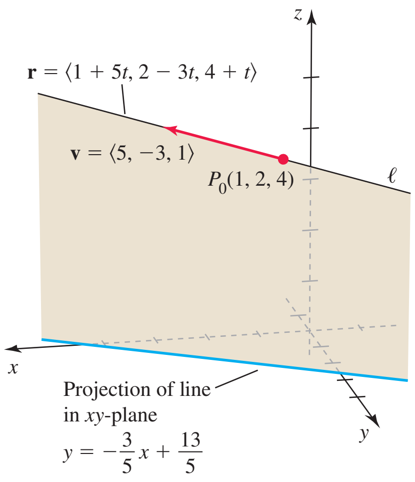


Figure 13.67

EXAMPLE 1 **Equations of lines** Find a vector equation of the line ℓ that passes through the point $P_0(1, 2, 4)$ in the direction of $\mathbf{v} = \langle 5, -3, 1 \rangle$, and then find the corresponding parametric equations of ℓ.

SOLUTION We are given $\mathbf{r}_0 = \langle 1, 2, 4 \rangle$. Therefore, an equation of the line is

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v} = \langle 1, 2, 4 \rangle + t\langle 5, -3, 1 \rangle = \langle 1 + 5t, 2 - 3t, 4 + t \rangle,$$

for $-\infty < t < \infty$ (Figure 13.67). The corresponding parametric equations are

$$x = 1 + 5t, \quad y = 2 - 3t, \quad z = 4 + t, \quad \text{for} \quad -\infty < t < \infty.$$

The line is easier to visualize if it is plotted together with its projection in the xy-plane. Setting $z = 0$ (the equation of the xy-plane), parametric equations of the projection line are $x = 1 + 5t$, $y = 2 - 3t$, and $z = 0$. Eliminating t from these equations, an equation of the projection line is $y = -\frac{3}{5}x + \frac{13}{5}$ (Figure 13.67).

Related Exercises 11–12 ◄

EXAMPLE 2 **Equation of a line and a line segment** Let ℓ be the line that passes through the points $P_0(-3, 5, 8)$ and $P_1(4, 2, -1)$.

a. Find an equation of ℓ.

b. Find parametric equations of the line segment that extends from P_0 to P_1.

SOLUTION

a. The direction of the line is

$$\mathbf{v} = \overrightarrow{P_0P_1} = \langle 4 - (-3), 2 - 5, -1 - 8 \rangle = \langle 7, -3, -9 \rangle.$$

Therefore, with $\mathbf{r}_0 = \langle -3, 5, 8 \rangle$, a vector equation of ℓ is

$$\begin{aligned} \mathbf{r} &= \mathbf{r}_0 + t\mathbf{v}, \quad \text{or} \\ \langle x, y, z \rangle &= \langle -3, 5, 8 \rangle + t\langle 7, -3, -9 \rangle \\ &= \langle -3 + 7t, 5 - 3t, 8 - 9t \rangle. \end{aligned}$$

b. Parametric equations for ℓ are

$$x = -3 + 7t, \quad y = 5 - 3t, \quad z = 8 - 9t, \quad \text{for } -\infty < t < \infty.$$

To generate only the line segment from P_0 to P_1, we simply restrict the values of the parameter t. Notice that $t = 0$ corresponds to $P_0(-3, 5, 8)$, and $t = 1$ corresponds to $P_1(4, 2, -1)$. Letting t vary from 0 to 1 generates the line segment from P_0 to P_1. Therefore, parametric equations of the line segment are

$$x = -3 + 7t, \quad y = 5 - 3t, \quad z = 8 - 9t, \quad \text{for } 0 \le t \le 1.$$

The graph of ℓ, which includes the line segment from P_0 to P_1, is shown in Figure 13.68, along with the projection of ℓ in the xz-plane. The parametric equations of the projection line are found by setting $y = 0$, which is the equation of the xz-plane.

Related Exercises 16, 29–30 ◄

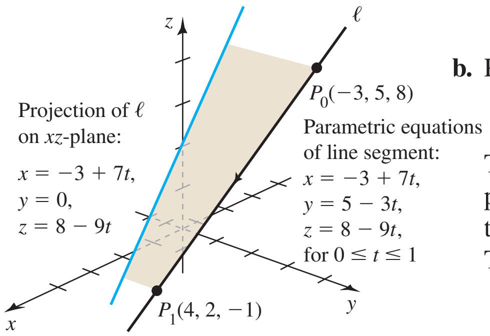


Figure 13.68

QUICK CHECK 2 In the equation of the line

$$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t\langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle,$$

what value of t corresponds to the point $P_0(x_0, y_0, z_0)$? What value of t corresponds to the point $P_1(x_1, y_1, z_1)$? ◄

Distance from a Point to a Line

Three-dimensional geometry has practical applications in such diverse fields as orbital mechanics, ballistics, computer graphics, and regression analysis. For example, determining the distance from a point to a line is an important calculation in problems such as programming video games. We first derive a formula for this distance and then illustrate how the formula is used to determine whether (virtual) billiard balls collide in a video game.

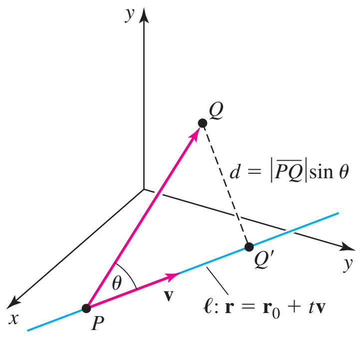


Figure 13.69

Consider a point Q and a line ℓ containing the point P, where ℓ is given by the vector equation $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$. Our goal is to find the distance d between Q and ℓ. The geometry of the problem is shown in Figure 13.69, where we have placed the tail of $\mathbf{v}$, which is a vector parallel to ℓ, at the point P. We drop a perpendicular from Q to the point Q' on ℓ to form the right triangle PQQ'; the shortest distance d from Q to ℓ is the distance from Q to Q'. From trigonometry, we know that $d = |\overrightarrow{PQ}| \sin\theta$, where θ is the angle between $\mathbf{v}$ and $\overrightarrow{PQ}$. Using the definition of the magnitude of the cross product, we can also write

$$|\mathbf{v} \times \overrightarrow{PQ}| = |\mathbf{v}| \underbrace{|\overrightarrow{PQ}| \sin\theta}_{d} = |\mathbf{v}|d. \quad \text{Cross product definition}$$

Dividing both sides of this equation by $|\mathbf{v}|$ leads to the desired result,

$$d = \frac{|\mathbf{v} \times \overrightarrow{PQ}|}{|\mathbf{v}|}.$$

QUICK CHECK 3 Find the distance between the point $Q(1, 0, 3)$ and the line $\langle x, y, z\rangle = t\langle 2, 1, 2\rangle$. Note that $P(0, 0, 0)$ lies on the line and $\mathbf{v} = \langle 2, 1, 2\rangle$ is parallel to the line. ◄

Distance Between a Point and a Line

The distance d between the point Q and the line $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$ is

$$d = \frac{|\mathbf{v} \times \overrightarrow{PQ}|}{|\mathbf{v}|},$$

where P is any point on the line and $\mathbf{v}$ is a vector parallel to the line.

The computer program for a billiards video game must keep track of the locations of all the balls on a two-dimensional screen. Although it is possible to write code that tracks each pixel in every ball, it is much simpler to track only the center pixel of each ball. As explained in the next example, the question of whether two balls collide during the game is answered by computing the distance between a point and a line.

EXAMPLE 3 **Video game calculation** Arria is playing a billiards video game on her iPad. The playing surface is represented in the video game by the rectangle in the first quadrant with opposite corners at the origin and the point $(100, 50)$ (Figure 13.70). Suppose the cue ball is located at $P(25, 16)$, and Arria shoots the ball with an angle of $30°$ above the x-axis, aiming for a target ball located at $Q(75, 46)$. If the cue ball is struck with sufficient force, will it collide with the target ball? Assume the diameter of each ball is 2.25.

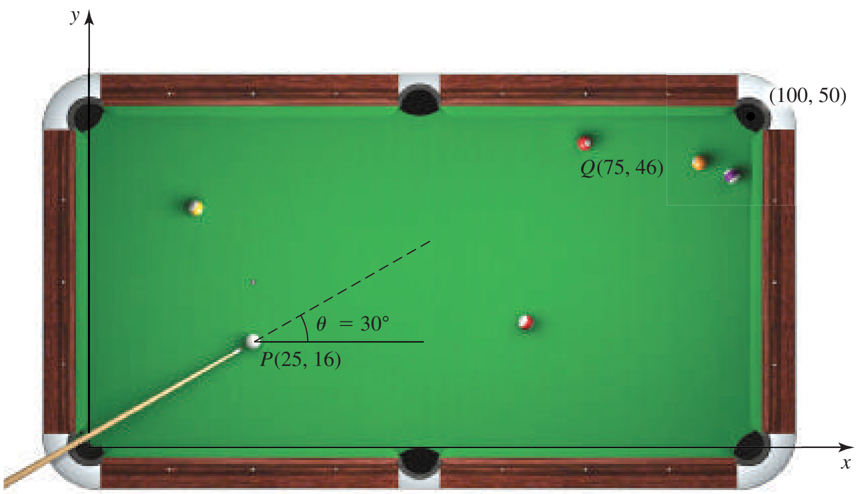


Figure 13.70

SOLUTION We assume the video game stores the locations of all the balls by representing each ball with the coordinates of its center, and that the path of a pool shot is represented by the equation of a line. To determine whether two balls collide, it helps to look first at the situation where the cue ball barely touches the target ball. As illustrated in Figure 13.71, the balls will *not* collide when the distance between their centers (which lie on a line perpendicular to the line of the shot) is greater than the diameter of the balls, or equivalently, when the distance d between Q and the line of the shot is greater than 2.25. Stating this result in another way leads to a useful test for the programmers of the game: If $d < 2.25$, the balls *will* collide.

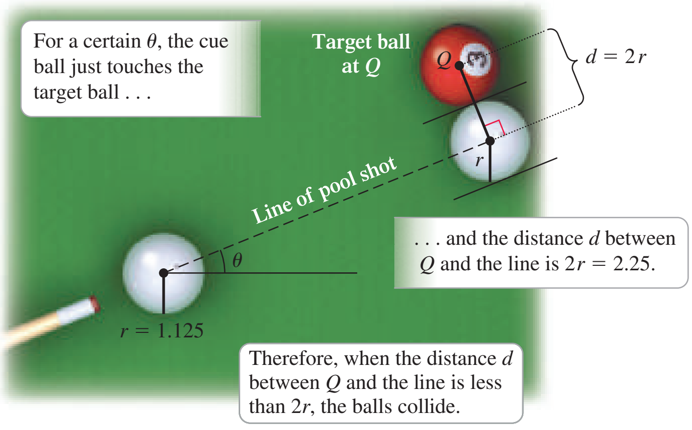


Figure 13.71

To find d for Arria's attempt, we need a vector parallel to the line of the shot, and we need the vector from the cue ball at $P(25, 16)$ to the target ball at $Q(75, 46)$. Because the cue ball is aimed at an angle 30° above the x-axis, a vector parallel to the line of the shot is $\mathbf{v} = \langle \cos 30°, \sin 30° \rangle = \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle$. Notice that $\overrightarrow{PQ} = \langle 75, 46 \rangle - \langle 25, 16 \rangle = \langle 50, 30 \rangle$. Therefore, the distance between $Q(75, 46)$ and the line of the pool shot is

$$d = \frac{|\mathbf{v} \times \overrightarrow{PQ}|}{|\mathbf{v}|} = \frac{\left| \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2}, 0 \right\rangle \times \langle 50, 30, 0 \rangle \right|}{\left| \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2}, 0 \right\rangle \right|}.$$

➤ Example 3 is a two-dimensional problem while the cross product is defined for three-dimensional vectors. Therefore, we must embed the 2D vectors of the example into three dimensions by adding a z-component of 0.

The cross product in the numerator is

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sqrt{3}/2 & 1/2 & 0 \\ 50 & 30 & 0 \end{vmatrix} = \left(\frac{\sqrt{3}}{2} 30 - \frac{1}{2} 50 \right) \mathbf{k} \approx 0.98\mathbf{k}.$$

Because $\mathbf{v}$ is a unit vector, its length is 1, and we find that $d = |0.98\mathbf{k}| = 0.98 < 2.25$; therefore, the balls collide. Additional calculations enable the programmers to determine the directions in which the two balls travel after the collision.

Related Exercise 41 ◀

Determining whether two balls collide in a video game may not seem very important. However, the same mathematical methods used to create realistic video games are employed in flight simulators that train military and civilian pilots. The principles used to design video games are also used to train surgeons and to assist them in performing surgery. Example 4 looks at another crucial calculation used in designing the virtual-world tools that are becoming part of our daily lives: finding points of intersection.

EXAMPLE 4 **Points of intersection** Determine whether the lines ℓ_1 and ℓ_2 intersect. If they do, find the point of intersection.

a. $\ell_1: x = 2 + 3t, y = 3t, z = 1 - t$ $\quad$ $\ell_2: x = 4 + 2s, y = -3 + 3s, z = -2s$

b. $\ell_1: x = 3 + t, y = 4 - t, z = 5 + 3t$ $\quad$ $\ell_2: x = 2s, y = -1 + 2s, z = 4s$

SOLUTION

a. Let's first check whether the lines are parallel; if they are, there is no point of intersection (unless the lines are identical). Reading the coefficients in the parametric equations for each line, we find that $\mathbf{v}_1 = \langle 3, 3, -1 \rangle$ is parallel to ℓ_1 and that $\mathbf{v}_2 = \langle 2, 3, -2 \rangle$ is parallel to ℓ_2. Because $\mathbf{v}_1$ is not a constant multiple of $\mathbf{v}_2$, the lines are not parallel. In $\mathbb{R}^3$, this fact alone does not guarantee that the lines intersect. Two lines that are neither parallel nor intersecting are said to be **skew**.

Determining whether two lines intersect amounts to solving a system of three linear equations in two variables. We set the x-, y-, and z-components of each line equal to one another, which results in the following system:

$$\begin{aligned} 2 + 3t &= 4 + 2s & (1) &\quad \text{Equate the } x\text{-components.} \\ 3t &= -3 + 3s & (2) &\quad \text{Equate the } y\text{-components.} \\ 1 - t &= -2s. & (3) &\quad \text{Equate the } z\text{-components.} \end{aligned}$$

➤ Given two skew lines in $\mathbb{R}^3$, one can always find two parallel planes in which the lines lie: one line in one plane, and the other line in a plane parallel to the first.

When equation (2) is subtracted from equation (1), the result is $2 = 7 - s$, or $s = 5$. Substituting $s = 5$ into equation (1) or (2) yields $t = 4$. However, when these values are substituted into (3), a false statement results, which implies that the system of equations is *inconsistent* and the lines do not intersect. We conclude that ℓ_1 and ℓ_2 are skew.

b. Proceeding as we did in part (a), we note that the lines are not parallel and solve the system

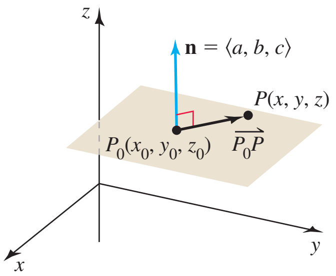
$$\begin{aligned} 3 + t &= 2s \\ 4 - t &= -1 + 2s \\ 5 + 3t &= 4s. \end{aligned}$$

When the first two equations are added to eliminate the variable t, we find that $s = 2$, which implies that $t = 1$. When these values are substituted into the last equation, a true statement results, which means we have a solution to the system and the lines intersect. To find the point of intersection, we substitute $t = 1$ into the parametric equations for ℓ_1 and arrive at $(4, 3, 8)$. You can verify that when $s = 2$ is substituted into the equations for ℓ_2, the same point of intersection results.

Related Exercises 32–33 ◄

Equations of Planes

Intuitively, a plane is a flat surface with infinite extent in all directions. Three noncollinear points (not all on the same line) determine a unique plane in $\mathbb{R}^3$. A plane in $\mathbb{R}^3$ is also uniquely determined by one point in the plane and any nonzero vector orthogonal (perpendicular) to the plane. Such a vector, called a *normal vector*, specifies the orientation of the plane.



The orientation of a plane is specified by a normal vector $\mathbf{n}$. All vectors $\overrightarrow{P_0P}$ in the plane are orthogonal to $\mathbf{n}$.

Figure 13.72

> **DEFINITION** **Plane in $\mathbb{R}^3$**
>
> Given a fixed point P_0 and a nonzero **normal vector n**, the set of points P in $\mathbb{R}^3$ for which $\overrightarrow{P_0P}$ is orthogonal to $\mathbf{n}$ is called a **plane** (Figure 13.72).

We now derive an equation of the plane passing through the point $P_0(x_0, y_0, z_0)$ with nonzero normal vector $\mathbf{n} = \langle a, b, c \rangle$. Notice that for any point $P(x, y, z)$ in the plane,

➤ Just as the slope determines the orientation of a line in $\mathbb{R}^2$, a normal vector determines the orientation of a plane in $\mathbb{R}^3$.

the vector $\overrightarrow{P_0P} = \langle x - x_0, y - y_0, z - z_0 \rangle$ lies in the plane and is orthogonal to $\mathbf{n}$. This orthogonality relationship is written and simplified as follows:

$$\begin{aligned}
\mathbf{n} \cdot \overrightarrow{P_0P} &= 0 && \text{Dot product of orthogonal vectors} \\
\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle &= 0 && \text{Substitute vector components.} \\
a(x - x_0) + b(y - y_0) + c(z - z_0) &= 0 && \text{Expand the dot product.} \\
ax + by + cz &= d. && d = ax_0 + by_0 + cz_0
\end{aligned}$$

This important result states that the most general linear equation in three variables, $ax + by + cz = d$, describes a plane in $\mathbb{R}^3$.

➤ A vector $\mathbf{n} = \langle a, b, c \rangle$ is used to describe a *plane* by specifying a direction *orthogonal* to the plane. By contrast, a vector $\mathbf{v} = \langle a, b, c \rangle$ is used to describe a *line* by specifying a direction *parallel* to the line.

General Equation of a Plane in $\mathbb{R}^3$

The plane passing through the point $P_0(x_0, y_0, z_0)$ with a nonzero normal vector $\mathbf{n} = \langle a, b, c \rangle$ is described by the equation

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \quad \text{or} \quad ax + by + cz = d,$$

where $d = ax_0 + by_0 + cz_0$.

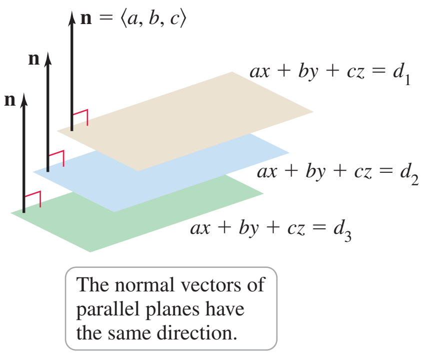


Figure 13.73

The coefficients a, b, and c in the equation of a plane determine the *orientation* of the plane, while the constant term d determines the *location* of the plane. If a, b, and c are held constant and d is varied, a family of parallel planes is generated, all with the same orientation (**Figure 13.73**).

QUICK CHECK 4 Consider the equation of a plane in the form $\mathbf{n} \cdot \overrightarrow{P_0P} = 0$. Explain why the equation of the plane depends only on the direction, not on the length, of the normal vector $\mathbf{n}$. ◄

EXAMPLE 5 Equation of a plane

a. Find an equation of the plane passing through $P_0(2, -3, 4)$ with a normal vector $\mathbf{n} = \langle -1, 2, 3 \rangle$.

b. Find an equation of the plane passing through $P_0(2, -3, 4)$ that is perpendicular to the line $x = 3 + 2t$, $y = -4t$, $z = 1 - 6t$.

SOLUTION

a. Substituting the components of $\mathbf{n}$ ($a = -1$, $b = 2$, and $c = 3$) and the coordinates of P_0 ($x_0 = 2$, $y_0 = -3$, and $z_0 = 4$) into the equation of a plane, we have

$$\begin{aligned}
a(x - x_0) + b(y - y_0) + c(z - z_0) &= 0 && \text{General equation of a plane} \\
(-1)(x - 2) + 2(y - (-3)) + 3(z - 4) &= 0 && \text{Substitute.} \\
-x + 2y + 3z &= 4. && \text{Simplify.}
\end{aligned}$$

The plane is shown in **Figure 13.74**.

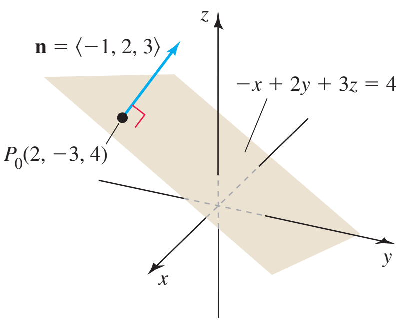


Figure 13.74

b. Note that $\mathbf{v} = \langle 2, -4, -6 \rangle$ is parallel to the given line and therefore perpendicular to the plane, so we have a vector normal to the plane. We could carry out calculations similar to those found in part (a) to find the equation of the plane, but here is an easier solution. Observe that $\mathbf{v}$ is a multiple of the normal vector in part (a) ($\mathbf{v} = -2\mathbf{n}$), and therefore $\mathbf{v}$ and $\mathbf{n}$ are parallel, which implies both planes are oriented in the same direction. Because both planes pass through P_0, we conclude that the planes are identical.

Related Exercises 43–44 ◄

EXAMPLE 6 A plane through three points

Find an equation of the plane that passes through the (noncollinear) points $P(2, -1, 3)$, $Q(1, 4, 0)$, and $R(0, -1, 5)$.

SOLUTION To write an equation of the plane, we must find a normal vector. Because P, Q, and R lie in the plane, the vectors $\overrightarrow{PQ} = \langle -1, 5, -3 \rangle$ and $\overrightarrow{PR} = \langle -2, 0, 2 \rangle$ also lie in the plane. The cross product $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to both $\overrightarrow{PQ}$ and $\overrightarrow{PR}$; therefore, a vector normal to the plane is

$$\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 5 & -3 \\ -2 & 0 & 2 \end{vmatrix} = 10\mathbf{i} + 8\mathbf{j} + 10\mathbf{k}.$$

➤ Three points P, Q, and R determine a plane provided they are not collinear. If P, Q, and R *are* collinear, then the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are parallel, which implies that $\overrightarrow{PQ} \times \overrightarrow{PR} = \mathbf{0}$.

QUICK CHECK 5 Verify that in Example 6, the same equation for the plane results if either Q or R is used as the fixed point in the plane. ◄

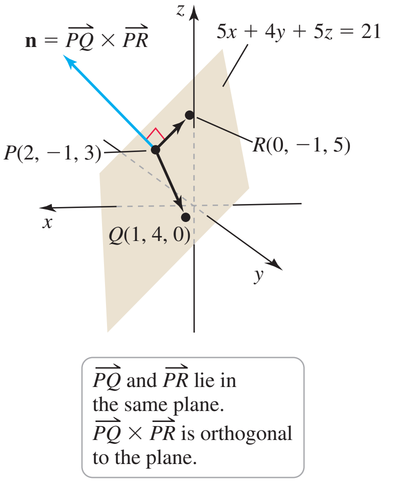


Figure 13.75

Any nonzero scalar multiple of $\mathbf{n}$ may be used as the normal vector. Choosing $\mathbf{n} = \langle 5, 4, 5 \rangle$ and $P_0(2, -1, 3)$ as the fixed point in the plane (**Figure 13.75**), an equation of the plane is

$$5(x - 2) + 4(y - (-1)) + 5(z - 3) = 0 \quad \text{or} \quad 5x + 4y + 5z = 21.$$

Using either Q or R as the fixed point in the plane leads to an equivalent equation of the plane.

Related Exercises 49–50 ◄

EXAMPLE 7 **Properties of a plane** Let Q be the plane described by the equation $2x - 3y - z = 6$.

a. Find a vector normal to Q.

b. Find the points at which Q intersects the coordinate axes and plot Q.

c. Describe the sets of points at which Q intersects the yz-plane, the xz-plane, and the xy-plane.

SOLUTION

a. The coefficients of x, y, and z in the equation of Q are the components of a vector normal to Q. Therefore, a normal vector is $\mathbf{n} = \langle 2, -3, -1 \rangle$ (or any nonzero multiple of $\mathbf{n}$).

b. The point (x, y, z) at which Q intersects the x-axis must have $y = z = 0$. Substituting $y = z = 0$ into the equation of Q gives $x = 3$, so Q intersects the x-axis at $(3, 0, 0)$. Similarly, Q intersects the y-axis at $(0, -2, 0)$, and Q intersects the z-axis at $(0, 0, -6)$. Connecting the three intercepts with straight lines allows us to visualize the plane (**Figure 13.76**).

c. All points in the yz-plane have $x = 0$. Setting $x = 0$ in the equation of Q gives the equation $-3y - z = 6$, which, with the condition $x = 0$, describes a line in the yz-plane. If we set $y = 0$, then Q intersects the xz-plane in the line $2x - z = 6$, where $y = 0$. If $z = 0$, then Q intersects the xy-plane in the line $2x - 3y = 6$, where $z = 0$ (Figure 13.76).

➤ There is a possibility for confusion here. Working in $\mathbb{R}^3$ with no other restrictions, the equation $-3y - z = 6$ describes a plane that is parallel to the x-axis (because x is unspecified). To make it clear that $-3y - z = 6$ is a line in the yz-plane, the condition $x = 0$ is included.

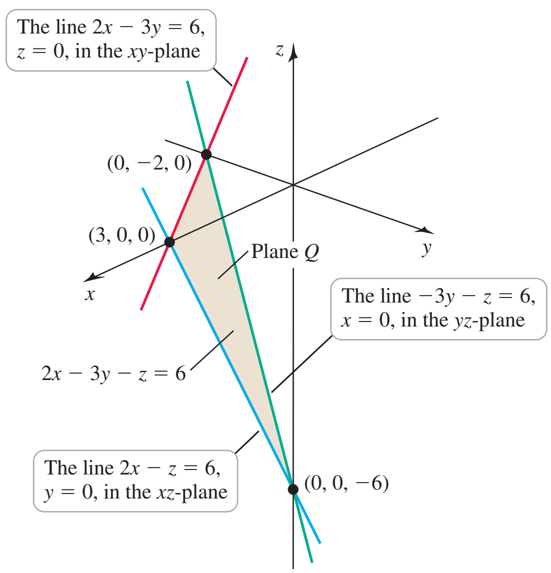


Figure 13.76

Related Exercise 61 ◄

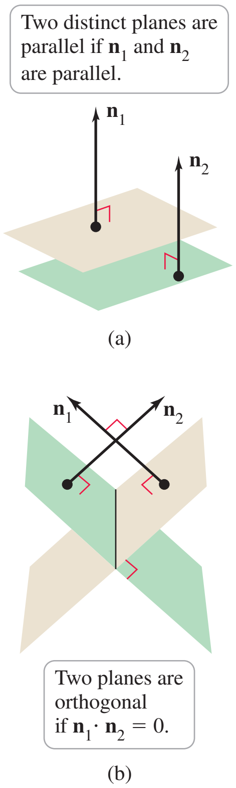


Figure 13.77

Parallel and Orthogonal Planes

The normal vectors of distinct planes tell us about the relative orientation of the planes. Two cases are of particular interest: Two distinct planes may be *parallel* (**Figure 13.77a**) and two intersecting planes may be *orthogonal* (**Figure 13.77b**).

> **DEFINITION Parallel and Orthogonal Planes**
>
> Two distinct planes are **parallel** if their respective normal vectors are parallel (that is, the normal vectors are scalar multiples of each other). Two planes are **orthogonal** if their respective normal vectors are orthogonal (that is, the dot product of the normal vectors is zero).

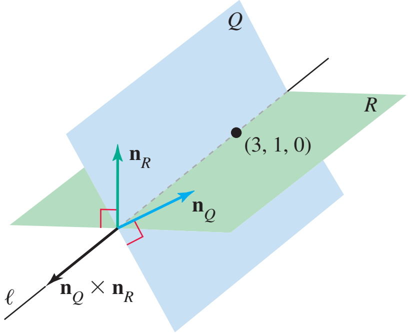
QUICK CHECK 6 Determine whether the planes $2x - 3y + 6z = 12$ and $6x + 8y + 2z = 1$ are parallel, orthogonal, or neither. ◀

EXAMPLE 8 Parallel planes Find an equation of the plane Q that passes through the point $(-2, 4, 1)$ and is parallel to the plane R: $3x - 2y + z = 4$.

SOLUTION The vector $\mathbf{n} = \langle 3, -2, 1 \rangle$ is normal to R. Because Q and R are parallel, $\mathbf{n}$ is also normal to Q. We conclude that an equation of Q (which passes through $(-2, 4, 1)$ and has a normal vector $\langle 3, -2, 1 \rangle$) is given by

$$3(x + 2) - 2(y - 4) + (z - 1) = 0 \quad \text{or} \quad 3x - 2y + z = -13.$$

Related Exercises 51–52 ◀



$\mathbf{n}_Q \times \mathbf{n}_R$ is a vector perpendicular to $\mathbf{n}_Q$ and $\mathbf{n}_R$.
Line ℓ is perpendicular to $\mathbf{n}_Q$ and $\mathbf{n}_R$.
Therefore, ℓ and $\mathbf{n}_Q \times \mathbf{n}_R$ are parallel to each other.

Figure 13.78

EXAMPLE 9 Intersecting planes Find an equation of the line of intersection of the planes Q: $x + 2y + z = 5$ and R: $2x + y - z = 7$.

SOLUTION First note that the vectors normal to the planes, $\mathbf{n}_Q = \langle 1, 2, 1 \rangle$ and $\mathbf{n}_R = \langle 2, 1, -1 \rangle$, are *not* multiples of each other. Therefore, the planes are not parallel and they must intersect in a line; call it ℓ. To find an equation of ℓ, we need two pieces of information: a point on ℓ and a vector pointing in the direction of ℓ. Here is one of several ways to find a point on ℓ. Setting $z = 0$ in the equations of the planes gives equations of the lines in which the planes intersect the xy-plane:

$$\begin{aligned} x + 2y &= 5 \\ 2x + y &= 7. \end{aligned}$$

➤ By setting $z = 0$ and solving the two resulting equations, we find the point that lies on both planes *and* lies in the xy-plane $(z = 0)$.

Solving these equations simultaneously, we find that $x = 3$ and $y = 1$. Combining this result with $z = 0$, we see that $(3, 1, 0)$ is a point on ℓ (Figure 13.78).

We next find a vector parallel to ℓ. Because ℓ lies in Q and R, it is orthogonal to the normal vectors $\mathbf{n}_Q$ and $\mathbf{n}_R$. Therefore, the cross product of $\mathbf{n}_Q$ and $\mathbf{n}_R$ is a vector parallel to ℓ (Figure 13.78). In this case, the cross product is

$$\mathbf{n}_Q \times \mathbf{n}_R = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 2 & 1 & -1 \end{vmatrix} = -3\mathbf{i} + 3\mathbf{j} - 3\mathbf{k} = \langle -3, 3, -3 \rangle.$$

An equation of the line ℓ in the direction of the vector $\langle -3, 3, -3 \rangle$ passing through the point $(3, 1, 0)$ is

$$\begin{aligned} \mathbf{r} &= \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle && \text{Equation of a line} \\ &= \langle 3, 1, 0 \rangle + t\langle -3, 3, -3 \rangle && \text{Substitute.} \\ &= \langle 3 - 3t, 1 + 3t, -3t \rangle, && \text{Simplify.} \end{aligned}$$

➤ Any nonzero scalar multiple of $\langle -3, 3, -3 \rangle$ can be used for the direction of ℓ. For example, another equation of ℓ is $\mathbf{r} = \langle 3 + t, 1 - t, t \rangle$.

where $-\infty < t < \infty$. You can check that any point (x, y, z) with $x = 3 - 3t$, $y = 1 + 3t$, and $z = -3t$ satisfies the equations of both planes.

Related Exercise 74 ◀

SECTION 13.5 EXERCISES

Getting Started

1. Find a position vector that is parallel to the line $x = 2 + 4t, y = 5 - 8t, z = 9t$.

2. Find the parametric equations of the line $\mathbf{r} = \langle 1, 2, 3 \rangle + t\langle 4, 0, -6 \rangle$.

3. Explain how to find a vector in the direction of the line segment from $P_0(x_0, y_0, z_0)$ to $P_1(x_1, y_1, z_1)$.

4. Find the vector equation of the line through the points $P_0(x_0, y_0, z_0)$ and $P_1(x_1, y_1, z_1)$.

5. Determine whether the plane $x + y + z = 9$ and the line $x = t, y = t + 1, z = t + 2$ are parallel, perpendicular, or neither. Be careful.

6. Determine whether the plane $x + y + z = 9$ and the line $x = t, y = -2t + 1, z = t + 2$ are parallel, perpendicular, or neither.

7. Give two pieces of information that, taken together, uniquely determine a plane.

8. Find a vector normal to the plane $-2x - 3y = 12 - 4z$.

9. Where does the plane $-2x - 3y + 4z = 12$ intersect the coordinate axes?

10. Give an equation of the plane with a normal vector $\mathbf{n} = \langle 1, 1, 1 \rangle$ that passes through the point $(1, 0, 0)$.

Practice Exercises

11–26. Equations of lines *Find both the parametric and the vector equations of the following lines.*

11. The line through $(0, 0, 1)$ in the direction of the vector $\mathbf{v} = \langle 4, 7, 0 \rangle$

12. The line through $(-3, 2, -1)$ in the direction of the vector $\mathbf{v} = \langle 1, -2, 0 \rangle$

13. The line through $(0, 0, 1)$ parallel to the y-axis

14. The line through $(-2, 4, 3)$ parallel to the x-axis

15. The line through $(0, 0, 0)$ and $(1, 2, 3)$

16. The line through $(-3, 4, 6)$ and $(5, -1, 0)$

17. The line through $(0, 0, 0)$ that is parallel to the line $\mathbf{r} = \langle 3 - 2t, 5 + 8t, 7 - 4t \rangle$

18. The line through $(1, -3, 4)$ that is parallel to the line $x = 3 + 4t, y = 5 - t, z = 7$

19. The line through $(0, 0, 0)$ that is perpendicular to both $\mathbf{u} = \langle 1, 0, 2 \rangle$ and $\mathbf{v} = \langle 0, 1, 1 \rangle$

20. The line through $(-3, 4, 2)$ that is perpendicular to both $\mathbf{u} = \langle 1, 1, -5 \rangle$ and $\mathbf{v} = \langle 0, 4, 0 \rangle$

21. The line through $(-2, 5, 3)$ that is perpendicular to both $\mathbf{u} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and the x-axis

22. The line through $(0, 2, 1)$ that is perpendicular to both $\mathbf{u} = 4\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$ and the z-axis

23. The line through $(1, 2, 3)$ that is perpendicular to the lines $x = 3 - 2t, y = 5 + 8t, z = 7 - 4t$ and $x = -2t, y = 5 + t, z = 7 - t$

24. The line through $(1, 0, -1)$ that is perpendicular to the lines $x = 3 + 2t, y = 3t, z = -4t$ and $x = t, y = t, z = -t$

25. The line that is perpendicular to the lines $\mathbf{r} = \langle 4t, 1 + 2t, 3t \rangle$ and $\mathbf{R} = \langle -1 + s, -7 + 2s, -12 + 3s \rangle$, and passes through the point of intersection of the lines $\mathbf{r}$ and $\mathbf{R}$

26. The line that is perpendicular to the lines $\mathbf{r} = \langle -2 + 3t, 2t, 3t \rangle$ and $\mathbf{R} = \langle -6 + s, -8 + 2s, -12 + 3s \rangle$, and passes through the point of intersection of the lines $\mathbf{r}$ and $\mathbf{R}$

27–30. Line segments *Find parametric equations for the line segment joining the first point to the second point.*

27. $(0, 0, 0)$ and $(1, 2, 3)$ **28.** $(1, 0, 1)$ and $(0, -2, 1)$

29. $(2, 4, 8)$ and $(7, 5, 3)$ **30.** $(-1, -8, 4)$ and $(-9, 5, -3)$

31–37. Parallel, intersecting, or skew lines *Determine whether the following pairs of lines are parallel, intersect at a single point, or are skew. If the lines are parallel, determine whether they are the same line (and thus intersect at all points). If the lines intersect at a single point, determine the point of intersection.*

31. $\mathbf{r} = \langle 1, 3, 2 \rangle + t\langle 6, -7, 1 \rangle$; $\mathbf{R} = \langle 10, 6, 14 \rangle + s\langle 3, 1, 4 \rangle$

32. $x = 2t, y = t + 2, z = 3t - 1$ and $x = 5s - 2, y = s + 4, z = 5s + 1$

33. $x = 4, y = 6 - t, z = 1 + t$ and $x = -3 - 7s, y = 1 + 4s, z = 4 - s$

34. $x = 4 + 5t, y = -2t, z = 1 + 3t$ and $x = 10s, y = 6 + 4s, z = 4 + 6s$

35. $x = 1 + 2t, y = 7 - 3t, z = 6 + t$ and $x = -9 + 6t, y = 22 - 9t, z = 1 + 3t$

36. $\mathbf{r} = \langle 3, 1, 0 \rangle + t\langle 4, -6, 4 \rangle$; $\mathbf{R} = \langle 0, 5, 4 \rangle + s\langle -2, 3, -2 \rangle$

37. $\mathbf{r} = \langle 4 + t, -2t, 1 + 3t \rangle$; $\mathbf{R} = \langle 1 - 7s, 6 + 14s, 4 - 21s \rangle$

38. Intersecting lines and colliding particles Consider the lines

$$\mathbf{r} = \langle 2 + 2t, 8 + t, 10 + 3t \rangle \quad \text{and}$$
$$\mathbf{R} = \langle 6 + s, 10 - 2s, 16 - s \rangle.$$

a. Determine whether the lines intersect (have a common point), and if so, find the coordinates of that point.

b. If $\mathbf{r}$ and $\mathbf{R}$ describe the paths of two particles, do the particles collide? Assume $t \geq 0$ and $s \geq 0$ measure time in seconds, and that motion starts at $s = t = 0$.

39–40. Distance from a point to a line *Find the distance between the given point Q and the given line.*

39. $Q(-5, 2, 9)$; $x = 5t + 7, y = 2 - t, z = 12t + 4$

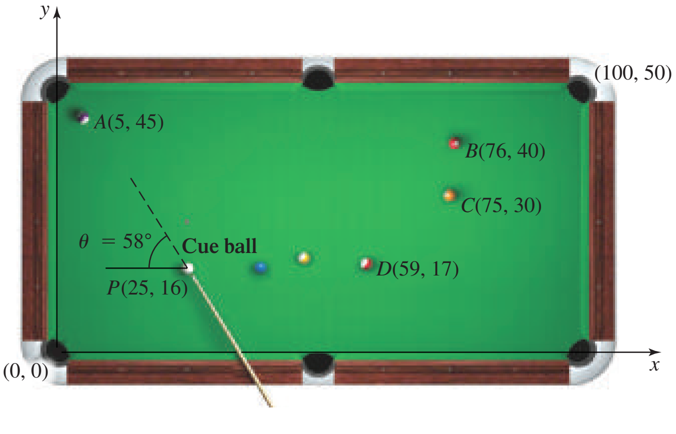
40. $Q(5, 6, 1)$; $x = 1 + 3t, y = 3 - 4t, z = t + 1$

T 41. Billiards shot A cue ball in a billiards video game lies at $P(25, 16)$ (see figure). Refer to Example 3, where we assume the diameter of each ball is 2.25 screen units, and pool balls are represented by the point at their center.

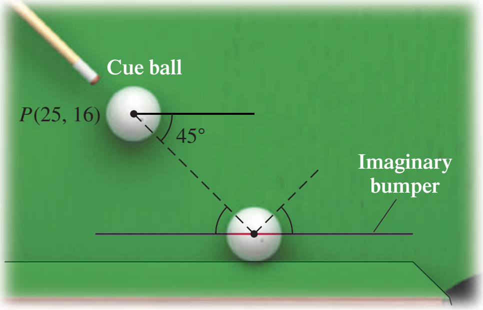
a. The cue ball is aimed at an angle of 58° above the negative x-axis toward a target ball at $A(5, 45)$. Do the balls collide?

b. The cue ball is aimed at the point $(50, 25)$ in an attempt to hit a target ball at $B(76, 40)$. Do the balls collide?

c. The cue ball is aimed at an angle θ above the x-axis in the general direction of a target ball at $C(75, 30)$. What range of angles (for $0 \leq \theta \leq \pi/2$) will result in a collision? Express your answer in degrees.



T **42. Bank shot** Refer to the figure in Exercise 41. The cue ball lies at $P(25, 16)$ and Jerrod hopes to hit ball $D(59, 17)$; a direct shot isn't an option with other balls blocking the path. Instead, he attempts a bank shot, aiming the cue ball at an angle 45° below the x-axis. Will the balls collide? Assume the angles at which the cue ball meets and leaves the bumper are equal and that the diameter of each ball is 2 screen units. (*Hint:* The cue ball will bounce off the bumper when its center hits an "imaginary bumper" one unit above the bumper; see following figure.)



43–58. Equations of planes *Find an equation of the following planes.*

43. The plane passing through the point $P_0(0, 2, -2)$ with a normal vector $\mathbf{n} = \langle 1, 1, -1 \rangle$

44. The plane passing through the point $P_0(2, 3, 0)$ with a normal vector $\mathbf{n} = \langle -1, 2, -3 \rangle$

45. The plane that is parallel to the vectors $\langle 1, 0, 1 \rangle$ and $\langle 0, 2, 1 \rangle$, passing through the point $(1, 2, 3)$

46. The plane that is parallel to the vectors $\langle 1, -3, 1 \rangle$ and $\langle 4, 2, 0 \rangle$, passing through the point $(3, 0, -2)$

47. The plane passing through the origin that is perpendicular to the line $x = t, y = 1 + 4t, z = 7t$

48. The plane passing through the point $(2, -3, 5)$ that is perpendicular to the line $x = 2t, y = 1 + 3t, z = 5 + 4t$

49. The plane passing through the points $(1, 0, 3)$, $(0, 4, 2)$, and $(1, 1, 1)$

50. The plane passing through the points $(2, -1, 4)$, $(1, 1, -1)$, and $(-4, 1, 1)$

51. The plane passing through the point $P_0(1, 0, 4)$ that is parallel to the plane $-x + 2y - 4z = 1$

52. The plane passing through the point $P_0(0, 2, -2)$ that is parallel to the plane $2x + y - z = 1$

53. The plane containing the x-axis and the point $P_0(1, 2, 3)$

54. The plane containing the z-axis and the point $P_0(3, -1, 2)$

55. The plane passing through the origin and containing the line $x = t - 1, y = 2t, z = 3t + 4$

56. The plane passing though the point $P_0(1, -2, 3)$ and containing the line $\mathbf{r} = \langle t, -t, 2t \rangle$

57. The plane passing though the point $P_0(-4, 1, 2)$ and containing the line $\mathbf{r} = \langle 2t - 2, -2t, -4t + 1 \rangle$

58. The plane passing through the origin that contains the line of intersection of the planes $x + y + 2z = 0$ and $x - y = 4$

59. Parallel planes Is the line $x = t + 1, y = 2t + 3, z = 4t + 5$ parallel to the plane $2x - y = -2$? If so, explain why, and then find an equation of the plane containing the line that is parallel to the plane $2x - y = -2$.

60. Do the lines $x = t, y = 2t + 1, z = 3t + 4$ and $x = 2s - 2, y = 2s - 1, z = 3s + 1$ intersect each other at only one point? If so, find a plane that contains both lines.

61–64. Properties of planes *Find the points at which the following planes intersect the coordinate axes, and find equations of the lines where the planes intersect the coordinate planes. Sketch a graph of the plane.*

61. $3x - 2y + z = 6$

62. $-4x + 8z = 16$

63. $x + 3y - 5z - 30 = 0$

64. $12x - 9y + 4z + 72 = 0$

65–68. Pairs of planes *Determine whether the following pairs of planes are parallel, orthogonal, or neither.*

65. $x + y + 4z = 10$ and $-x - 3y + z = 10$

66. $2x + 2y - 3z = 10$ and $-10x - 10y + 15z = 10$

67. $3x + 2y - 3z = 10$ and $-6x - 10y + z = 10$

68. $3x + 2y + 2z = 10$ and $-6x - 10y + 19z = 10$

69–70. Equations of planes *For the following sets of planes, determine which pairs of planes in the set are parallel, which pairs are orthogonal, and which pairs are identical.*

69. Q: $3x - 2y + z = 12$; R: $-x + 2y/3 - z/3 = 0$; S: $-x + 2y + 7z = 1$; T: $3x/2 - y + z/2 = 6$

70. Q: $x + y - z = 0$; R: $y + z = 0$; S: $x - y = 0$; T: $x + y + z = 0$

71–72. Lines normal to planes *Find an equation of the following lines.*

71. The line passing through the point $P_0(2, 1, 3)$ that is normal to the plane $2x - 4y + z = 10$

72. The line passing through the point $P_0(0, -10, -3)$ that is normal to the plane $x + 4z = 2$

73–76. Intersecting planes *Find an equation of the line of intersection of the planes Q and R.*

73. Q: $-x + 2y + z = 1$; R: $x + y + z = 0$

74. Q: $x + 2y - z = 1$; R: $x + y + z = 1$

75. Q: $2x - y + 3z - 1 = 0$; R: $-x + 3y + z - 4 = 0$

76. Q: $x - y - 2z = 1$; R: $x + y + z = -1$

77–80. Line-plane intersections *Find the point (if it exists) at which the following planes and lines intersect.*

77. $x = 3$ and $\mathbf{r} = \langle t, t, t \rangle$

78. $y = -2$ and $\mathbf{r} = \langle 2t + 1, -t + 4, t - 6 \rangle$

79. $3x + 2y - 4z = -3$ and $x = -2t + 5, y = 3t - 5, z = 4t - 6$

80. $2x - 3y + 3z = 2$ and $x = 3t, y = t, z = -t$

81. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. The line $\mathbf{r} = \langle 3, -1, 4 \rangle + t\langle 6, -2, 8 \rangle$ passes through the origin.

b. Any two nonparallel lines in $\mathbb{R}^3$ intersect.

c. The plane $x + y + z = 0$ and the line $x = t, y = t, z = t$ are parallel.

d. The vector equations $\mathbf{r} = \langle 1, 2, 3 \rangle + t\langle 1, 1, 1 \rangle$ and $\mathbf{R} = \langle 1, 2, 3 \rangle + t\langle -2, -2, -2 \rangle$ describe the same line.
e. The equations $x + y - z = 1$ and $-x - y + z = 1$ describe the same plane.
f. Any two distinct lines in $\mathbb{R}^3$ determine a unique plane.
g. The vector $\langle -1, -5, 7 \rangle$ is perpendicular to both the line $x = 1 + 5t, y = 3 - t, z = 1$ and the line $x = 7t, y = 3, z = 3 + t$.

82. Distance from a point to a plane Suppose P is a point in the plane $ax + by + cz = d$. The distance from any point Q to the plane equals the length of the orthogonal projection of $\overrightarrow{PQ}$ onto a vector $\mathbf{n} = \langle a, b, c \rangle$ normal to the plane. Use this information to show that the distance from Q to the plane is $|\overrightarrow{PQ} \cdot \mathbf{n}|/|\mathbf{n}|$.

83. Find the distance from the point $Q(6, -2, 4)$ to the plane $2x - y + 2z = 4$.

84. Find the distance from the point $Q(1, 2, -4)$ to the plane $2x - 5z = 5$.

Explorations and Challenges

85–86. Symmetric equations for a line *If we solve for t in the parametric equations of the line $x = x_0 + at, y = y_0 + bt, z = z_0 + ct$, we obtain the symmetric equations*

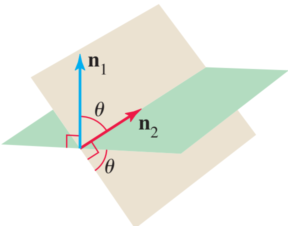
$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c},$$

provided a, b, and c do not equal 0.

85. Find symmetric equations of the line $\mathbf{r} = \langle 1, 2, 0 \rangle + t\langle 4, 7, 2 \rangle$.

86. Find parametric and symmetric equations of the line passing through the points $P(1, -2, 3)$ and $Q(2, 3, -1)$.

T **87. Angle between planes** The angle between two planes is the smallest angle θ between the normal vectors of the planes, where the directions of the normal vectors are chosen so that $0 \le \theta \le \pi/2$. Find the angle between the planes $5x + 2y - z = 0$ and $-3x + y + 2z = 0$.



88. Intercepts Let a, b, c, and d be constants. Find the points at which the plane $ax + by + cz = d$ intersects the x-, y-, and z-axes.

89. A family of orthogonal planes Find an equation for a family of planes that are orthogonal to the planes $2x + 3y = 4$ and $-x - y + 2z = 8$.

90. Orthogonal plane Find an equation of the plane passing through $(0, -2, 4)$ that is orthogonal to the planes $2x + 5y - 3z = 0$ and $-x + 5y + 2z = 8$.

91. Three intersecting planes Describe the set of all points (if any) at which all three planes $x + 3z = 3$, $y + 4z = 6$, and $x + y + 6z = 9$ intersect.

92. Three intersecting planes Describe the set of all points (if any) at which all three planes $x + 2y + 2z = 3$, $y + 4z = 6$, and $x + 2y + 8z = 9$ intersect.

93. T-shirt profits A clothing company makes a profit of \$10 on its long-sleeved T-shirts and a profit of \$5 on its short-sleeved T-shirts. Assuming there is a \$200 setup cost, the profit on T-shirt sales is $z = 10x + 5y - 200$, where x is the number of long-sleeved T-shirts sold and y is the number of short-sleeved T-shirts sold. Assume x and y are nonnegative.

a. Graph the plane that gives the profit using the window $[0, 40] \times [0, 40] \times [-400, 400]$.
b. If $x = 20$ and $y = 10$, is the profit positive or negative?
c. Describe the values of x and y for which the company breaks even (for which the profit is zero). Mark this set on your graph.

QUICK CHECK ANSWERS

1. The z-axis; the line $y = x$ in the xy-plane **2.** When $t = 0$, the point on the line is P_0; when $t = 1$, the point on the line is P_1. **3.** $d = \sqrt{26}/3$ **4.** Because the right side of the equation is 0, the equation can be multiplied by any nonzero constant (changing the length of $\mathbf{n}$) without changing the graph. **6.** The planes are orthogonal because $\langle 2, -3, 6 \rangle \cdot \langle 6, 8, 2 \rangle = 0$. ◄

13.6 Cylinders and Quadric Surfaces

In Section 13.5, we discovered that lines in three-dimensional space are described by parametric equations (or vector equations) that are linear in the variable. We also saw that planes are described with linear equations in three variables. In this section, we take this progression one step further and investigate the geometry of three-dimensional objects described by quadratic equations in three variables. The result is a collection of *quadric surfaces* that you will encounter frequently throughout the remainder of the text. You saw one such surface in Section 13.2: A sphere with radius a centered at the origin with an equation of $x^2 + y^2 + z^2 = a^2$ is an example of a quadric surface. We also introduce a family of surfaces called *cylinders*, some of which are quadric surfaces.

Cylinders and Traces

In everyday language, we use the word *cylinder* to describe the surface that forms, say, the curved wall of a paint can. In the context of three-dimensional surfaces, the term *cylinder* refers to a surface that is parallel to a line. In this text, we focus on cylinders that are parallel to one of the coordinate axes. Equations for such cylinders are easy to identify: The variable corresponding to the coordinate axis parallel to the cylinder is missing from the equation.

For example, working in $\mathbb{R}^3$, the equation $y = x^2$ does not include z, which means that z is arbitrary and can take on all values. Therefore, $y = x^2$ describes the cylinder consisting of all lines parallel to the z-axis that pass through the parabola $y = x^2$ in the xy-plane (Figure 13.79a). In a similar way, the equation $y = z^2$ in $\mathbb{R}^3$ is missing the variable x, so it describes a cylinder parallel to the x-axis. The cylinder consists of lines parallel to the x-axis that pass through the parabola $y = z^2$ in the yz-plane (Figure 13.79b).

➤ The parabolic cylinder in Figure 13.79a can also be described as the surface swept out by translating the plane curve $y = x^2$ up and down (or parallel to) the z-axis.

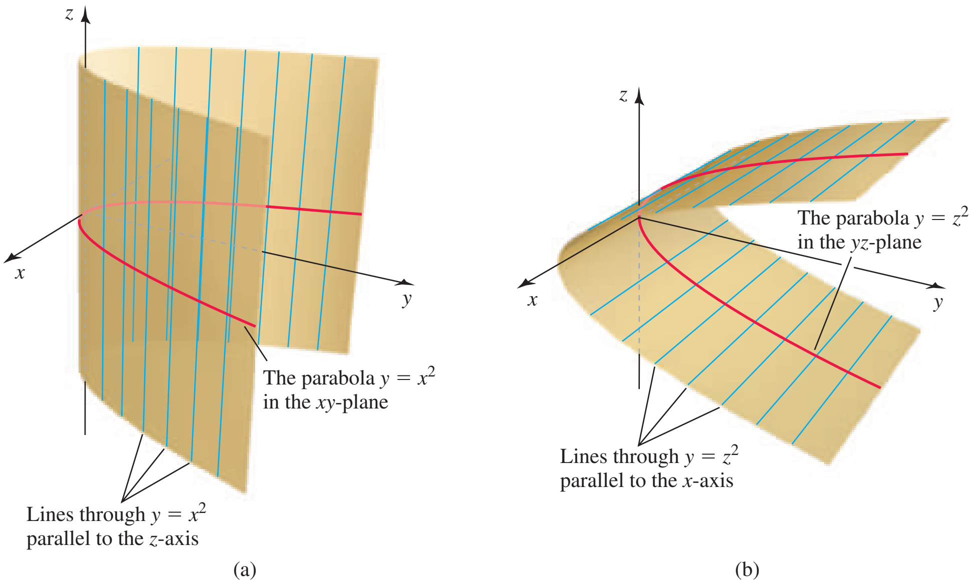


Figure 13.79

QUICK CHECK 1 To which coordinate axis in $\mathbb{R}^3$ is the cylinder $z - 2 \ln x = 0$ parallel? To which coordinate axis in $\mathbb{R}^3$ is the cylinder $y = 4z^2 - 1$ parallel? ◄

Graphing surfaces—and cylinders in particular—is facilitated by identifying the *traces* of the surface.

> **DEFINITION Trace**
>
> A **trace** of a surface is the set of points at which the surface intersects a plane that is parallel to one of the coordinate planes. The traces in the coordinate planes are called the ***xy*-trace**, the ***yz*-trace**, and the ***xz*-trace** (Figure 13.80).

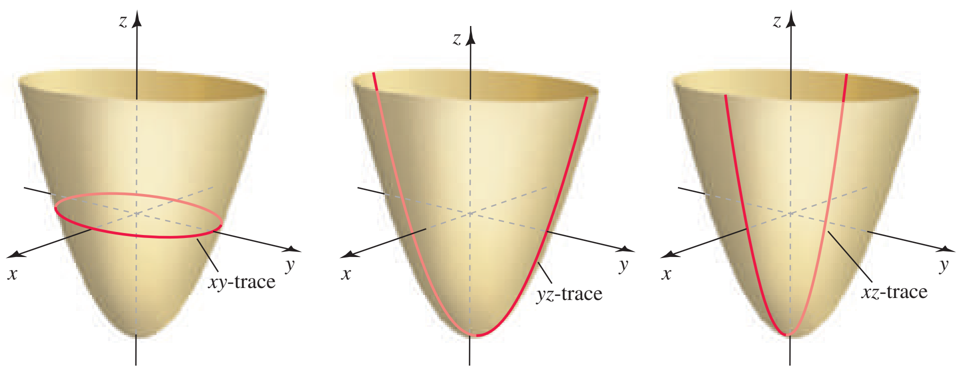


Figure 13.80

EXAMPLE 1 **Graphing cylinders** Sketch the graphs of the following cylinders in $\mathbb{R}^3$. Identify the axis to which each cylinder is parallel.

a. $x^2 + 4y^2 = 16$ **b.** $x - \sin z = 0$

SOLUTION

a. As an equation in $\mathbb{R}^3$, the variable z is absent. Therefore, z assumes all real values and the graph is a cylinder consisting of lines parallel to the z-axis passing through the curve $x^2 + 4y^2 = 16$ in the xy-plane. You can sketch the cylinder in the following steps.

1. Rewriting the given equation as $\frac{x^2}{4^2} + \frac{y^2}{2^2} = 1$, we see that the trace of the cylinder in the xy-plane (the xy-trace) is an ellipse. We begin by drawing this ellipse.
2. Next draw a second trace (a copy of the ellipse in Step 1) in a plane parallel to the xy-plane.
3. Now draw lines parallel to the z-axis through the two traces to fill out the cylinder (Figure 13.81a).

The resulting surface, called an *elliptic cylinder*, runs parallel to the z-axis (Figure 13.81b).

b. As an equation in $\mathbb{R}^3$, $x - \sin z = 0$ is missing the variable y. Therefore, y assumes all real values and the graph is a cylinder consisting of lines parallel to the y-axis passing through the curve $x = \sin z$ in the xz-plane. You can sketch the cylinder in the following steps.

1. Graph the curve $x = \sin z$ in the xz-plane, which is the xz-trace of the surface.
2. Draw a second trace (a copy of the curve in Step 1) in a plane parallel to the xz-plane.
3. Draw lines parallel to the y-axis passing through the two traces (Figure 13.82a).

The result is a cylinder, running parallel to the y-axis, consisting of copies of the curve $x = \sin z$ (Figure 13.82b).

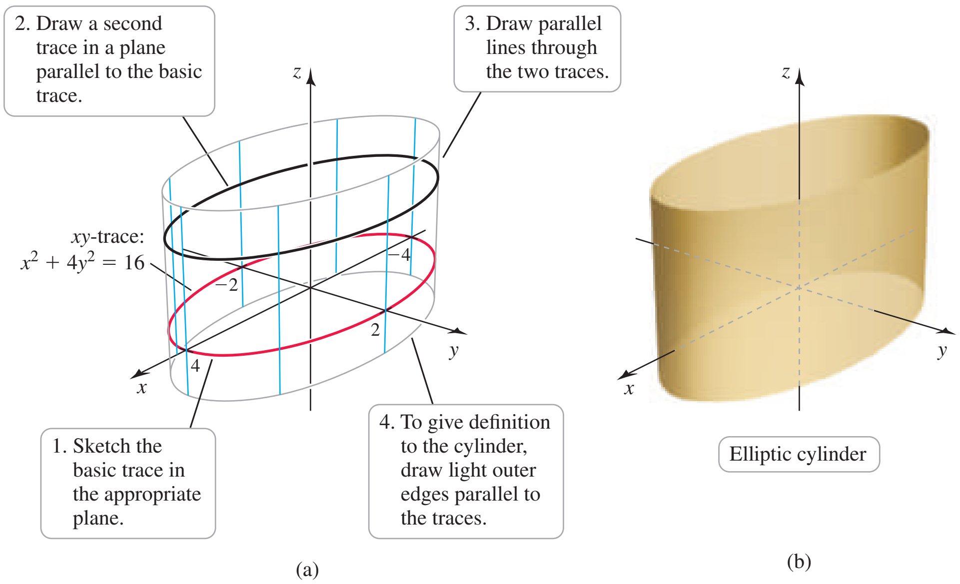


Figure 13.81

Related Exercises 8, 13 ◄

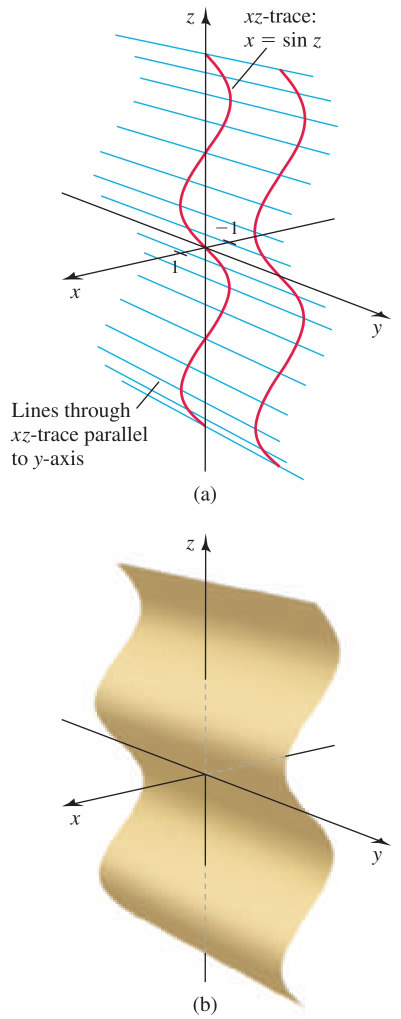


Figure 13.82

Quadric Surfaces

Quadric surfaces are described by the general quadratic (second-degree) equation in three variables,

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0,$$

where the coefficients $A, \ldots, J$ are constants and not all of A, B, C, D, E, and F are zero. We do not attempt a detailed study of this large family of surfaces. However, a few standard surfaces are worth investigating.

Apart from their mathematical interest, quadric surfaces have a variety of practical uses. Paraboloids (defined in Example 3) share the reflective properties of their two-dimensional counterparts (Section 12.4) and are used to design satellite dishes, headlamps, and mirrors in telescopes. Cooling towers for nuclear power plants have the shape of hyperboloids of one sheet. Ellipsoids appear in the design of water tanks and gears.

Making hand sketches of quadric surfaces can be challenging. Here are a few general ideas to keep in mind as you sketch their graphs.

1. **Intercepts** Determine the points, if any, where the surface intersects the coordinate axes. To find these intercepts, set x, y, and z equal to zero in pairs in the equation of the surface, and solve for the third coordinate.
2. **Traces** As illustrated in the following examples, finding traces of the surface helps visualize the surface. For example, setting $z = 0$ or $z = z_0$ (a constant) gives the traces in planes parallel to the xy-plane.
3. **Completing the figure** Sketch at least two traces in parallel planes (for example, traces with $z = 0$ and $z = \pm 1$). Then draw smooth curves that pass through the traces to fill out the surface.

➤ Working with quadric surfaces requires familiarity with conic sections (Section 12.4).

QUICK CHECK 2 Explain why the elliptic cylinder discussed in Example 1a is a quadric surface. ➤

EXAMPLE 2 **An ellipsoid** The surface defined by the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is an *ellipsoid*. Graph the ellipsoid with $a = 3$, $b = 4$, and $c = 5$.

SOLUTION Setting x, y, and z equal to zero in pairs gives the intercepts $(\pm 3, 0, 0)$, $(0, \pm 4, 0)$, and $(0, 0, \pm 5)$. Note that points in $\mathbb{R}^3$ with $|x| > 3$ or $|y| > 4$ or $|z| > 5$ do not satisfy the equation of the surface (because the left side of the equation is a sum of nonnegative terms that cannot exceed 1). Therefore, the entire surface is contained in the rectangular box defined by $|x| \le 3$, $|y| \le 4$, and $|z| \le 5$.

The trace in the horizontal plane $z = z_0$ is found by substituting $z = z_0$ into the equation of the ellipsoid, which gives

$$\frac{x^2}{9} + \frac{y^2}{16} + \frac{z_0^{\,2}}{25} = 1 \quad \text{or} \quad \frac{x^2}{9} + \frac{y^2}{16} = 1 - \frac{z_0^{\,2}}{25}.$$

➤ The name *ellipsoid* is used in Example 2 because all traces of this surface, when they exist, are ellipses.

If $|z_0| < 5$, then $1 - \frac{z_0^{\,2}}{25} > 0$, and the equation describes an ellipse in the horizontal plane $z = z_0$. The largest ellipse parallel to the xy-plane occurs with $z_0 = 0$; it is the xy-trace, which is the ellipse $\frac{x^2}{9} + \frac{y^2}{16} = 1$ with axes of length 6 and 8 (Figure 13.83a). You can check that the yz-trace, found by setting $x = 0$, is the ellipse $\frac{y^2}{16} + \frac{z^2}{25} = 1$. The xz-trace (set $y = 0$) is the ellipse $\frac{x^2}{9} + \frac{z^2}{25} = 1$ (Figure 13.83b). When we sketch the xy-, xz-, and yz-traces, an outline of the ellipsoid emerges (Figure 13.83c).

QUICK CHECK 3 Assume $0 < c < b < a$ in the general equation of an ellipsoid. Along which coordinate axis does the ellipsoid have its longest axis? Its shortest axis? ➤

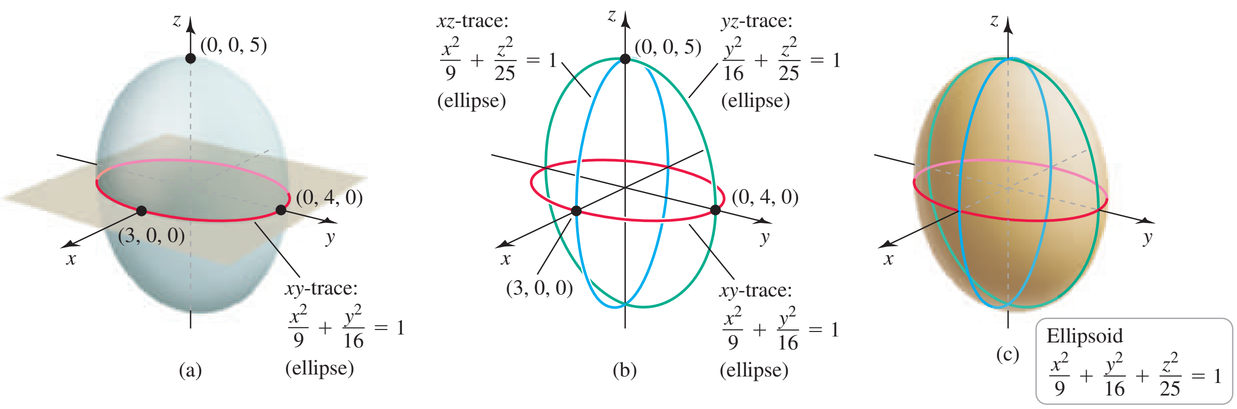


Figure 13.83

Related Exercise 29 ➤

EXAMPLE 3 **An elliptic paraboloid** The surface defined by the equation $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ is an *elliptic paraboloid*. Graph the elliptic paraboloid with $a = 4$ and $b = 2$.

SOLUTION Note that the only intercept of the coordinate axes is $(0, 0, 0)$, which is the *vertex* of the paraboloid. The trace in the horizontal plane $z = z_0$, where $z_0 > 0$, satisfies the equation $\frac{x^2}{16} + \frac{y^2}{4} = z_0$, which describes an ellipse; there is no horizontal trace when $z_0 < 0$ (Figure 13.84a). The trace in the vertical plane $x = x_0$ is the parabola $z = \frac{x_0^2}{16} + \frac{y^2}{4}$ (Figure 13.84b); the trace in the vertical plane $y = y_0$ is the parabola $z = \frac{x^2}{16} + \frac{y_0^2}{4}$ (Figure 13.84c).

To graph the surface, we sketch the xz-trace $z = \frac{x^2}{16}$ (setting $y = 0$) and the yz-trace $z = \frac{y^2}{4}$ (setting $x = 0$). When these traces are combined with an elliptical trace $\frac{x^2}{16} + \frac{y^2}{4} = z_0$ in a plane $z = z_0$, an outline of the surface appears (Figure 13.84d).

➤ The name *elliptic paraboloid* reflects the fact that the traces of this surface are parabolas and ellipses. Two of the three traces in the coordinate planes are parabolas, so this surface is called a paraboloid rather than an ellipsoid.

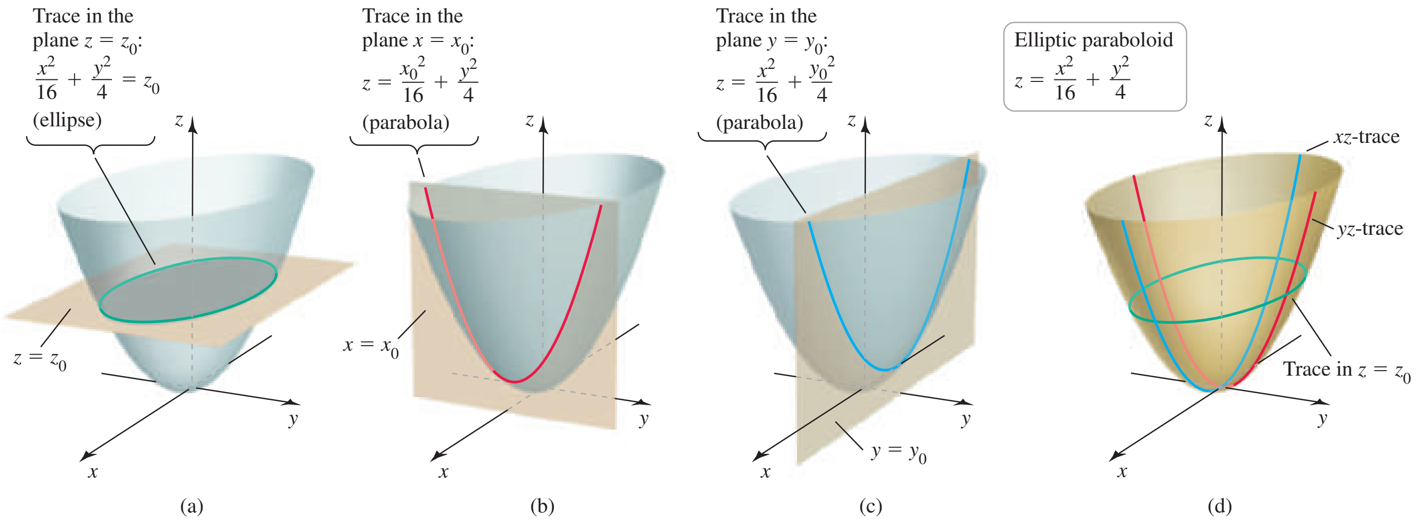


Figure 13.84

Related Exercise 32 ◄

QUICK CHECK 4 The elliptic paraboloid $x = \frac{y^2}{3} + \frac{z^2}{7}$ is a bowl-shaped surface. Along which axis does the bowl open? ◄

EXAMPLE 4 **A hyperboloid of one sheet** Graph the surface defined by the equation $\frac{x^2}{4} + \frac{y^2}{9} - z^2 = 1$.

SOLUTION The intercepts of the coordinate axes are $(0, \pm 3, 0)$ and $(\pm 2, 0, 0)$. Setting $z = z_0$, the traces in horizontal planes are ellipses of the form $\frac{x^2}{4} + \frac{y^2}{9} = 1 + z_0^2$. This equation has solutions for all choices of z_0, so the surface has traces in all horizontal planes. These elliptical traces increase in size as $|z_0|$ increases (Figure 13.85a), with the smallest trace being the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ in the xy-plane. Setting $y = 0$, the xz-trace is the hyperbola $\frac{x^2}{4} - z^2 = 1$; with $x = 0$, the yz-trace is the hyperbola $\frac{y^2}{9} - z^2 = 1$ (Figure 13.85b, c). In fact, the intersection of the surface with any vertical plane is a hyperbola. The resulting surface is a *hyperboloid of one sheet* (Figure 13.85d).

➤ To be completely accurate, this surface should be called an *elliptic hyperboloid of one sheet* because the traces are ellipses and hyperbolas.

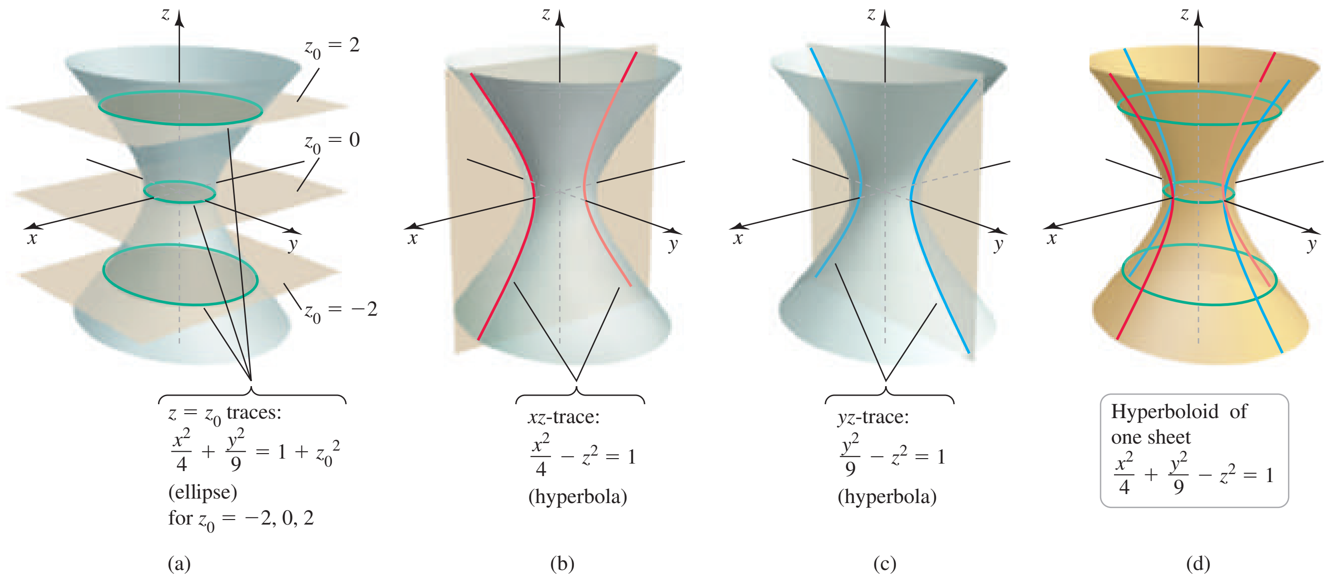


Figure 13.85

Related Exercise 33 ◀

QUICK CHECK 5 Which coordinate axis is the axis of the hyperboloid $\frac{y^2}{a^2} + \frac{z^2}{b^2} - \frac{x^2}{c^2} = 1$? ◀

➤ The name *hyperbolic paraboloid* tells us that the traces are hyperbolas and parabolas. Two of the three traces in the coordinate planes are parabolas, so this surface is a paraboloid rather than a hyperboloid.

➤ The hyperbolic paraboloid has a feature called a *saddle point*. For the surface in Example 5, if you walk from the saddle point at the origin in the direction of the x-axis, you move uphill. If you walk from the saddle point in the direction of the y-axis, you move downhill. Saddle points are examined in detail in Section 15.7.

EXAMPLE 5 **A hyperbolic paraboloid** Graph the surface defined by the equation $z = x^2 - \frac{y^2}{4}$.

SOLUTION Setting $z = 0$ in the equation of the surface, we see that the xy-trace consists of the two lines $y = \pm 2x$. However, slicing the surface with any other horizontal plane $z = z_0$ produces a hyperbola $x^2 - \frac{y^2}{4} = z_0$. If $z_0 > 0$, then the axis of the hyperbola is parallel to the x-axis. On the other hand, if $z_0 < 0$, then the axis of the hyperbola is parallel to the y-axis (**Figure 13.86a**). Setting $x = x_0$ produces the trace $z = {x_0}^2 - \frac{y^2}{4}$,

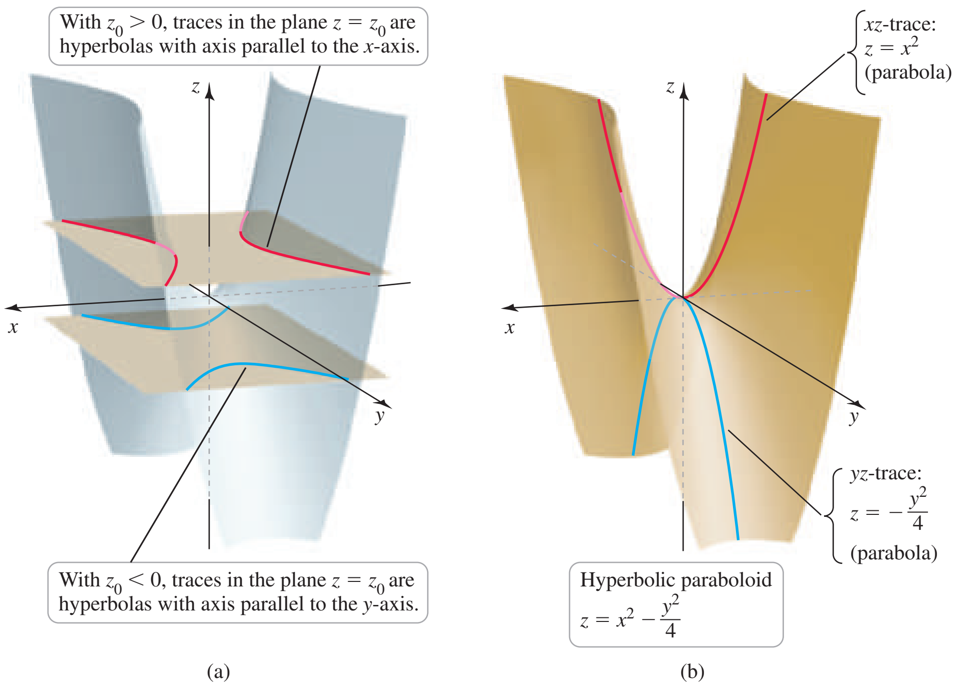


Figure 13.86

which is the equation of a parabola that opens downward in a plane parallel to the yz-plane. You can check that traces in planes parallel to the xz-plane are parabolas that open upward. The resulting surface is a *hyperbolic paraboloid* (**Figure 13.86b**).

Related Exercise 35 ◄

EXAMPLE 6 **Elliptic cones** Graph the surface defined by the equation $\frac{y^2}{4} + z^2 = 4x^2$.

SOLUTION The only point at which the surface intersects the coordinate axes is $(0, 0, 0)$. Traces in the planes $x = x_0$ are ellipses of the form $\frac{y^2}{4} + z^2 = 4x_0{}^2$ that shrink in size as x_0 approaches 0. Setting $y = 0$, the xz-trace satisfies the equation $z^2 = 4x^2$ or $z = \pm 2x$, which are equations of two lines in the xz-plane that intersect at the origin. Setting $z = 0$, the xy-trace satisfies $y^2 = 16x^2$ or $y = \pm 4x$, which describes two lines in the xy-plane that intersect at the origin (**Figure 13.87a**). The complete surface consists of two *cones* opening in opposite directions along the x-axis with a common vertex at the origin (**Figure 13.87b**).

Related Exercise 38 ◄

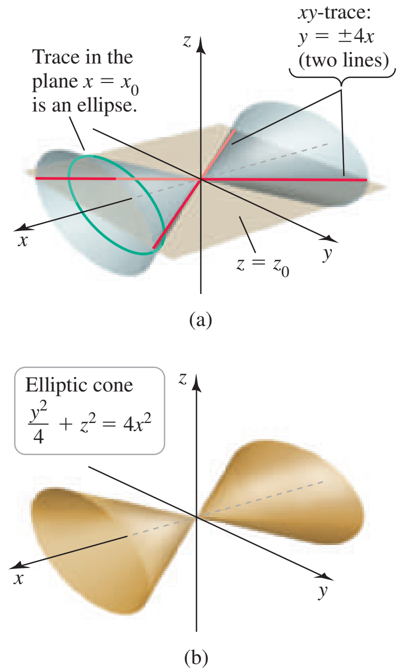


Figure 13.87

EXAMPLE 7 **A hyperboloid of two sheets** Graph the surface defined by the equation

$$-16x^2 - 4y^2 + z^2 + 64x - 80 = 0.$$

SOLUTION We first regroup terms, which yields

$$\underbrace{-16(x^2 - 4x)}_{\text{complete the square}} - 4y^2 + z^2 - 80 = 0,$$

and then complete the square in x:

$$-16(\underbrace{x^2 - 4x + 4}_{(x-2)^2} - 4) - 4y^2 + z^2 - 80 = 0.$$

Collecting terms and dividing by 16 gives the equation

$$-(x-2)^2 - \frac{y^2}{4} + \frac{z^2}{16} = 1.$$

Notice that if $z = 0$, the equation has no solution, so the surface does not intersect the xy-plane. The traces in planes parallel to the xz- and yz-planes are hyperbolas. If $|z_0| \geq 4$, the trace in the plane $z = z_0$ is an ellipse. This equation describes a *hyperboloid of two sheets*, with its axis parallel to the z-axis and shifted 2 units in the positive x-direction (**Figure 13.88**).

Related Exercise 56 ◄

➤ The equation $-x^2 - \frac{y^2}{4} + \frac{z^2}{16} = 1$ describes a hyperboloid of two sheets with its axis on the z-axis. Therefore, the equation in Example 7 describes the same surface shifted 2 units in the positive x-direction.

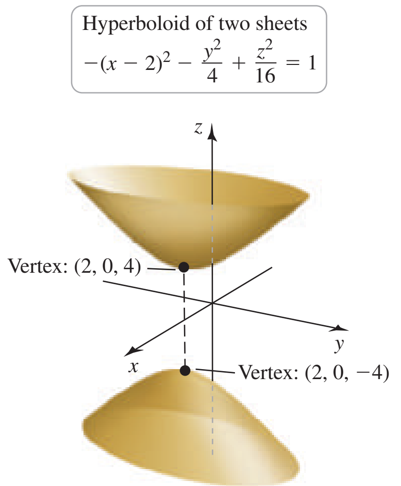
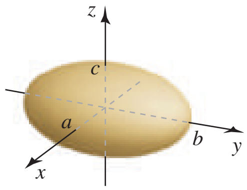
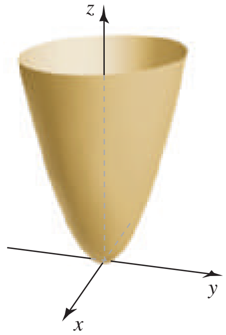
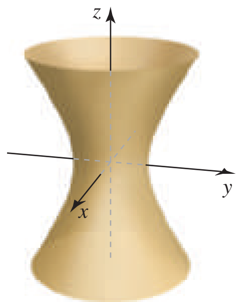
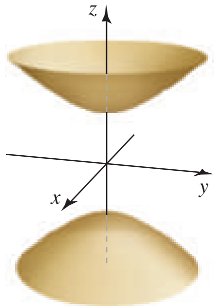
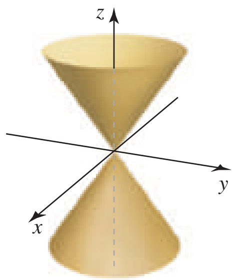
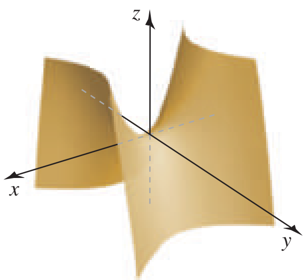


Figure 13.88

QUICK CHECK 6 In which variable(s) should you complete the square to identify the surface $x = y^2 + 2y + z^2 - 4z + 16$? Name and describe the surface. ◄

Table 13.1 (where a, b, and c are nonzero real numbers) summarizes the standard quadric surfaces. It is important to note that the same surfaces with different orientations are obtained when the roles of the variables are interchanged. For this reason, Table 13.1 summarizes many more surfaces than those listed.

Table 13.1

Name	Standard Equation	Features	Graph
Ellipsoid	$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$	All traces are ellipses.	
Elliptic paraboloid	$z = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$	Traces with $z = z_0 > 0$ are ellipses. Traces with $x = x_0$ or $y = y_0$ are parabolas.	
Hyperboloid of one sheet	$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$	Traces with $z = z_0$ are ellipses for all z_0. Traces with $x = x_0$ or $y = y_0$ are hyperbolas.	
Hyperboloid of two sheets	$-\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$	Traces with $z = z_0$ with $\lvert z_0\rvert > \lvert c\rvert$ are ellipses. Traces with $x = x_0$ and $y = y_0$ are hyperbolas.	
Elliptic cone	$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{z^2}{c^2}$	Traces with $z = z_0 \neq 0$ are ellipses. Traces with $x = x_0$ or $y = y_0$ are hyperbolas or intersecting lines.	
Hyperbolic paraboloid	$z = \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}$	Traces with $z = z_0 \neq 0$ are hyperbolas. Traces with $x = x_0$ or $y = y_0$ are parabolas.	

SECTION 13.6 EXERCISES

Getting Started

1. To which coordinate axes are the following cylinders in $\mathbb{R}^3$ parallel: $x^2 + 2y^2 = 8$, $z^2 + 2y^2 = 8$, and $x^2 + 2z^2 = 8$?

2. Describe the graph of $x = z^2$ in $\mathbb{R}^3$.

3. What is a trace of a surface?

4. What is the name of the surface defined by the equation $y = \frac{x^2}{4} + \frac{z^2}{8}$?

5. What is the name of the surface defined by the equation $x^2 + \frac{y^2}{3} + 2z^2 = 1$?

6. What is the name of the surface defined by the equation $-y^2 - \frac{z^2}{2} + x^2 = 1$?

Practice Exercises

7–14. Cylinders in $\mathbb{R}^3$ *Consider the following cylinders in $\mathbb{R}^3$.*

a. *Identify the coordinate axis to which the cylinder is parallel.*
b. *Sketch the cylinder.*

7. $z = y^2$
8. $x^2 + 4y^2 = 4$
9. $x^2 + z^2 = 4$
10. $x = z^2 - 4$
11. $y - x^3 = 0$
12. $x - 2z^2 = 0$
13. $z - \ln y = 0$
14. $x - \frac{1}{y} = 0$

15–20. Identifying quadric surfaces *Identify the following quadric surfaces by name. Find and describe the xy-, xz-, and yz-traces, when they exist.*

15. $25x^2 + 25y^2 + z^2 = 25$
16. $25x^2 + 25y^2 - z^2 = 25$
17. $25x^2 + 25y^2 - z = 0$
18. $25x^2 - 25y^2 - z = 0$
19. $-25x^2 - 25y^2 + z^2 = 25$
20. $-25x^2 - 25y^2 + z^2 = 0$

21–28. Identifying surfaces *Identify the following surfaces by name.*

21. $y = 4z^2 - x^2$
22. $-y^2 - 9z^2 + \frac{x^2}{4} = 1$
23. $y = \frac{x^2}{6} + \frac{z^2}{16}$
24. $z^2 + 4y^2 - x^2 = 1$
25. $y^2 - z^2 = 2$
26. $x^2 + 4y^2 = 1$
27. $9x^2 + 4z^2 - 36y = 0$
28. $9y^2 + 4z^2 - 36x^2 = 0$

29–51. Quadric surfaces *Consider the following equations of quadric surfaces.*

a. *Find the intercepts with the three coordinate axes, when they exist.*
b. *Find the equations of the xy-, xz-, and yz-traces, when they exist.*
c. *Identify and sketch a graph of the surface.*

29. $x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1$
30. $4x^2 + y^2 + \frac{z^2}{2} = 1$
31. $x = y^2 + z^2$
32. $z = \frac{x^2}{4} + \frac{y^2}{9}$
33. $\frac{x^2}{25} + \frac{y^2}{9} - z^2 = 1$
34. $\frac{y^2}{4} + \frac{z^2}{9} - \frac{x^2}{16} = 1$
35. $z = \frac{x^2}{9} - y^2$
36. $y = \frac{x^2}{16} - 4z^2$
37. $x^2 + \frac{y^2}{4} = z^2$
38. $4y^2 + z^2 = x^2$
39. $\frac{x^2}{3} + 3y^2 + \frac{z^2}{12} = 3$
40. $\frac{x^2}{6} + 24y^2 + \frac{z^2}{24} - 6 = 0$
41. $9x - 81y^2 - \frac{z^2}{4} = 0$
42. $2y - \frac{x^2}{8} - \frac{z^2}{18} = 0$
43. $\frac{y^2}{16} + 36z^2 - \frac{x^2}{4} - 9 = 0$
44. $9z^2 + x^2 - \frac{y^2}{3} - 1 = 0$
45. $5x - \frac{y^2}{5} + \frac{z^2}{20} = 0$
46. $6y + \frac{x^2}{6} - \frac{z^2}{24} = 0$
47. $\frac{z^2}{32} + \frac{y^2}{18} = 2x^2$
48. $\frac{x^2}{3} + \frac{z^2}{12} = 3y^2$
49. $-x^2 + \frac{y^2}{4} - \frac{z^2}{9} = 1$
50. $-\frac{x^2}{6} - 24y^2 + \frac{z^2}{24} - 6 = 0$
51. $-\frac{x^2}{3} + 3y^2 - \frac{z^2}{12} = 1$

52. Describe the relationship between the graphs of the quadric surfaces $x^2 + y^2 - z^2 + 2z = 1$ and $x^2 + y^2 - z^2 = 0$, and state the names of the surfaces.

53. Describe the relationship between the graphs of $x^2 + 4y^2 + 9z^2 = 100$ and $x^2 + 4y^2 + 9z^2 + 54z = 19$, and state the names of the surfaces.

54–58. Identifying surfaces *Identify and briefly describe the surfaces defined by the following equations.*

54. $x^2 + y^2 + 4z^2 + 2x = 0$
55. $9x^2 + y^2 - 4z^2 + 2y = 0$
56. $-x^2 - y^2 + \frac{z^2}{9} + 6x - 8y = 26$
57. $\frac{x^2}{4} + y^2 - 2x - 10y - z^2 + 41 = 0$
58. $z = -x^2 - y^2$

59. **Explain why or why not** Determine whether the following statements are true and give an explanation or counterexample.
   a. The graph of the equation $y = z^2$ in $\mathbb{R}^3$ is both a cylinder and a quadric surface.
   b. The xy-traces of the ellipsoid $x^2 + 2y^2 + 3z^2 = 16$ and the cylinder $x^2 + 2y^2 = 16$ are identical.
   c. Traces of the surface $y = 3x^2 - z^2$ in planes parallel to the xy-plane are parabolas.
   d. Traces of the surface $y = 3x^2 - z^2$ in planes parallel to the xz-plane are parabolas.
   e. The graph of the ellipsoid $x^2 + 2y^2 + 3(z - 4)^2 = 25$ is obtained by shifting the graph of the ellipsoid $x^2 + 2y^2 + 3z^2 = 25$ down 4 units.

60. Matching graphs with equations Match equations a–f with surfaces A–F.

a. $y - z^2 = 0$

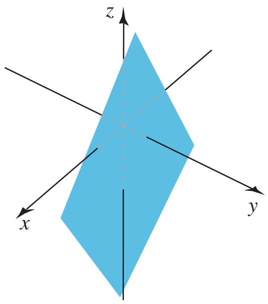
b. $2x + 3y - z = 5$

c. $4x^2 + \dfrac{y^2}{9} + z^2 = 1$

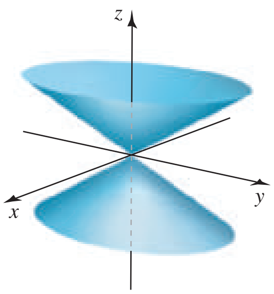
d. $x^2 + \dfrac{y^2}{9} - z^2 = 1$

e. $x^2 + \dfrac{y^2}{9} = z^2$

f. $y = |x|$



(A)



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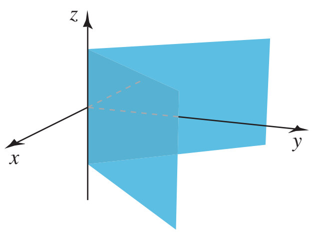

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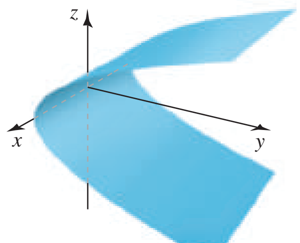

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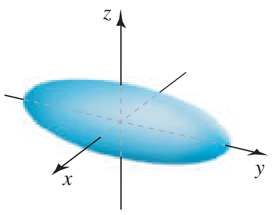

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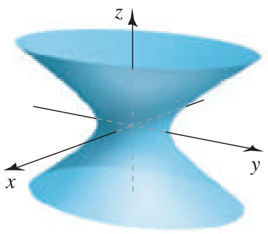


(F)

Explorations and Challenges

61. Solids of revolution Which of the quadric surfaces in Table 13.1 can be generated by revolving a curve in one of the coordinate planes about a coordinate axis, assuming $a = b = c \neq 0$?

62. Solids of revolution Consider the ellipse $x^2 + 4y^2 = 1$ in the xy-plane.

a. If this ellipse is revolved about the x-axis, what is the equation of the resulting ellipsoid?

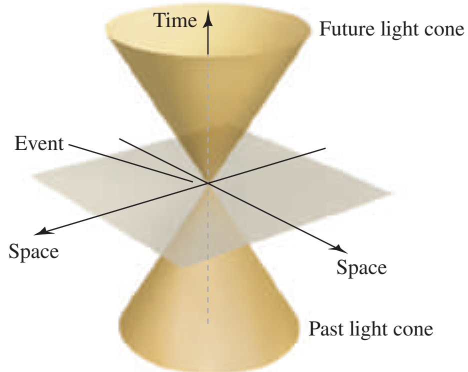
b. If this ellipse is revolved about the y-axis, what is the equation of the resulting ellipsoid?

63. Volume Find the volume of the solid that is bounded between the planes $z = 0$ and $z = 3$ and the cylinders $y = x^2$ and $y = 2 - x^2$.

64. Light cones The idea of a *light cone* appears in the Special Theory of Relativity. The xy-plane (see figure) represents all of three-dimensional space, and the z-axis is the time axis (t-axis). If an event E occurs at the origin, the interior of the future light cone ($t > 0$) represents all events in the future that could be affected by E, assuming no signal travels faster than the speed of light. The interior of the past light cone ($t < 0$) represents all events in the past that could have affected E, again assuming no signal travels faster than the speed of light.

a. If time is measured in seconds and distance (x and y) is measured in light-seconds (the distance light travels in 1 s), the light cone makes a 45° angle with the xy-plane. Write the equation of the light cone in this case.

b. Suppose distance is measured in meters and time is measured in seconds. Write the equation of the light cone in this case, given that the speed of light is 3×10^8 m/s.



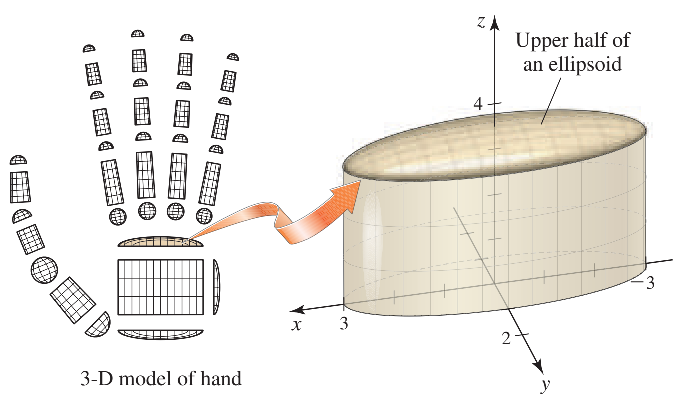
65. Designing an NFL football A *prolate spheroid* is a surface of revolution obtained by rotating an ellipse about its major axis.

a. Explain why one possible equation for a prolate spheroid is $\dfrac{x^2 + z^2}{a^2} + \dfrac{y^2}{b^2} = 1$, where $b > a > 0$.

b. According to the National Football League (NFL) rulebook, the shape of an NFL football is required to be a prolate spheroid with a long axis between 11 and 11.25 inches long and a short circumference (the circumference of the xz-trace) between 21 and 21.25 inches. Find an equation for the shape of the football if the long axis is 11.1 inches and the short circumference is 21.1 inches.

66. Hand tracking Researchers are developing hand tracking software that will allow computers to track and recognize detailed hand movements for better human-computer interaction. One three-dimensional hand model under investigation is constructed from a set of truncated quadrics (see figure). For example, the palm of the hand consists of a truncated elliptic cylinder, capped off by the upper half of an ellipsoid. Suppose the palm of the hand is modeled by the truncated cylinder $4x^2/9 + 4y^2 = 1$, for $0 \leq z \leq 3$. Find an equation of the upper half of an ellipsoid, whose bottom corresponds with the top of the cylinder, if the distance from the top of the truncated cylinder to the top of the ellipsoid is $1/2$.

(*Source: Computer Vision and Pattern Recognition,* 2, Dec 2001)



67. Designing a snow cone A surface, having the shape of an oblong snow cone, consists of a truncated cone, $\frac{x^2}{2} + y^2 = \frac{z^2}{8}$, for $0 \le z \le 3$, capped off by the upper half of an ellipsoid. Find an equation for the upper half of the ellipsoid so that the bottom edge of the truncated ellipsoid and the top edge of the cone coincide, and the distance from the top of the cone to the top of the ellipsoid is $3/2$.

68. Designing a glass The outer, lateral side of a 6-inch-tall glass has the shape of the truncated hyperboloid of one sheet $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$, for $0 \le z \le 6$. If the base of the glass has a radius of 1 inch and the top of the glass has a radius of 2 inches, find the values of a^2, b^2, and c^2 that satisfy these conditions. Assume horizontal traces of the glass are circular.

QUICK CHECK ANSWERS

1. y-axis; x-axis **2.** The equation $x^2 + 4y^2 = 16$ is a special case of the general equation for quadric surfaces; all the coefficients except A, B, and J are zero. **3.** x-axis; z-axis **4.** Positive x-axis **5.** x-axis **6.** Complete the square in y and z; elliptic paraboloid with its axis parallel to the x-axis ◄

CHAPTER 13 REVIEW EXERCISES

1. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. Given two vectors $\mathbf{u}$ and $\mathbf{v}$, it is always true that $2\mathbf{u} + \mathbf{v} = \mathbf{v} + 2\mathbf{u}$.

b. The vector in the direction of $\mathbf{u}$ with the length of $\mathbf{v}$ equals the vector in the direction of $\mathbf{v}$ with the length of $\mathbf{u}$.

c. If $\mathbf{u} \ne \mathbf{0}$ and $\mathbf{u} + \mathbf{v} = \mathbf{0}$, then $\mathbf{u}$ and $\mathbf{v}$ are parallel.

d. The lines $x = 3 + t, y = 4 + 2t, z = 2 - t$ and $x = 2t, y = 4t, z = t$ are parallel.

e. The lines $x = 3 + t, y = 4 + 2t, z = 2 - t$ and the plane $x + 2y + 5z = 3$ are parallel.

f. There is always a plane orthogonal to both of two distinct intersecting planes.

2–5. Working with vectors *Let* $\mathbf{u} = \langle 3, -4 \rangle$ *and* $\mathbf{v} = \langle -1, 2 \rangle$. *Evaluate each of the following.*

2. $\mathbf{u} - \mathbf{v}$

3. $-3\mathbf{v}$

4. $\mathbf{u} + 2\mathbf{v}$

5. $2\mathbf{v} - \mathbf{u}$

6–15. Working with vectors *Let* $\mathbf{u} = \langle 2, 4, -5 \rangle$, $\mathbf{v} = \langle -6, 10, 2 \rangle$, *and* $\mathbf{w} = \langle 4, -8, 8 \rangle$.

6. Compute $\mathbf{u} - 3\mathbf{v}$.

7. Compute $|\mathbf{u} + \mathbf{v}|$.

8. Find the unit vector with the same direction as $\mathbf{u}$.

9. Write the vector $\mathbf{w}$ as a product of its magnitude and a unit vector in the direction of $\mathbf{w}$.

10. Find a vector in the direction of $\mathbf{w}$ that is 10 times as long as $\mathbf{w}$.

11. Find a vector in the direction of $\mathbf{w}$ with a length of 10.

12. Compute $\mathbf{u} \cdot \mathbf{v}$

13. Compute $\mathbf{u} \times \mathbf{v}$

14. For what value of a is the vector $\mathbf{v}$ orthogonal to $\mathbf{y} = \langle a, 1, -3 \rangle$?

15. For what value of a is the vector $\mathbf{w}$ parallel to $\mathbf{y} = \langle a, 6, -6 \rangle$?

16. Scalar multiples Find scalars a, b, and c such that

$$\langle 2, 2, 2 \rangle = a\langle 1, 1, 0 \rangle + b\langle 0, 1, 1 \rangle + c\langle 1, 0, 1 \rangle.$$

17. Velocity vectors Assume the positive x-axis points east and the positive y-axis points north.

a. An airliner flies northwest at a constant altitude at 550 mi/hr in calm air. Find a and b such that its velocity may be expressed in the form $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$.

b. An airliner flies northwest at a constant altitude at 550 mi/hr relative to the air in a southerly crosswind $\mathbf{w} = \langle 0, 40 \rangle$. Find the velocity of the airliner relative to the ground.

18. Position vectors Let $\overrightarrow{PQ}$ extend from $P(2, 0, 6)$ to $Q(2, -8, 5)$.

a. Find the position vector equal to $\overrightarrow{PQ}$.

b. Find the midpoint M of the line segment PQ. Then find the magnitude of $\overrightarrow{PM}$.

c. Find a vector of length 8 with direction opposite to that of $\overrightarrow{PQ}$.

19–21. Spheres and balls *Use set notation to describe the following sets.*

19. The sphere of radius 4 centered at $(1, 0, -1)$

20. The points inside the sphere of radius 10 centered at $(2, 4, -3)$

21. The points outside the sphere of radius 2 centered at $(0, 1, 0)$

22–25. Identifying sets *Give a geometric description of the following sets of points.*

22. $x^2 - 6x + y^2 + 8y + z^2 - 2z - 23 = 0$

23. $x^2 - x + y^2 + 4y + z^2 - 6z + 11 \le 0$

24. $x^2 + y^2 - 10y + z^2 - 6z = -34$

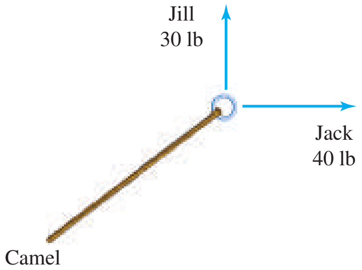
25. $x^2 - 6x + y^2 + z^2 - 20z + 9 > 0$

26. Combined force An object at the origin is acted on by the forces $\mathbf{F}_1 = -10\mathbf{i} + 20\mathbf{k}$, $\mathbf{F}_2 = 40\mathbf{j} + 10\mathbf{k}$, and $\mathbf{F}_3 = -50\mathbf{i} + 20\mathbf{j}$. Find the magnitude of the combined force, and use a sketch to illustrate the direction of the combined force.

27. Falling probe A remote sensing probe falls vertically with a terminal velocity of 60 m/s when it encounters a horizontal crosswind blowing north at 4 m/s and an updraft blowing vertically at 10 m/s. Find the magnitude and direction of the resulting velocity relative to the ground.

28. Crosswinds A small plane is flying north in calm air at 250 mi/hr when it is hit by a horizontal crosswind blowing northeast at 40 mi/hr and a 25 mi/hr downdraft. Find the resulting velocity and speed of the plane.

29. Net force Jack pulls east on a rope attached to a camel with a force of 40 lb. Jill pulls north on a rope attached to the same camel with a force of 30 lb. What is the magnitude and direction of the force on the camel? Assume the vectors lie in a horizontal plane.



30. Canoe in a current A woman in a canoe paddles due west at 4 mi/hr relative to the water in a current that flows northwest at 2 mi/hr. Find the speed and direction of the canoe relative to the shore.

31. Set of points Describe the set of points satisfying both the equations $x^2 + z^2 = 1$ and $y = 2$.

32–33. Angles and projections

a. *Find the angle between* $\mathbf{u}$ *and* $\mathbf{v}$.
b. *Compute* $\text{proj}_{\mathbf{v}}\mathbf{u}$ *and* $\text{scal}_{\mathbf{v}}\mathbf{u}$.
c. *Compute* $\text{proj}_{\mathbf{u}}\mathbf{v}$ *and* $\text{scal}_{\mathbf{u}}\mathbf{v}$.

32. $\mathbf{u} = -3\mathbf{j} + 4\mathbf{k}$, $\mathbf{v} = -4\mathbf{i} + \mathbf{j} + 5\mathbf{k}$

33. $\mathbf{u} = -\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$, $\mathbf{v} = 3\mathbf{i} + 6\mathbf{j} + 6\mathbf{k}$

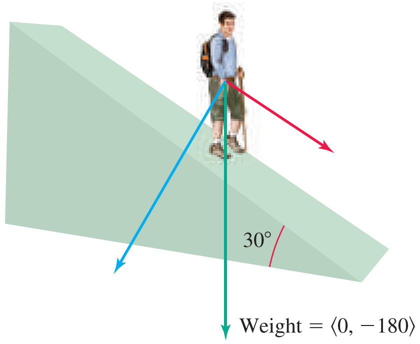
34. Parallelepiped Find the volume of a parallelepiped determined by the position vectors $\mathbf{u} = \langle 2, 4, -5 \rangle$, $\mathbf{v} = \langle -6, 10, 2 \rangle$, and $\mathbf{w} = \langle 4, -8, 8 \rangle$ (see Exercise 63 in Section 13.4).

35–36. Computing work *Calculate the work done in the following situations.*

35. A suitcase is pulled 25 ft along a horizontal sidewalk with a constant force of 20 lb at an angle of 45° above the horizontal.

36. A constant force $\mathbf{F} = \langle 2, 3, 4 \rangle$ (in newtons) moves an object from $(0, 0, 0)$ to $(2, 1, 6)$. (Distance is measured in meters.)

37–38. Inclined plane *A* 180-lb *man stands on a hillside that makes an angle of* 30° *with the horizontal, producing a force of* $\mathbf{W} = \langle 0, -180 \rangle$.



37. Find the component of his weight in the downward direction perpendicular to the hillside and in the downward direction parallel to the hillside.

38. How much work is done when the man moves 10 ft up the hillside?

39. Area of a parallelogram Find the area of the parallelogram with the vertices $(1, 2, 3)$, $(1, 0, 6)$, and $(4, 2, 4)$.

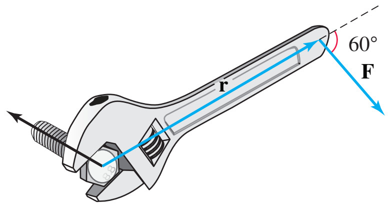
40. Area of a triangle Find the area of the triangle with the vertices $(1, 0, 3)$, $(5, 0, -1)$, and $(0, 2, -2)$.

41. Unit normal vector Find unit vectors normal to the vectors $\langle 2, -6, 9 \rangle$ and $\langle -1, 0, 6 \rangle$.

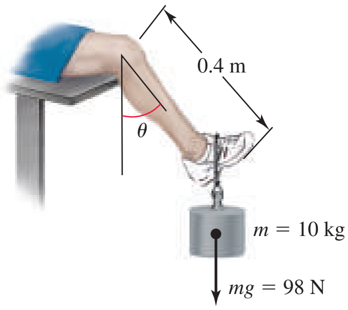
42. Angle in two ways Find the angle between $\langle 2, 0, -2 \rangle$ and $\langle 2, 2, 0 \rangle$ using (a) the dot product and (b) the cross product.

43. Let $\mathbf{r} = \overrightarrow{OP} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$. A force $\mathbf{F} = \langle 10, 10, 0 \rangle$ is applied at P. Find the torque about O that is produced.

44. Suppose you apply a force of $|\mathbf{F}| = 50$ N near the end of a wrench attached to a bolt (see figure). Determine the magnitude of the torque when the force is applied at an angle of 60° to the wrench. Assume the distance along the wrench from the center of the bolt to the point where the force is applied is $|\mathbf{r}| = 0.25$ m.



45. Knee torque Jan does leg lifts with a 10-kg weight attached to her foot, so the resulting force is $mg \approx 98$ N directed vertically downward (see figure). If the distance from her knee to the weight is 0.4 m and her lower leg makes an angle of θ to the vertical, find the magnitude of the torque about her knee as her leg is lifted (as a function of θ). What are the minimum and maximum magnitudes of the torque? Does the direction of the torque change as her leg is lifted?



46–50. Lines in space *Find an equation of the following lines or line segments.*

46. The line that passes through the points $(2, 6, -1)$ and $(-6, 4, 0)$

47. The line segment that joins the points $(0, -3, 9)$ and $(2, -8, 1)$

48. The line through the point $(0, 1, 1)$ and parallel to the line $\mathbf{R} = \langle 1 + 2t, 3 - 5t, 7 + 6t \rangle$.

49. The line through the point $(0, 1, 1)$ that is orthogonal to both $\langle 0, -1, 3 \rangle$ and $\langle 2, -1, 2 \rangle$.

50. The line through the point $(0, 1, 4)$ and orthogonal to the vector $\langle -2, 1, 7 \rangle$ and the y-axis.

51. **Equations of planes** Consider the plane passing through the points $(0, 0, 3)$, $(1, 0, -6)$, and $(1, 2, 3)$.
 a. Find an equation of the plane.
 b. Find the intercepts of the plane with the three coordinate axes.
 c. Make a sketch of the plane.

52–53. Intersecting planes *Find an equation of the line of intersection of the planes Q and R.*

52. $Q: 2x + y - z = 0,\ R: -x + y + z = 1$

53. $Q: -3x + y + 2z = 0,\ R: 3x + 3y + 4z - 12 = 0$

54–57. Equations of planes *Find an equation of the following planes.*

54. The plane passing through $(5, 0, 2)$ that is parallel to the plane $2x + y - z = 0$

55. The plane containing the lines $x = 5 + t, y = 3 - 2t, z = 1$ and $x = 4s, y = 5s, z = 3 - 2s$, if possible

56. The plane passing through $(2, -3, 1)$ normal to the line $\langle x, y, z\rangle = \langle 2 + t, 3t, 2 - 3t\rangle$

57. The plane passing through $(-2, 3, 1)$, $(1, 1, 0)$, and $(-1, 0, 1)$

58. **Distance from a point to a line** Find the distance from the point $(1, 2, 3)$ to the line $x = 2 + t, y = 3, z = 1 - 3t$.

59. **Distance from a point to a plane** Find the distance from the point $(2, 2, 2)$ to the plane $x + 2y + 2z = 1$.

60–74. Identifying surfaces *Consider the surfaces defined by the following equations.*

a. Identify and briefly describe the surface.
b. Find the xy-, xz-, and yz-traces, when they exist.
c. Find the intercepts with the three coordinate axes, when they exist.
d. Sketch the surface.

60. $z - \sqrt{x} = 0$

61. $3z = \dfrac{x^2}{12} - \dfrac{y^2}{48}$

62. $\dfrac{x^2}{100} + 4y^2 + \dfrac{z^2}{16} = 1$

63. $y^2 = 4x^2 + \dfrac{z^2}{25}$

64. $\dfrac{4x^2}{9} + \dfrac{9z^2}{4} = y^2$

65. $4z = \dfrac{x^2}{4} + \dfrac{y^2}{9}$

66. $\dfrac{x^2}{16} + \dfrac{z^2}{36} - \dfrac{y^2}{100} = 1$

67. $y^2 + 4z^2 - 2x^2 = 1$

68. $-\dfrac{x^2}{16} + \dfrac{z^2}{36} - \dfrac{y^2}{25} = 4$

69. $\dfrac{x^2}{4} + \dfrac{y^2}{16} - z^2 = 4$

70. $x = \dfrac{y^2}{64} - \dfrac{z^2}{9}$

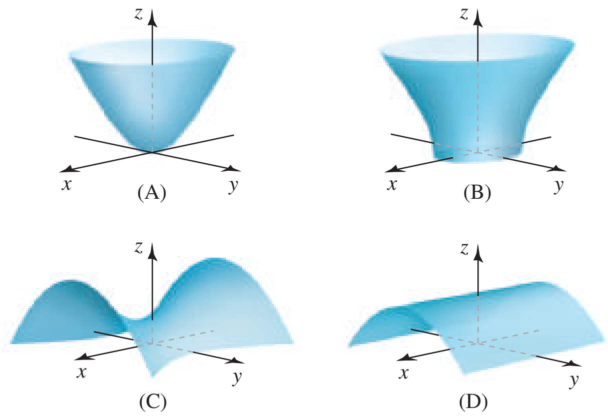
71. $\dfrac{x^2}{4} + \dfrac{y^2}{16} + z^2 = 4$

72. $y - e^{-x} = 0$

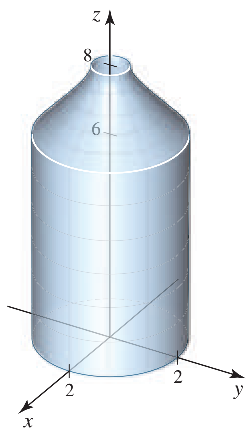
73. $\dfrac{y^2}{49} + \dfrac{x^2}{9} = \dfrac{z^2}{64}$

74. $y = 4x^2 + \dfrac{z^2}{9}$

75. **Matching surfaces** Match equations a–d with surfaces A–D.
 a. $z = \sqrt{2x^2 + 3y^2 + 1} - 1$ **b.** $z = -3y^2$
 c. $z = 2x^2 - 3y^2 + 1$ **d.** $z = \sqrt{2x^2 + 3y^2 - 1}$



76. **Designing a water bottle** The lateral surface of a water bottle consists of a circular cylinder of radius 2 and height 6, topped off by a truncated hyperboloid of one sheet of height 2 (see figure). Assume the top of the truncated hyperboloid has a radius of 1/2. Find two equations that, when graphed together, form the lateral surface of the bottle. Answers may vary.



Chapter 13 Guided Projects

Applications of the material in this chapter and related topics can be found in the following Guided Projects. For additional information, see the Preface.

- Intercepting a UFO
- CORDIC algorithms: How your calculator works

14

Vector-Valued Functions

Chapter Preview In Chapter 13, we used vectors to represent static quantities, such as the constant force applied to the end of a wrench or the constant velocity of a boat in a current. In this chapter, we put vectors in motion by introducing *vector-valued functions*, or simply *vector functions.* Our first task is to investigate the graphs of vector-valued functions and to study them in the setting of calculus. Everything you already know about limits, derivatives, and integrals applies to this new family of functions. Also, with the calculus of vector functions, we can solve a wealth of practical problems involving the motion of objects in space. The chapter closes with an exploration of arc length, curvature, and tangent and normal vectors, all important features of space curves.

14.1 Vector-Valued Functions

Imagine a projectile moving along a path in three-dimensional space; it could be an electron or a comet, a soccer ball or a rocket. If you take a snapshot of the object, its position is described by a static position vector $\mathbf{r} = \langle x, y, z \rangle$. However, if you want to describe the full trajectory of the object as it unfolds in time, you must represent the object's position with a *vector-valued function* such as $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ whose components change in time (Figure 14.1). The goal of this section is to describe continuous motion using vector-valued functions.

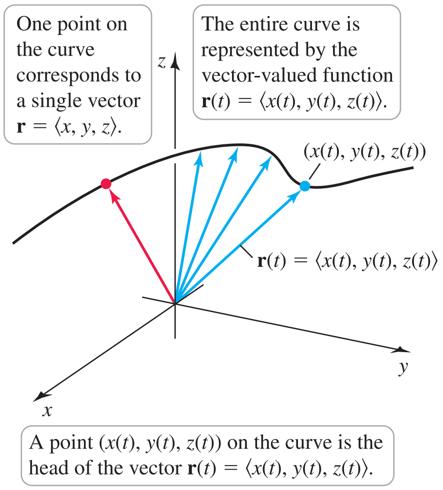


Figure 14.1

Vector-Valued Functions

A function of the form $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ may be viewed in two ways.

- It is a set of three parametric equations that describe a curve in space.
- It is also a **vector-valued function**, which means that the three dependent variables (x, y, and z) are the components of $\mathbf{r}$, and each component varies with respect to a single independent variable t (that often represents time).

Here is the connection between these perspectives: As t varies, a point $(x(t), y(t), z(t))$ on a parametric curve is also the head of the position vector $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$. In other words, a vector-valued function is a set of parametric equations written in vector form. It is useful to keep both of these interpretations in mind as you work with vector-valued functions.

Although our focus is on vector functions whose graphs lie in three-dimensional space, vector functions can be given in any number of dimensions. In fact, you became acquainted with the essential ideas behind two-dimensional vector functions in Section 12.1 when you studied parametric equations. For example, recall that the parametric equations

$$x = a\cos t,\ \ y = a\sin t, \quad \text{for } 0 \le t \le 2\pi$$

describe a circle of radius a centered at the origin. The corresponding vector function is

$$\mathbf{r}(t) = \langle a\cos t, a\sin t\rangle, \quad \text{for } 0 \le t \le 2\pi.$$

All the plane curves described by parametric equations in Section 12.1 are easily converted to vector functions in the same manner.

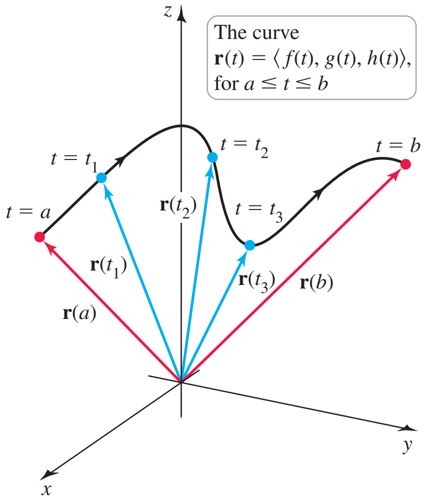


Figure 14.2

Curves in Space

We now explore general vector-valued functions of the form

$$\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k},$$

where f, g, and h are defined on an interval $a \le t \le b$. The **domain** of $\mathbf{r}$ is the largest set of values of t on which all of f, g, and h are defined.

Figure 14.2 illustrates how a parameterized curve is generated by such a function. As the parameter t varies over the interval $a \le t \le b$, each value of t produces a position vector that corresponds to a point on the curve, starting at the initial vector $\mathbf{r}(a)$ and ending at the terminal vector $\mathbf{r}(b)$. The resulting parameterized curve can either have finite length or extend indefinitely. The curve may also cross itself or close and retrace itself. As shown in Example 1, when f, g, and h are linear functions of t, the resulting curve is a line or line segment.

EXAMPLE 1 **Lines as vector-valued functions** Find a vector function for the line that passes through the points $P(2, -1, 4)$ and $Q(3, 0, 6)$.

SOLUTION Recall from Section 13.5 that parametric equations of the line parallel to the vector $\mathbf{v} = \langle a, b, c\rangle$ and passing through the point $P_0(x_0, y_0, z_0)$ are

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct.$$

The vector $\mathbf{v} = \overrightarrow{PQ} = \langle 3 - 2, 0 - (-1), 6 - 4\rangle = \langle 1, 1, 2\rangle$ is parallel to the line, and we let $P_0 = P(2, -1, 4)$. Therefore, parametric equations for the line are

$$x = 2 + t, \quad y = -1 + t, \quad z = 4 + 2t,$$

and the corresponding vector function for the line is

$$\mathbf{r}(t) = \langle 2 + t, -1 + t, 4 + 2t\rangle,$$

with a domain of all real numbers. As t increases, the line is generated in the direction of $\overrightarrow{PQ}$. Just as we did with parametric equations, we can restrict the domain to a finite interval to produce a vector function for a line segment (see Quick Check 1).

Related Exercises 9, 13 ◄

QUICK CHECK 1 Restrict the domain of the vector function in Example 1 to produce a line segment that goes from $P(2, -1, 4)$ to $R(5, 1, 8)$. ◄

Orientation of Curves If a smooth curve C is viewed only as a set of points, then at any point of C, it is possible to draw tangent vectors in two directions (Figure 14.3a). On the other hand, a parameterized curve described by the function $\mathbf{r}(t)$, where $a \le t \le b$, has a natural direction, or **orientation**. The *positive* orientation is the direction in which the curve is generated as the parameter increases from a to b. For example, the positive orientation of the circle $\mathbf{r}(t) = \langle \cos t, \sin t\rangle$, for $0 \le t \le 2\pi$, is counterclockwise (Figure 14.3b), and the positive orientation of the line in Example 1 is in the direction of the vector $\overrightarrow{PQ}$. An important property of all parameterized curves is the relationship between the orientation of a given curve and its tangent vectors (to be defined precisely in Section 14.2): At all points, the tangent vectors point in the direction of the positive orientation of the curve.

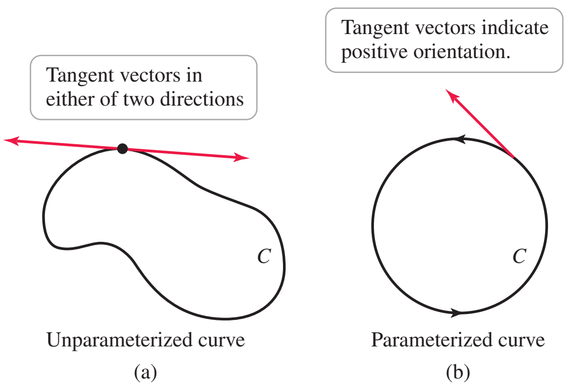


Figure 14.3

EXAMPLE 2 **A spiral** Graph the curve described by the equation

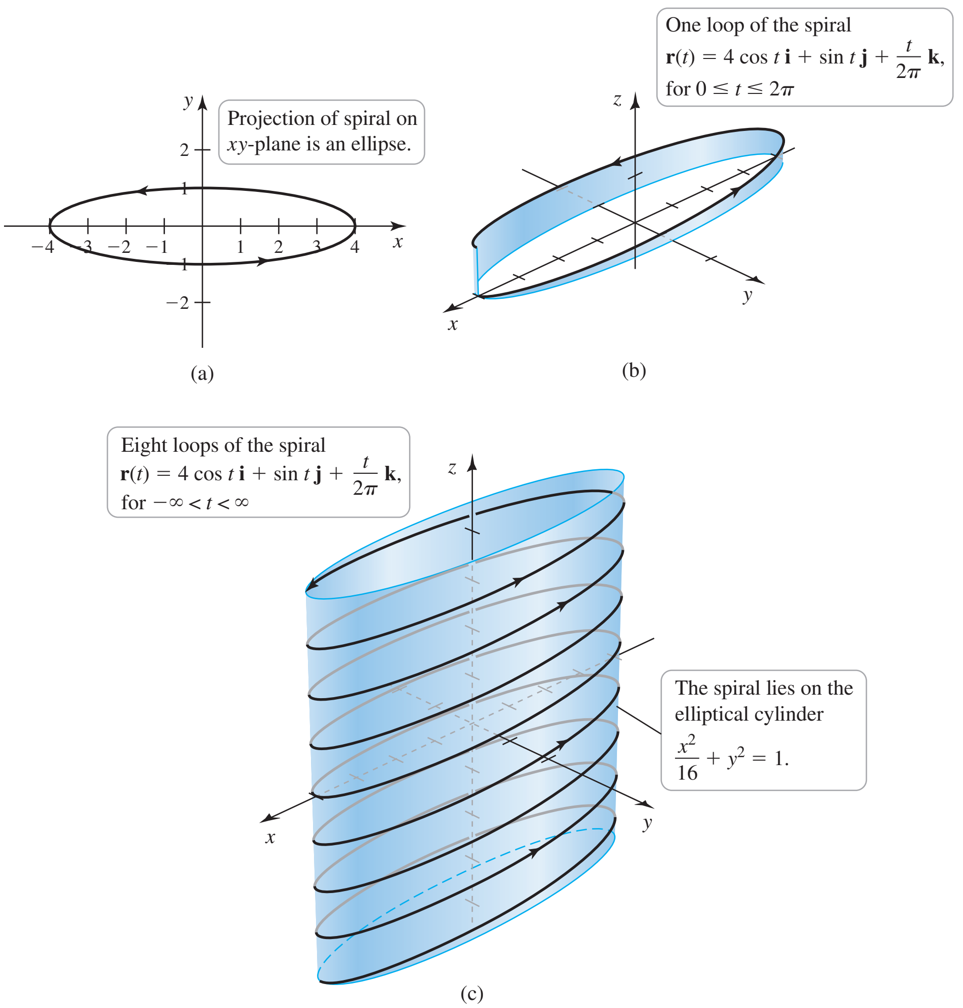
$$\mathbf{r}(t) = 4\cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \frac{t}{2\pi}\mathbf{k},$$

where (a) $0 \le t \le 2\pi$ and (b) $-\infty < t < \infty$.

SOLUTION

a. We begin by setting $z = 0$ to determine the projection of the curve in the xy-plane. The resulting function $\mathbf{r}(t) = 4\cos t\,\mathbf{i} + \sin t\,\mathbf{j}$ implies that $x = 4\cos t$ and $y = \sin t$; these equations describe an ellipse in the xy-plane whose positive direction is counterclockwise (Figure 14.4a). Because $z = \frac{t}{2\pi}$, the value of z increases from 0 to 1 as t increases from 0 to 2π. Therefore, the curve rises out of the xy-plane to create an elliptical spiral (or coil). Over the interval $[0, 2\pi]$, the spiral begins at $(4, 0, 0)$, circles the z-axis once, and ends at $(4, 0, 1)$ (Figure 14.4b).

b. Letting the parameter vary over the interval $-\infty < t < \infty$ generates a spiral that winds around the z-axis endlessly in both directions. The positive orientation is in the upward direction (increasing z-direction). Noticing once more that $x = 4\cos t$ and $y = \sin t$ are x- and y-components of $\mathbf{r}$, we see that the spiral lies on the elliptical cylinder $\left(\frac{x}{4}\right)^2 + y^2 = \cos^2 t + \sin^2 t = 1$ (Figure 14.4c).



➤ Recall that the functions $\sin at$ and $\cos at$ oscillate a times over the interval $[0, 2\pi]$. Therefore, their period is $2\pi/a$.

Figure 14.4

Related Exercise 24 ◀

EXAMPLE 3 **Roller coaster curve** Graph the curve

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + 0.4 \sin 2t\,\mathbf{k}, \quad \text{for } 0 \le t \le 2\pi.$$

SOLUTION Without the z-component, the resulting function $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$ describes a circle of radius 1 in the xy-plane. The z-component of the function varies between -0.4 and 0.4 with a period of π units. Therefore, on the interval $[0, 2\pi]$, the z-coordinates of points on the curve oscillate twice between -0.4 and 0.4, while the x- and y-coordinates describe a circle. The result is a curve that circles the z-axis once in the counterclockwise direction with two peaks and two valleys (Figure 14.5a).

The space curve in this example is not particularly complicated, but visualizing a given curve is easier when we determine the surface(s) on which it lies. Writing the vector function $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + 0.4 \sin 2t\,\mathbf{k}$ in parametric form, we have

$$x = \cos t, \quad y = \sin t, \quad z = 0.4 \sin 2t, \quad \text{for } 0 \le t \le 2\pi.$$

Noting that $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$, we conclude that the curve lies on the cylinder $x^2 + y^2 = 1$. In this case, we can also eliminate the parameter by writing

$$\begin{aligned} z &= 0.4 \sin 2t \\ &= 0.4(2\underbrace{\sin t}_{y}\underbrace{\cos t}_{x}) && \text{Double angle identity: } \sin 2t = 2 \sin t \cos t \\ &= 0.8xy, && x = \cos t,\ y = \sin t \end{aligned}$$

which implies that the curve also lies on the hyperbolic paraboloid $z = 0.8xy$ (see margin note). In fact, the roller coaster curve is the curve in which the surfaces $x^2 + y^2 = 1$ and $z = 0.8xy$ intersect, as shown in Figure 14.5b.

➤ The graph of $z = 0.8xy$ is a rotation of the quadric surface $z = 0.4(x^2 - y^2)$, which we recognize as a hyperbolic paraboloid. See the Guided Project *Translation and rotation of axes*.

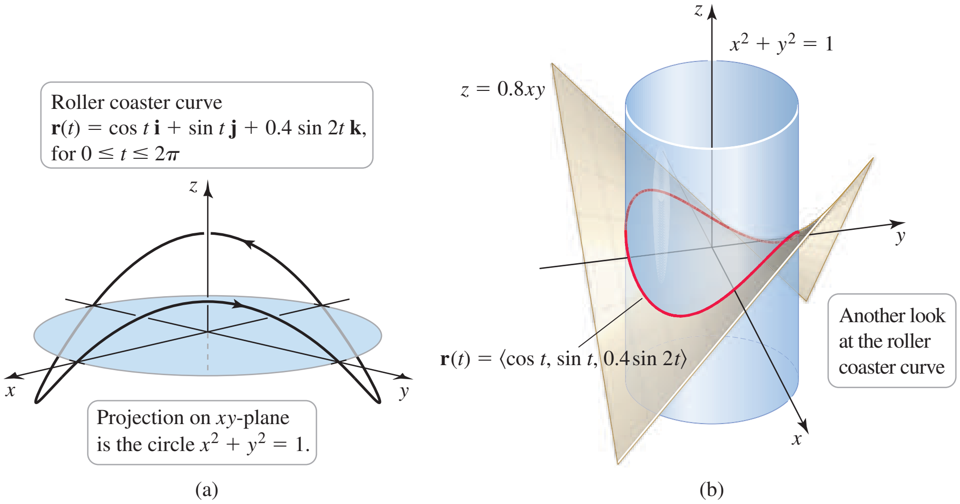


Figure 14.5

Related Exercises 28, 55 ◀

EXAMPLE 4 **Slinky curve** Use a graphing utility to graph the curve

$$\mathbf{r}(t) = (3 + \cos 15t) \cos t\,\mathbf{i} + (3 + \cos 15t) \sin t\,\mathbf{j} + \sin 15t\,\mathbf{k},$$

for $0 \le t \le 2\pi$, and discuss its properties.

SOLUTION The factor $A(t) = 3 + \cos 15t$ that appears in the x- and y-components is a varying amplitude for $\cos t\,\mathbf{i}$ and $\sin t\,\mathbf{j}$. Its effect is seen in the graph of the x-component $A(t) \cos t$ (Figure 14.6a). For $0 \le t \le 2\pi$, the curve consists of one period of $3 \cos t$ with 15 small oscillations superimposed on it. As a result, the x-component of $\mathbf{r}$ varies from -4 to 4 with 15 small oscillations along the way. A similar behavior is seen in the y-component of $\mathbf{r}$. Finally, the z-component of $\mathbf{r}$, which is $\sin 15t$, oscillates between -1 and 1 fifteen times over $[0, 2\pi]$. Combining these effects, we discover a coil-shaped curve that circles the z-axis in the counterclockwise direction and closes on itself. Figures 14.6b and 14.6c show two views, one looking along the xy-plane and the other from overhead on the z-axis. It can be shown (Exercise 56) that eliminating the parameter from the parametric equations defining $\mathbf{r}$ leads to a standard equation of a torus in Cartesian coordinates—in this case, $(3 - \sqrt{x^2 + y^2})^2 + z^2 = 1$—and therefore, the curve lies on this torus, as seen in Figure 14.6b.

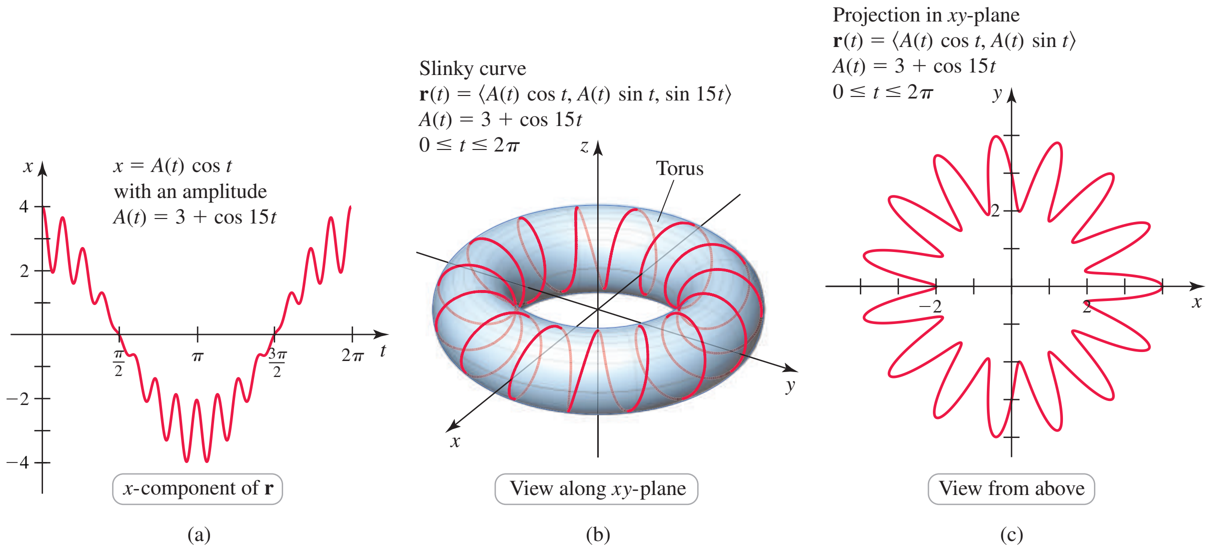


Figure 14.6

Related Exercises 27, 56 ◄

Limits and Continuity for Vector-Valued Functions

We have presented vector valued functions and established their relationship to parametric equations. The next step is to investigate the calculus of vector-valued functions. The concepts of limits, derivatives, and integrals of vector-valued functions are direct extensions of what you have already learned.

The limit of a vector-valued function $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ is defined much as it is for scalar-valued functions. If there is a vector $\mathbf{L}$ such that the scalar function $|\mathbf{r}(t) - \mathbf{L}|$ can be made arbitrarily small by taking t sufficiently close to a, then we write $\lim_{t \to a} \mathbf{r}(t) = \mathbf{L}$ and say the limit of $\mathbf{r}$ as t approaches a is $\mathbf{L}$.

> **DEFINITION Limit of a Vector-Valued Function**
>
> A vector-valued function $\mathbf{r}$ approaches the limit $\mathbf{L}$ as t approaches a, written $\lim_{t \to a} \mathbf{r}(t) = \mathbf{L}$, provided $\lim_{t \to a} |\mathbf{r}(t) - \mathbf{L}| = 0$.

Notice that while $\mathbf{r}$ is vector valued, $|\mathbf{r}(t) - \mathbf{L}|$ is a function of the single variable t, to which our familiar limit theorems apply. Therefore, this definition and a short calculation (Exercise 66) lead to a straightforward method for computing limits of the vector-valued function $\mathbf{r} = \langle f, g, h \rangle$. Suppose

$$\lim_{t \to a} f(t) = L_1, \qquad \lim_{t \to a} g(t) = L_2, \qquad \text{and} \qquad \lim_{t \to a} h(t) = L_3.$$

Then

$$\lim_{t \to a} \mathbf{r}(t) = \left\langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \right\rangle = \langle L_1, L_2, L_3 \rangle.$$

In other words, the limit of $\mathbf{r}$ is determined by computing the limits of its components.

The limits laws in Chapter 2 have analogs for vector-valued functions. For example, if $\lim_{t \to a} \mathbf{r}(t)$ and $\lim_{t \to a} \mathbf{s}(t)$ exist and c is a scalar, then

$$\lim_{t \to a} (\mathbf{r}(t) + \mathbf{s}(t)) = \lim_{t \to a} \mathbf{r}(t) + \lim_{t \to a} \mathbf{s}(t) \quad \text{and} \quad \lim_{t \to a} c\mathbf{r}(t) = c \lim_{t \to a} \mathbf{r}(t).$$

The idea of continuity also extends directly to vector-valued functions. A function $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ is **continuous at *a*** provided $\lim_{t \to a} \mathbf{r}(t) = \mathbf{r}(a)$. Specifically,

➤ Continuity is often taken as part of the definition of a parameterized curve.

if the component functions f, g, and h are continuous at a, then $\mathbf{r}$ is also continuous at a, and vice versa. The function $\mathbf{r}$ is **continuous on an interval I** if it is continuous for all t in I.

Continuity has the same intuitive meaning in this setting as it does for scalar-valued functions. If $\mathbf{r}$ is continuous on an interval, the curve it describes has no breaks or gaps, which is an important property when $\mathbf{r}$ describes the trajectory of an object.

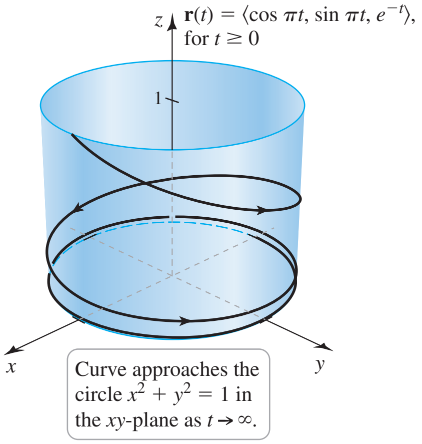


Figure 14.7

QUICK CHECK 2 Explain why the curve in Example 5 lies on the cylinder $x^2 + y^2 = 1$, as shown in Figure 14.7. ◄

EXAMPLE 5 **Limits and continuity** Consider the function

$$\mathbf{r}(t) = \cos \pi t\, \mathbf{i} + \sin \pi t\, \mathbf{j} + e^{-t}\, \mathbf{k}, \quad \text{for } t \geq 0.$$

a. Evaluate $\lim_{t \to 2} \mathbf{r}(t)$.

b. Evaluate $\lim_{t \to \infty} \mathbf{r}(t)$.

c. At what points is $\mathbf{r}$ continuous?

SOLUTION

a. We evaluate the limit of each component of $\mathbf{r}$:

$$\lim_{t \to 2} \mathbf{r}(t) = \lim_{t \to 2} (\underbrace{\cos \pi t}_{\to 1}\, \mathbf{i} + \underbrace{\sin \pi t}_{\to 0}\, \mathbf{j} + \underbrace{e^{-t}}_{\to e^{-2}}\mathbf{k}) = \mathbf{i} + e^{-2}\mathbf{k}.$$

b. Note that although $\lim_{t \to \infty} e^{-t} = 0$, $\lim_{t \to \infty} \cos t$ and $\lim_{t \to \infty} \sin t$ do not exist. Therefore, $\lim_{t \to \infty} \mathbf{r}(t)$ does not exist. As shown in Figure 14.7, the curve is a coil that approaches the unit circle in the xy-plane.

c. Because the components of $\mathbf{r}$ are continuous for all t, $\mathbf{r}$ is also continuous for all t.

Related Exercise 31 ◄

SECTION 14.1 EXERCISES

Getting Started

1. How many independent variables does the function $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ have?

2. How many dependent scalar variables does the function $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ have?

3. Why is $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ called a vector-valued function?

4. In what plane does the curve $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{k}$ lie?

5. How do you evaluate $\lim_{t \to a} \mathbf{r}(t)$, where $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$?

6. How do you determine whether $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ is continuous at $t = a$?

7. Find a function $\mathbf{r}(t)$ for the line passing through the points $P(0, 0, 0)$ and $Q(1, 2, 3)$. Express your answer in terms of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

8. Find a function $\mathbf{r}(t)$ whose graph is a circle of radius 1 parallel to the xy-plane and centered at $(0, 0, 10)$.

Practice Exercises

9–14. Lines and line segments *Find a function* $\mathbf{r}(t)$ *that describes the given line or line segment.*

9. The line through $P(2, 3, 7)$ and $Q(4, 6, 3)$

10. The line through $P(0, -3, 2)$ that is parallel to the line $\mathbf{r}(t) = \langle 4, 6 - t, 1 + t \rangle$

11. The line through $P(3, 4, 5)$ that is orthogonal to the plane $2x - z = 4$

12. The line of intersection of the planes $2x + 3y + 4z = 7$ and $2x + 3y + 5z = 8$

13. The line segment from $P(1, 2, 1)$ to $Q(0, 2, 3)$

14. The line segment from $P(-4, -2, 1)$ to $Q(-2, -2, 3)$

15–26. Graphing curves *Graph the curves described by the following functions, indicating the positive orientation.*

15. $\mathbf{r}(t) = \langle 2 \cos t, 2 \sin t \rangle$, for $0 \leq t \leq 2\pi$

16. $\mathbf{r}(t) = \langle 1 + \cos t, 2 + \sin t \rangle$, for $0 \leq t \leq 2\pi$

17. $\mathbf{r}(t) = \langle t, 2t \rangle$, for $0 \leq t \leq 1$

18. $\mathbf{r}(t) = \langle 3 \cos t, 2 \sin t \rangle$, for $0 \leq t \leq 2\pi$

19. $\mathbf{r}(t) = \langle \cos t, 0, \sin t \rangle$, for $0 \leq t \leq 2\pi$

20. $\mathbf{r}(t) = \langle 0, 4 \cos t, 16 \sin t \rangle$, for $0 \leq t \leq 2\pi$

21. $\mathbf{r}(t) = \cos t\,\mathbf{i} + \mathbf{j} + \sin t\,\mathbf{k}$, for $0 \leq t \leq 2\pi$

22. $\mathbf{r}(t) = 2 \cos t\,\mathbf{i} + 2 \sin t\,\mathbf{j} + 2\,\mathbf{k}$, for $0 \leq t \leq 2\pi$

T **23.** $\mathbf{r}(t) = t \cos t\,\mathbf{i} + t \sin t\,\mathbf{j} + t\,\mathbf{k}$, for $0 \leq t \leq 6\pi$

T **24.** $\mathbf{r}(t) = 4 \sin t\,\mathbf{i} + 4 \cos t\,\mathbf{j} + e^{-t/10}\,\mathbf{k}$, for $0 \leq t < \infty$

T **25.** $\mathbf{r}(t) = e^{-t/20} \sin t\,\mathbf{i} + e^{-t/20} \cos t\,\mathbf{j} + t\,\mathbf{k}$, for $0 \leq t < \infty$

T **26.** $\mathbf{r}(t) = e^{-t/10}\,\mathbf{i} + 3 \cos t\,\mathbf{j} + 3 \sin t\,\mathbf{k}$, for $0 \leq t < \infty$

T **27–30. Exotic curves** *Graph the curves described by the following functions. Use analysis to anticipate the shape of the curve before using a graphing utility.*

27. $\mathbf{r}(t) = \cos 15t\,\mathbf{i} + (4 + \sin 15t) \cos t\,\mathbf{j} + (4 + \sin 15t) \sin t\,\mathbf{k}$, for $0 \leq t \leq 2\pi$

28. $\mathbf{r}(t) = 2 \cos t\,\mathbf{i} + 4 \sin t\,\mathbf{j} + \cos 10t\,\mathbf{k}$, for $0 \leq t \leq 2\pi$

29. $\mathbf{r}(t) = \sin t\,\mathbf{i} + \sin^2 t\,\mathbf{j} + \dfrac{t}{5\pi}\mathbf{k}$, for $0 \le t \le 10\pi$

30. $\mathbf{r}(t) = \cos t \sin 3t\,\mathbf{i} + \sin t \sin 3t\,\mathbf{j} + \sqrt{t}\,\mathbf{k}$, for $0 \le t \le 9$

31–36. Limits *Evaluate the following limits.*

31. $\displaystyle\lim_{t\to\pi/2}\left(\cos 2t\,\mathbf{i} - 4\sin t\,\mathbf{j} + \frac{2t}{\pi}\mathbf{k}\right)$

32. $\displaystyle\lim_{t\to\ln 2}\left(2e^t\,\mathbf{i} + 6e^{-t}\,\mathbf{j} - 4e^{-2t}\,\mathbf{k}\right)$

33. $\displaystyle\lim_{t\to\infty}\left(e^{-t}\,\mathbf{i} - \frac{2t}{t+1}\mathbf{j} + \tan^{-1} t\,\mathbf{k}\right)$

34. $\displaystyle\lim_{t\to 2}\left(\frac{t}{t^2+1}\mathbf{i} - 4e^{-t}\sin \pi t\,\mathbf{j} + \frac{1}{\sqrt{4t+1}}\mathbf{k}\right)$

35. $\displaystyle\lim_{t\to 0}\left(\frac{\sin t}{t}\mathbf{i} - \frac{e^t - t - 1}{t}\mathbf{j} + \frac{\cos t + t^2/2 - 1}{t^2}\mathbf{k}\right)$

36. $\displaystyle\lim_{t\to 0}\left(\frac{\tan t}{t}\mathbf{i} - \frac{3t}{\sin t}\mathbf{j} + \sqrt{t+1}\,\mathbf{k}\right)$

37. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. The projection of the curve $\mathbf{r}(t) = \langle t, \cos t, t^2\rangle$ in the xz-plane is a parabola.

b. The curve $\mathbf{r}(t) = \langle \sin t, \cos t, \sin t\rangle$ lies on a unit sphere.

c. The curve $\mathbf{r}(t) = \langle e^{-t}, \sin t, -\cos t\rangle$ approaches a circle as $t \to \infty$.

d. If $\mathbf{r}(t) = e^{-t^2}\langle 1, 1, 1\rangle$, then $\displaystyle\lim_{t\to\infty}\mathbf{r}(t) = \lim_{t\to-\infty}\mathbf{r}(t)$.

38–41. Domains *Find the domain of the following vector-valued functions.*

38. $\mathbf{r}(t) = \dfrac{2}{t-1}\mathbf{i} + \dfrac{3}{t+2}\mathbf{j}$

39. $\mathbf{r}(t) = \sqrt{t+2}\,\mathbf{i} + \sqrt{2-t}\,\mathbf{j}$

40. $\mathbf{r}(t) = \cos 2t\,\mathbf{i} + e^{\sqrt{t}}\,\mathbf{j} + \dfrac{12}{t}\mathbf{k}$

41. $\mathbf{r}(t) = \sqrt{4-t^2}\,\mathbf{i} + \sqrt{t}\,\mathbf{j} - \dfrac{2}{\sqrt{1+t}}\mathbf{k}$

42–44. Curve-plane intersections *Find the points (if they exist) at which the following planes and curves intersect.*

42. $y = 1$; $\mathbf{r}(t) = \langle 10\cos t, 2\sin t, 1\rangle$, for $0 \le t \le 2\pi$

43. $z = 16$; $\mathbf{r}(t) = \langle t, 2t, 4 + 3t\rangle$, for $-\infty < t < \infty$

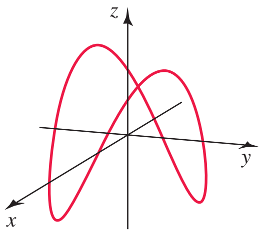
44. $y + x = 0$; $\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle$, for $0 \le t \le 4\pi$

45. Matching functions with graphs Match functions a–f with the appropriate graphs A–F.

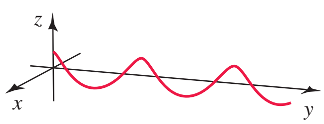
a. $\mathbf{r}(t) = \langle t, -t, t\rangle$ **b.** $\mathbf{r}(t) = \langle t^2, t, t\rangle$

c. $\mathbf{r}(t) = \langle 4\cos t, 4\sin t, 2\rangle$ **d.** $\mathbf{r}(t) = \langle 2t, \sin t, \cos t\rangle$

e. $\mathbf{r}(t) = \langle \sin t, \cos t, \sin 2t\rangle$ **f.** $\mathbf{r}(t) = \langle \sin t, 2t, \cos t\rangle$



(A)



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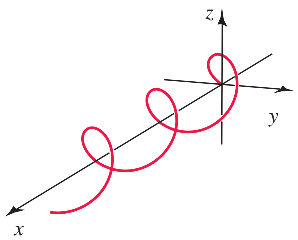

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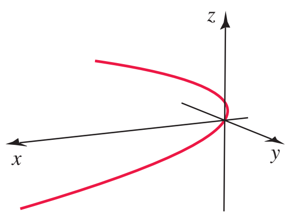

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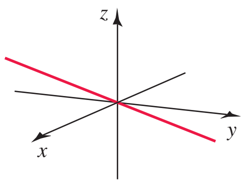
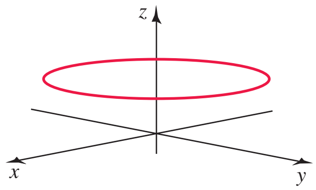


(E)



(F)

46. Upward path Consider the curve described by the vector function $\mathbf{r}(t) = (50e^{-t}\cos t)\mathbf{i} + (50e^{-t}\sin t)\mathbf{j} + (5 - 5e^{-t})\mathbf{k}$, for $t \ge 0$.

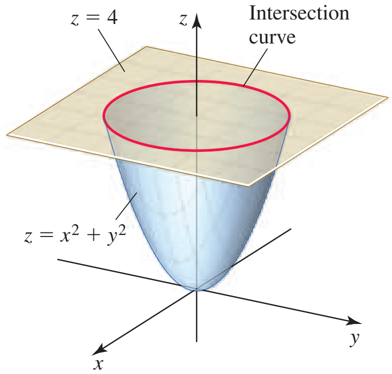
a. What is the initial point of the path corresponding to $\mathbf{r}(0)$?

b. What is $\displaystyle\lim_{t\to\infty}\mathbf{r}(t)$?

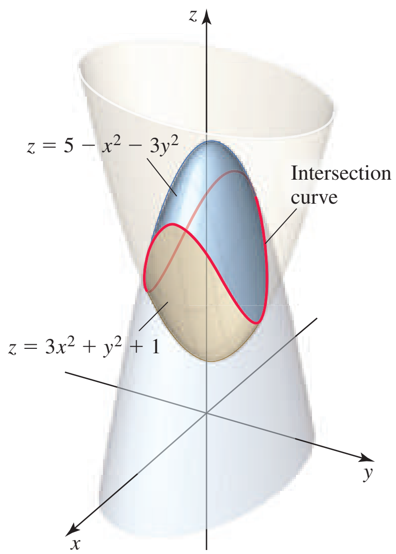
c. Eliminate the parameter t to show that the curve $\mathbf{r}(t)$ lies on the surface $z = 5 - r/10$, where $r^2 = x^2 + y^2$.

47–50. Curve of intersection *Find a function $\mathbf{r}(t)$ that describes the curve where the following surfaces intersect. Answers are not unique.*

47. $z = 4$; $z = x^2 + y^2$



48. $z = 3x^2 + y^2 + 1$; $z = 5 - x^2 - 3y^2$



49. $x^2 + y^2 = 25; z = 2x + 2y$

50. $z = y + 1; z = x^2 + 1$

T 51. Golf slice A golfer launches a tee shot down a horizontal fairway; it follows a path given by $\mathbf{r}(t) = \langle at, (75 - 0.1a)t, -5t^2 + 80t \rangle$, where $t \geq 0$ measures time in seconds and $\mathbf{r}$ has units of feet. The y-axis points straight down the fairway and the z-axis points vertically upward. The parameter a is the slice factor that determines how much the shot deviates from a straight path down the fairway.

- **a.** With no slice ($a = 0$), describe the shot. How far does the ball travel horizontally (the distance between the point where the ball leaves the ground and the point where it first strikes the ground)?
- **b.** With a slice ($a = 0.2$), how far does the ball travel horizontally?
- **c.** How far does the ball travel horizontally with $a = 2.5$?

52–56. Curves on surfaces *Verify that the curve* $\mathbf{r}(t)$ *lies on the given surface. Give the name of the surface.*

52. $\mathbf{r}(t) = (t \cos t)\mathbf{i} + (t \sin t)\mathbf{j} + t\,\mathbf{k}; x^2 + y^2 = z^2$

53. $\mathbf{r}(t) = \left(\sqrt{t^2 + 1} \cos t\right)\mathbf{i} + \left(\sqrt{t^2 + 1} \sin t\right)\mathbf{j} + t\,\mathbf{k};$ $x^2 + y^2 - z^2 = 1$

54. $\mathbf{r}(t) = \langle \sqrt{t} \cos t, \sqrt{t} \sin t, t \rangle; z = x^2 + y^2$

55. $\mathbf{r}(t) = \langle 0, 2 \cos t, 3 \sin t \rangle; x^2 + \dfrac{y^2}{4} + \dfrac{z^2}{9} = 1$

56. $\mathbf{r}(t) = \langle (3 + \cos 15t) \cos t, (3 + \cos 15t) \sin t, \sin 15t \rangle;$ $\left(3 - \sqrt{x^2 + y^2}\right)^2 + z^2 = 1$ (*Hint:* See Example 4.)

57–58. Closest point on a curve *Find the point P on the curve* $\mathbf{r}(t)$ *that lies closest to P_0 and state the distance between P_0 and P.*

57. $\mathbf{r}(t) = t^2\,\mathbf{i} + t\,\mathbf{j} + t\,\mathbf{k}; P_0(1, 1, 15)$

T 58. $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}; P_0(1, 1, 3)$

Explorations and Challenges

59–61. Curves on spheres

T 59. Graph the curve $\mathbf{r}(t) = \left\langle \dfrac{1}{2} \sin 2t, \dfrac{1}{2}(1 - \cos 2t), \cos t \right\rangle$ and prove that it lies on the surface of a sphere centered at the origin.

60. Prove that for integers m and n, the curve

$$\mathbf{r}(t) = \langle a \sin mt \cos nt, b \sin mt \sin nt, c \cos mt \rangle$$

lies on the surface of a sphere provided $a^2 = b^2 = c^2$.

61. Find the period of the function in Exercise 60; that is, in terms of m and n, find the smallest positive real number T such that $\mathbf{r}(t + T) = \mathbf{r}(t)$ for all t.

62–65. Closed plane curves *Consider the curve*

$$\mathbf{r}(t) = \langle a \cos t + b \sin t, c \cos t + d \sin t, e \cos t + f \sin t \rangle,$$

where a, b, c, d, e, and f are real numbers. It can be shown that this curve lies in a plane.

62. Assuming the curve lies in a plane, show that it is a circle centered at the origin with radius R provided $a^2 + c^2 + e^2 = b^2 + d^2 + f^2 = R^2$ and $ab + cd + ef = 0$.

T 63. Graph the following curve and describe it.

$$\mathbf{r}(t) = \left(\frac{1}{\sqrt{2}} \cos t + \frac{1}{\sqrt{3}} \sin t\right)\mathbf{i} + \left(-\frac{1}{\sqrt{2}} \cos t + \frac{1}{\sqrt{3}} \sin t\right)\mathbf{j} + \left(\frac{1}{\sqrt{3}} \sin t\right)\mathbf{k}$$

T 64. Graph the following curve and describe it.

$$\mathbf{r}(t) = (2 \cos t + 2 \sin t)\mathbf{i} + (-\cos t + 2 \sin t)\mathbf{j} + (\cos t - 2 \sin t)\mathbf{k}$$

65. Find a general expression for a nonzero vector orthogonal to the plane containing the curve

$$\mathbf{r}(t) = \langle a \cos t + b \sin t, c \cos t + d \sin t, e \cos t + f \sin t \rangle,$$

where $\langle a, c, e \rangle \times \langle b, d, f \rangle \neq \mathbf{0}$.

66. Limits of vector functions Let $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$.

- **a.** Assume $\lim_{t \to a} \mathbf{r}(t) = \mathbf{L} = \langle L_1, L_2, L_3 \rangle$, which means that $\lim_{t \to a} |\mathbf{r}(t) - \mathbf{L}| = 0$. Prove that

$$\lim_{t \to a} f(t) = L_1, \quad \lim_{t \to a} g(t) = L_2, \quad \text{and} \quad \lim_{t \to a} h(t) = L_3.$$

- **b.** Assume $\lim_{t \to a} f(t) = L_1$, $\lim_{t \to a} g(t) = L_2$, and $\lim_{t \to a} h(t) = L_3$. Prove that $\lim_{t \to a} \mathbf{r}(t) = \mathbf{L} = \langle L_1, L_2, L_3 \rangle$, which means that $\lim_{t \to a} |\mathbf{r}(t) - \mathbf{L}| = 0$.

QUICK CHECK ANSWERS

1. $0 \leq t \leq 2$ **2.** The x- and y-components of the curve are $x = \cos \pi t$ and $y = \sin \pi t$, and $x^2 + y^2 = \cos^2 \pi t + \sin^2 \pi t = 1$. ➤

14.2 Calculus of Vector-Valued Functions

We now turn to the topic of ultimate interest in this chapter: the calculus of vector-valued functions. Everything you learned about differentiating and integrating functions of the form $y = f(x)$ carries over to vector-valued functions $\mathbf{r}(t)$; you simply apply the rules of differentiation and integration to the individual components of $\mathbf{r}$.

The Derivative and Tangent Vector

Consider the function $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$, where f, g, and h are differentiable functions on an interval $a < t < b$. The first task is to explain the meaning of the *derivative* of a vector-valued function and to show how to compute it. We begin with the definition of the derivative—now from a vector perspective:

$$\mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}.$$

Let's first look at the geometry of this limit. The function $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$ describes a parameterized curve in space. Let P be a point on that curve associated with the position vector $\mathbf{r}(t)$, and let Q be a nearby point associated with the position vector $\mathbf{r}(t + \Delta t)$, where $\Delta t > 0$ is a small increment in t (**Figure 14.8a**). The difference $\Delta \mathbf{r} = \mathbf{r}(t + \Delta t) - \mathbf{r}(t)$ is the vector $\overrightarrow{PQ}$, where we assume $\Delta \mathbf{r} \neq \mathbf{0}$. Because Δt is a scalar, the direction of $\Delta \mathbf{r}/\Delta t$ is the same as the direction of $\overrightarrow{PQ}$.

As Δt approaches 0, Q approaches P and the vector $\Delta \mathbf{r}/\Delta t$ approaches a limiting vector that we denote $\mathbf{r}'(t)$ (**Figure 14.8b**). This new vector $\mathbf{r}'(t)$ has two important interpretations.

➤ An analogous argument can be given for $\Delta t < 0$, with the same result. Figure 14.8 illustrates the tangent vector $\mathbf{r}'$ for $\Delta t > 0$.

- The vector $\mathbf{r}'(t)$ points in the direction of the curve at P. For this reason, $\mathbf{r}'(t)$ is a *tangent vector* at P (provided it is not the zero vector).
- The vector $\mathbf{r}'(t)$ is the *derivative* of $\mathbf{r}$ with respect to t; it gives the rate of change of the function $\mathbf{r}(t)$ at the point P. In fact, if $\mathbf{r}(t)$ is the position function of a moving object, then $\mathbf{r}'(t)$ is the velocity vector of the object, which always points in the direction of motion, and $|\mathbf{r}'(t)|$ is the speed of the object.

➤ Section 14.3 is devoted to problems of motion in two and three dimensions.

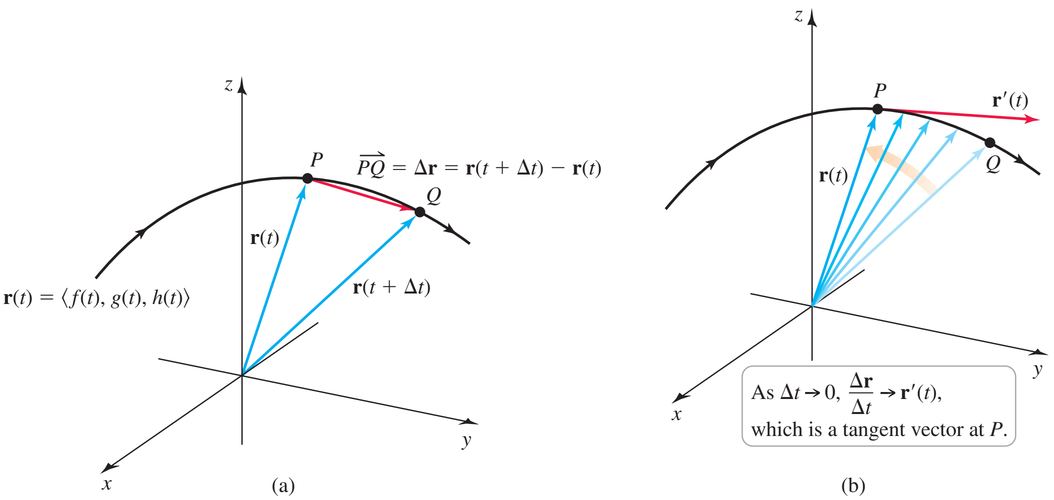


Figure 14.8

We now evaluate the limit that defines $\mathbf{r}'(t)$ by expressing $\mathbf{r}$ in terms of its components and using the properties of limits.

$$\begin{aligned}
\mathbf{r}'(t) &= \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{(f(t + \Delta t)\,\mathbf{i} + g(t + \Delta t)\,\mathbf{j} + h(t + \Delta t)\,\mathbf{k}) - (f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k})}{\Delta t} && \text{Substitute components of } \mathbf{r}. \\
&= \lim_{\Delta t \to 0} \left(\frac{f(t + \Delta t) - f(t)}{\Delta t}\,\mathbf{i} + \frac{g(t + \Delta t) - g(t)}{\Delta t}\,\mathbf{j} + \frac{h(t + \Delta t) - h(t)}{\Delta t}\,\mathbf{k} \right) && \text{Rearrange terms inside of limit.} \\
&= \underbrace{\lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}}_{f'(t)}\,\mathbf{i} + \underbrace{\lim_{\Delta t \to 0} \frac{g(t + \Delta t) - g(t)}{\Delta t}}_{g'(t)}\,\mathbf{j} + \underbrace{\lim_{\Delta t \to 0} \frac{h(t + \Delta t) - h(t)}{\Delta t}}_{h'(t)}\,\mathbf{k} && \text{Limit of sum equals sum of limits.}
\end{aligned}$$

Because f, g, and h are differentiable scalar-valued functions of the variable t, the three limits in the last step are identified as the derivatives of f, g, and h, respectively. Therefore, there are no surprises:

$$\mathbf{r}'(t) = f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j} + h'(t)\,\mathbf{k}.$$

In other words, to differentiate the vector-valued function $\mathbf{r}(t)$, we simply differentiate each of its components with respect to t.

DEFINITION Derivative and Tangent Vector

Let $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$, where f, g, and h are differentiable functions on (a, b). Then $\mathbf{r}$ has a **derivative** (or is **differentiable**) on (a, b) and

$$\mathbf{r}'(t) = f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j} + h'(t)\,\mathbf{k}.$$

Provided $\mathbf{r}'(t) \neq \mathbf{0}$, $\mathbf{r}'(t)$ is a **tangent vector** at the point corresponding to $\mathbf{r}(t)$.

EXAMPLE 1 Derivative of vector functions Compute the derivative of the following functions.

a. $\mathbf{r}(t) = \langle t^3, 3t^2, t^3/6 \rangle$ **b.** $\mathbf{r}(t) = e^{-t}\,\mathbf{i} + 10\sqrt{t}\,\mathbf{j} + 2\cos 3t\,\mathbf{k}$

SOLUTION

a. $\mathbf{r}'(t) = \langle 3t^2, 6t, t^2/2 \rangle$; note that $\mathbf{r}$ is differentiable for all t and $\mathbf{r}'(0) = \mathbf{0}$.

b. $\mathbf{r}'(t) = -e^{-t}\,\mathbf{i} + \dfrac{5}{\sqrt{t}}\,\mathbf{j} - 6\sin 3t\,\mathbf{k}$; the function $\mathbf{r}$ is differentiable for $t > 0$.

Related Exercises 11–12

QUICK CHECK 1 Let $\mathbf{r}(t) = \langle t, t, t \rangle$. Compute $\mathbf{r}'(t)$ and interpret the result.

The condition that $\mathbf{r}'(t) \neq \mathbf{0}$ in order for the tangent vector to be defined requires explanation. Consider the function $\mathbf{r}(t) = \langle t^3, 3t^2, t^3/6 \rangle$. As shown in Example 1a, $\mathbf{r}'(0) = \mathbf{0}$; that is, all three components of $\mathbf{r}'(t)$ are zero simultaneously when $t = 0$. We see in Figure 14.9 that this otherwise smooth curve has a *cusp*, or a sharp point, at the origin. If $\mathbf{r}$ describes the motion of an object, then $\mathbf{r}'(t) = \mathbf{0}$ means that the velocity (and speed) of the object is zero at a point. At such a stationary point, the object *may* change direction abruptly, creating a cusp in its trajectory. For this reason, we say a function $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ is **smooth** on an interval if f, g, and h are differentiable *and* $\mathbf{r}'(t) \neq \mathbf{0}$ on that interval. Smooth curves have no cusps or corners.

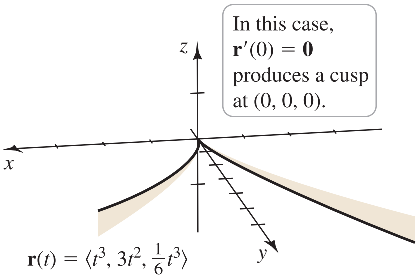


Figure 14.9

➤ If a curve has a cusp at a point, then $\mathbf{r}'(t) = \mathbf{0}$ at that point. However, the converse is not true; it may happen that $\mathbf{r}'(t) = \mathbf{0}$ at a point that is not a cusp (Exercise 95).

Unit Tangent Vector In situations in which only the direction (but not the length) of the tangent vector is of interest, we work with the *unit tangent vector*. It is the vector with magnitude 1, formed by dividing $\mathbf{r}'(t)$ by its length.

DEFINITION Unit Tangent Vector

Let $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$ be a smooth parameterized curve, for $a \leq t \leq b$. The **unit tangent vector** for a particular value of t is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}.$$

QUICK CHECK 2 Suppose $\mathbf{r}'(t)$ has units of m/s. Explain why $\mathbf{T}(t) = \mathbf{r}'(t)/|\mathbf{r}'(t)|$ is dimensionless (has no units) and carries information only about direction.

EXAMPLE 2 Unit tangent vectors Find the unit tangent vectors for the following parameterized curves.

a. $\mathbf{r}(t) = \langle t^2, 4t, 4\ln t \rangle$, for $t > 0$

b. $\mathbf{r}(t) = \langle 10, 3\cos t, 3\sin t \rangle$, for $0 \leq t \leq 2\pi$

SOLUTION

a. A tangent vector is $\mathbf{r}'(t) = \langle 2t, 4, 4/t \rangle$, which has a magnitude of

$$\begin{aligned} |\mathbf{r}'(t)| &= \sqrt{(2t)^2 + 4^2 + \left(\frac{4}{t}\right)^2} && \text{Definition of magnitude} \\ &= \sqrt{4t^2 + 16 + \frac{16}{t^2}} && \text{Expand.} \\ &= \sqrt{\left(2t + \frac{4}{t}\right)^2} && \text{Factor.} \\ &= 2t + \frac{4}{t}. && \text{Simplify.} \end{aligned}$$

Therefore, the unit tangent vector for a particular value of t is

$$\mathbf{T}(t) = \frac{\langle 2t, 4, 4/t \rangle}{2t + 4/t}.$$

As shown in **Figure 14.10**, the unit tangent vectors change direction along the curve but maintain unit length.

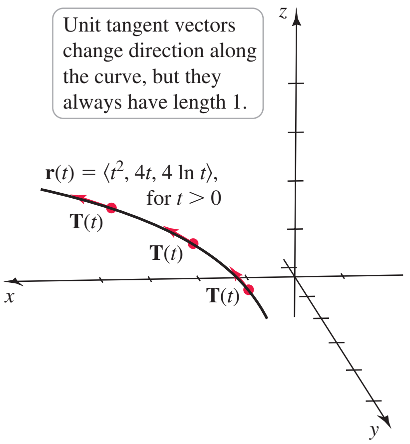


Figure 14.10

b. In this case, $\mathbf{r}'(t) = \langle 0, -3 \sin t, 3 \cos t \rangle$ and

$$|\mathbf{r}'(t)| = \sqrt{0^2 + (-3 \sin t)^2 + (3 \cos t)^2} = \sqrt{9\underbrace{(\sin^2 t + \cos^2 t)}_{1}} = 3.$$

Therefore, the unit tangent vector for a particular value of t is

$$\mathbf{T}(t) = \frac{1}{3}\langle 0, -3 \sin t, 3 \cos t \rangle = \langle 0, -\sin t, \cos t \rangle.$$

The direction of $\mathbf{T}$ changes along the curve, but its length remains 1.

Related Exercises 25, 27 ◄

Derivative Rules The rules for derivatives for single-variable functions either carry over directly to vector-valued functions or have close analogs. These rules are generally proved by working on the individual components of the vector function.

THEOREM 14.1 Derivative Rules

Let $\mathbf{u}$ and $\mathbf{v}$ be differentiable vector-valued functions, and let f be a differentiable scalar-valued function, all at a point t. Let $\mathbf{c}$ be a constant vector. The following rules apply.

1. $\frac{d}{dt}(\mathbf{c}) = \mathbf{0}$ Constant Rule
2. $\frac{d}{dt}(\mathbf{u}(t) + \mathbf{v}(t)) = \mathbf{u}'(t) + \mathbf{v}'(t)$ Sum Rule
3. $\frac{d}{dt}(f(t)\mathbf{u}(t)) = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$ Product Rule
4. $\frac{d}{dt}\big(\mathbf{u}(f(t))\big) = \mathbf{u}'(f(t))f'(t)$ Chain Rule
5. $\frac{d}{dt}(\mathbf{u}(t) \cdot \mathbf{v}(t)) = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$ Dot Product Rule
6. $\frac{d}{dt}(\mathbf{u}(t) \times \mathbf{v}(t)) = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$ Cross Product Rule

➤ With the exception of the Cross Product Rule, these rules apply to vector-valued functions with any number of components. Notice that we have three new product rules, all of which mimic the original Product Rule. In Rule 4, $\mathbf{u}$ must be differentiable at $f(t)$.

QUICK CHECK 3 Let $\mathbf{u}(t) = \langle t, t, t \rangle$ and $\mathbf{v}(t) = \langle 1, 1, 1 \rangle$. Compute $\frac{d}{dt}(\mathbf{u}(t) \cdot \mathbf{v}(t))$ using Derivative Rule 5, and show that it agrees with the result obtained by first computing the dot product and differentiating directly. ◄

The proofs of these rules are assigned in Exercises 92–94 with the exception of the following representative proofs.

Proof of the Chain Rule: Let $\mathbf{u}(t) = \langle u_1(t), u_2(t), u_3(t) \rangle$, which implies that

$$\mathbf{u}(f(t)) = u_1(f(t))\,\mathbf{i} + u_2(f(t))\,\mathbf{j} + u_3(f(t))\,\mathbf{k}.$$

We now apply the ordinary Chain Rule componentwise:

$$\begin{aligned}
\frac{d}{dt}\big(\mathbf{u}(f(t))\big) &= \frac{d}{dt}\big(u_1(f(t))\,\mathbf{i} + u_2(f(t))\,\mathbf{j} + u_3(f(t))\,\mathbf{k}\big) && \text{Components of } \mathbf{u}\\
&= \frac{d}{dt}\big(u_1(f(t))\big)\,\mathbf{i} + \frac{d}{dt}\big(u_2(f(t))\big)\,\mathbf{j} + \frac{d}{dt}\big(u_3(f(t))\big)\,\mathbf{k} && \text{Differentiate each component.}\\
&= u_1'(f(t))f'(t)\,\mathbf{i} + u_2'(f(t))f'(t)\,\mathbf{j} + u_3'(f(t))f'(t)\,\mathbf{k} && \text{Chain Rule}\\
&= \big(u_1'(f(t))\,\mathbf{i} + u_2'(f(t))\,\mathbf{j} + u_3'(f(t))\,\mathbf{k}\big)f'(t) && \text{Factor } f'(t).\\
&= \mathbf{u}'(f(t))\,f'(t). && \text{Definition of } \mathbf{u}'
\end{aligned}$$

Proof of the Dot Product Rule: We use the standard Product Rule on each component. Let $\mathbf{u}(t) = \langle u_1(t), u_2(t), u_3(t)\rangle$ and $\mathbf{v}(t) = \langle v_1(t), v_2(t), v_3(t)\rangle$. Then

$$\begin{aligned}
\frac{d}{dt}(\mathbf{u}\cdot\mathbf{v}) &= \frac{d}{dt}(u_1v_1 + u_2v_2 + u_3v_3) && \text{Definition of dot product}\\
&= u_1'\,v_1 + u_1\,v_1' + u_2'\,v_2 + u_2\,v_2' + u_3'\,v_3 + u_3\,v_3' && \text{Product Rule}\\
&= \underbrace{u_1'\,v_1 + u_2'\,v_2 + u_3'\,v_3}_{\mathbf{u}'\cdot\mathbf{v}} + \underbrace{u_1\,v_1' + u_2\,v_2' + u_3\,v_3'}_{\mathbf{u}\cdot\mathbf{v}'} && \text{Rearrange.}\\
&= \mathbf{u}'\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{v}'.
\end{aligned}$$

EXAMPLE 3 **Derivative rules** Compute the following derivatives, where

$$\mathbf{u}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} - t^3\,\mathbf{k} \quad\text{and}\quad \mathbf{v}(t) = \sin t\,\mathbf{i} + 2\cos t\,\mathbf{j} + \cos t\,\mathbf{k}.$$

a. $\dfrac{d}{dt}(\mathbf{v}(t^2))$ **b.** $\dfrac{d}{dt}(t^2\,\mathbf{v}(t))$ **c.** $\dfrac{d}{dt}(\mathbf{u}(t)\cdot\mathbf{v}(t))$

SOLUTION

a. Note that $\mathbf{v}'(t) = \cos t\,\mathbf{i} - 2\sin t\,\mathbf{j} - \sin t\,\mathbf{k}$. Using the Chain Rule, we have

$$\frac{d}{dt}(\mathbf{v}(t^2)) = \mathbf{v}'(t^2)\frac{d}{dt}(t^2) = \underbrace{(\cos t^2\,\mathbf{i} - 2\sin t^2\,\mathbf{j} - \sin t^2\,\mathbf{k})}_{\mathbf{v}'(t^2)}(2t).$$

b.
$$\begin{aligned}
\frac{d}{dt}(t^2\,\mathbf{v}(t)) &= \frac{d}{dt}(t^2)\mathbf{v}(t) + t^2\frac{d}{dt}(\mathbf{v}(t)) && \text{Product Rule}\\
&= 2t\,\mathbf{v}(t) + t^2\,\mathbf{v}'(t)\\
&= 2t\underbrace{(\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j} + \cos t\,\mathbf{k})}_{\mathbf{v}(t)} + t^2\underbrace{(\cos t\,\mathbf{i} - 2\sin t\,\mathbf{j} - \sin t\,\mathbf{k})}_{\mathbf{v}'(t)} && \text{Differentiate.}\\
&= (2t\sin t + t^2\cos t)\,\mathbf{i} + (4t\cos t - 2t^2\sin t)\,\mathbf{j} + (2t\cos t - t^2\sin t)\,\mathbf{k} && \text{Collect terms.}
\end{aligned}$$

c.
$$\begin{aligned}
\frac{d}{dt}(\mathbf{u}(t)\cdot\mathbf{v}(t)) &= \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t) && \text{Dot Product Rule}\\
&= (\mathbf{i} + 2t\,\mathbf{j} - 3t^2\,\mathbf{k})\cdot(\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j} + \cos t\,\mathbf{k})\\
&\quad + (t\,\mathbf{i} + t^2\,\mathbf{j} - t^3\,\mathbf{k})\cdot(\cos t\,\mathbf{i} - 2\sin t\,\mathbf{j} - \sin t\,\mathbf{k}) && \text{Differentiate.}\\
&= (\sin t + 4t\cos t - 3t^2\cos t) + (t\cos t - 2t^2\sin t + t^3\sin t) && \text{Dot products}\\
&= (1 - 2t^2 + t^3)\sin t + (5t - 3t^2)\cos t && \text{Simplify.}
\end{aligned}$$

Note that the result is a scalar. The same result is obtained if you first compute $\mathbf{u}\cdot\mathbf{v}$ and then differentiate.

Related Exercises 33, 36, 37

Higher-Order Derivatives Higher-order derivatives of vector-valued functions are computed in the expected way: We simply differentiate each component multiple times. Second derivatives feature prominently in the next section, playing the role of acceleration.

EXAMPLE 4 **Higher-order derivatives** Compute the first, second, and third derivative of $\mathbf{r}(t) = \langle t^2, 8 \ln t, 3e^{-2t} \rangle$.

SOLUTION Differentiating once, we have $\mathbf{r}'(t) = \langle 2t, 8/t, -6e^{-2t} \rangle$. Differentiating again produces $\mathbf{r}''(t) = \langle 2, -8/t^2, 12e^{-2t} \rangle$. Differentiating once more, we have $\mathbf{r}'''(t) = \langle 0, 16/t^3, -24e^{-2t} \rangle$.

Related Exercise 58 ◄

Integrals of Vector-Valued Functions

An **antiderivative** of the vector function $\mathbf{r}$ is a function $\mathbf{R}$ such that $\mathbf{R}' = \mathbf{r}$. If

$$\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k},$$

then an antiderivative of $\mathbf{r}$ is

$$\mathbf{R}(t) = F(t)\,\mathbf{i} + G(t)\,\mathbf{j} + H(t)\,\mathbf{k},$$

where F, G, and H are antiderivatives of f, g, and h, respectively. This fact follows by differentiating the components of $\mathbf{R}$ and verifying that $\mathbf{R}' = \mathbf{r}$. The collection of all antiderivatives of $\mathbf{r}$ is the *indefinite integral* of $\mathbf{r}$.

DEFINITION **Indefinite Integral of a Vector-Valued Function**

Let $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$ be a vector function, and let $\mathbf{R}(t) = F(t)\,\mathbf{i} + G(t)\,\mathbf{j} + H(t)\,\mathbf{k}$, where F, G, and H are antiderivatives of f, g, and h, respectively. The **indefinite integral** of $\mathbf{r}$ is

$$\int \mathbf{r}(t)\,dt = \mathbf{R}(t) + \mathbf{C},$$

where $\mathbf{C}$ is an arbitrary constant vector. Alternatively, in component form,

$$\int \langle f(t), g(t), h(t) \rangle\,dt = \langle F(t), G(t), H(t) \rangle + \langle C_1, C_2, C_3 \rangle.$$

EXAMPLE 5 **Indefinite integrals** Compute

$$\int \left(\frac{t}{\sqrt{t^2 + 2}}\,\mathbf{i} + e^{-3t}\,\mathbf{j} + (\sin 4t + 1)\,\mathbf{k} \right) dt.$$

➤ The substitution $u = t^2 + 2$ is used to evaluate the $\mathbf{i}$-component of the integral.

SOLUTION We compute the indefinite integral of each component:

$$\begin{aligned} &\int \left(\frac{t}{\sqrt{t^2 + 2}}\,\mathbf{i} + e^{-3t}\,\mathbf{j} + (\sin 4t + 1)\,\mathbf{k} \right) dt \\ &\quad = \left(\sqrt{t^2 + 2} + C_1\right)\mathbf{i} + \left(-\frac{1}{3}e^{-3t} + C_2\right)\mathbf{j} + \left(-\frac{1}{4}\cos 4t + t + C_3\right)\mathbf{k} \\ &\quad = \sqrt{t^2 + 2}\,\mathbf{i} - \frac{1}{3}e^{-3t}\,\mathbf{j} + \left(t - \frac{1}{4}\cos 4t\right)\mathbf{k} + \mathbf{C}. \quad \text{Let } \mathbf{C} = C_1\,\mathbf{i} + C_2\,\mathbf{j} + C_3\,\mathbf{k}. \end{aligned}$$

QUICK CHECK 4 Let $\mathbf{r}(t) = \langle 1, 2t, 3t^2 \rangle$. Compute $\int \mathbf{r}(t)\,dt$. ◄

The constants C_1, C_2, and C_3 are combined to form one vector constant $\mathbf{C}$ at the end of the calculation.

Related Exercise 63 ◄

EXAMPLE 6 **Finding one antiderivative** Find $\mathbf{r}(t)$ such that $\mathbf{r}'(t) = \langle 10, \sin t, t \rangle$ and $\mathbf{r}(0) = \mathbf{j}$.

SOLUTION The required function $\mathbf{r}$ is an antiderivative of $\langle 10, \sin t, t \rangle$:

$$\mathbf{r}(t) = \int \langle 10, \sin t, t \rangle\,dt = \left\langle 10t, -\cos t, \frac{t^2}{2} \right\rangle + \mathbf{C},$$

where $\mathbf{C}$ is an arbitrary constant vector. The condition $\mathbf{r}(0) = \mathbf{j}$ allows us to determine $\mathbf{C}$; substituting $t = 0$ implies that $\mathbf{r}(0) = \langle 0, -1, 0 \rangle + \mathbf{C} = \mathbf{j}$, where $\mathbf{j} = \langle 0, 1, 0 \rangle$. Solving for $\mathbf{C}$, we have $\mathbf{C} = \langle 0, 1, 0 \rangle - \langle 0, -1, 0 \rangle = \langle 0, 2, 0 \rangle$. Therefore,

$$\mathbf{r}(t) = \left\langle 10t, 2 - \cos t, \frac{t^2}{2} \right\rangle.$$

Related Exercise 65 ◀

Definite integrals are evaluated by applying the Fundamental Theorem of Calculus to each component of a vector-valued function.

DEFINITION Definite Integral of a Vector-Valued Function

Let $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$, where f, g, and h are integrable on the interval $[a, b]$. The **definite integral** of $\mathbf{r}$ on $[a, b]$ is

$$\int_a^b \mathbf{r}(t)\,dt = \left(\int_a^b f(t)\,dt\right)\mathbf{i} + \left(\int_a^b g(t)\,dt\right)\mathbf{j} + \left(\int_a^b h(t)\,dt\right)\mathbf{k}.$$

EXAMPLE 7 Definite integrals Evaluate

$$\int_0^{\pi} \left(\mathbf{i} + 3\cos\frac{t}{2}\,\mathbf{j} - 4t\,\mathbf{k}\right) dt.$$

SOLUTION

$$\int_0^{\pi} \left(\mathbf{i} + 3\cos\frac{t}{2}\,\mathbf{j} - 4t\,\mathbf{k}\right) dt = t\,\mathbf{i}\Big|_0^{\pi} + 6\sin\frac{t}{2}\,\mathbf{j}\Big|_0^{\pi} - 2t^2\,\mathbf{k}\Big|_0^{\pi}$$ Evaluate integrals for each component.

$$= \pi\,\mathbf{i} + 6\mathbf{j} - 2\pi^2\,\mathbf{k}$$ Simplify.

Related Exercise 75 ◀

With the tools of differentiation and integration in hand, we are prepared to tackle some practical problems, notably the motion of objects in space.

SECTION 14.2 EXERCISES

Getting Started

1. What is the derivative of $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$?
2. Explain the geometric meaning of $\mathbf{r}'(t)$.
3. Given a tangent vector on an oriented curve, how do you find the unit tangent vector?
4. Compute $\mathbf{r}''(t)$ when $\mathbf{r}(t) = \langle t^{10}, 8t, \cos t \rangle$.
5. How do you find the indefinite integral of $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$?
6. How do you evaluate $\int_a^b \mathbf{r}(t)\,dt$?
7. Find $\mathbf{C}$ if $\mathbf{r}(t) = \langle e^t, 3\cos t, t + 10 \rangle + \mathbf{C}$ and $\mathbf{r}(0) = \langle 0, 0, 0 \rangle$.
8. Find the unit tangent vector at $t = 0$ for the parameterized curve $\mathbf{r}(t)$ if $\mathbf{r}'(t) = \langle e^t + 5, \sin t + 2, \cos t + 2 \rangle$.

Practice Exercises

9–16. Derivatives of vector-valued functions *Differentiate the following functions.*

9. $\mathbf{r}(t) = \langle \cos t, t^2, \sin t \rangle$
10. $\mathbf{r}(t) = 4e^t\,\mathbf{i} + 5\,\mathbf{j} + \ln t\,\mathbf{k}$
11. $\mathbf{r}(t) = \left\langle 2t^3, 6\sqrt{t}, \frac{3}{t} \right\rangle$
12. $\mathbf{r}(t) = \langle 4, 3\cos 2t, 2\sin 3t \rangle$
13. $\mathbf{r}(t) = e^t\,\mathbf{i} + 2e^{-t}\,\mathbf{j} - 4e^{2t}\,\mathbf{k}$
14. $\mathbf{r}(t) = \tan t\,\mathbf{i} + \sec t\,\mathbf{j} + \cos^2 t\,\mathbf{k}$
15. $\mathbf{r}(t) = \langle te^{-t}, t\ln t, t\cos t \rangle$
16. $\mathbf{r}(t) = \langle (t+1)^{-1}, \tan^{-1} t, \ln(t+1) \rangle$

17–22. Tangent vectors *Find a tangent vector at the given value of t for the following parameterized curves.*

17. $\mathbf{r}(t) = \langle t, 3t^2, t^3 \rangle$, $t = 1$
18. $\mathbf{r}(t) = \langle e^t, e^{3t}, e^{5t} \rangle$, $t = 0$
19. $\mathbf{r}(t) = \langle t, \cos 2t, 2\sin t \rangle$, $t = \frac{\pi}{2}$
20. $\mathbf{r}(t) = \left\langle 2\sin t, 3\cos t, \sin\frac{t}{2} \right\rangle$, $t = \pi$
21. $\mathbf{r}(t) = 2t^4\,\mathbf{i} + 6t^{3/2}\,\mathbf{j} + \frac{10}{t}\,\mathbf{k}$, $t = 1$
22. $\mathbf{r}(t) = 2e^t\,\mathbf{i} + e^{-2t}\,\mathbf{j} + 4e^{2t}\,\mathbf{k}$, $t = \ln 3$

23–28. Unit tangent vectors *Find the unit tangent vector for the following parameterized curves.*

23. $\mathbf{r}(t) = \langle 2t, 2t, t \rangle$, for $0 \le t \le 1$
24. $\mathbf{r}(t) = \langle \cos t, \sin t, 2 \rangle$, for $0 \le t \le 2\pi$
25. $\mathbf{r}(t) = \langle 8, \cos 2t, 2\sin 2t \rangle$, for $0 \le t \le 2\pi$

26. $\mathbf{r}(t) = \langle \sin t, \cos t, \cos t \rangle$, for $0 \le t \le 2\pi$

27. $\mathbf{r}(t) = \left\langle t, 2, \frac{2}{t} \right\rangle$, for $t \ge 1$

28. $\mathbf{r}(t) = \langle e^{2t}, 2e^{2t}, 2e^{-3t} \rangle$, for $t \ge 0$

29–32. Unit tangent vectors at a point *Find the unit tangent vector at the given value of t for the following parameterized curves.*

29. $\mathbf{r}(t) = \langle \cos 2t, 4, 3 \sin 2t \rangle$; $t = \frac{\pi}{2}$

30. $\mathbf{r}(t) = \langle \sin t, \cos t, e^{-t} \rangle$; $t = 0$

31. $\mathbf{r}(t) = \left\langle 6t, 6, \frac{3}{t} \right\rangle$; $t = 1$

32. $\mathbf{r}(t) = \langle \sqrt{7}e^t, 3e^t, 3e^t \rangle$; $t = \ln 2$

33–38. Derivative rules *Let* $\mathbf{u}(t) = 2t^3\,\mathbf{i} + (t^2 - 1)\,\mathbf{j} - 8\,\mathbf{k}$ *and* $\mathbf{v}(t) = e^t\,\mathbf{i} + 2e^{-t}\,\mathbf{j} - e^{2t}\,\mathbf{k}$. *Compute the derivative of the following functions.*

33. $(t^{12} + 3t)\,\mathbf{u}(t)$

34. $(4t^8 - 6t^3)\,\mathbf{v}(t)$

35. $\mathbf{u}(t^4 - 2t)$

36. $\mathbf{v}(\sqrt{t})$

37. $\mathbf{u}(t) \cdot \mathbf{v}(t)$

38. $\mathbf{u}(t) \times \mathbf{v}(t)$

39–42. Derivative rules *Suppose* $\mathbf{u}$ *and* $\mathbf{v}$ *are differentiable functions at* $t = 0$ *with* $\mathbf{u}(0) = \langle 0, 1, 1 \rangle$, $\mathbf{u}'(0) = \langle 0, 7, 1 \rangle$, $\mathbf{v}(0) = \langle 0, 1, 1 \rangle$, *and* $\mathbf{v}'(0) = \langle 1, 1, 2 \rangle$. *Evaluate the following expressions.*

39. $\left.\frac{d}{dt}(\mathbf{u} \cdot \mathbf{v})\right|_{t=0}$

40. $\left.\frac{d}{dt}(\mathbf{u} \times \mathbf{v})\right|_{t=0}$

41. $\left.\frac{d}{dt}(\cos t\,\mathbf{u}(t))\right|_{t=0}$

42. $\left.\frac{d}{dt}(\mathbf{u}(\sin t))\right|_{t=0}$

43–48. Derivative rules *Let* $\mathbf{u}(t) = \langle 1, t, t^2 \rangle$, $\mathbf{v}(t) = \langle t^2, -2t, 1 \rangle$, *and* $g(t) = 2\sqrt{t}$. *Compute the derivatives of the following functions.*

43. $\mathbf{v}(e^t)$

44. $\mathbf{u}(t^3)$

45. $\mathbf{v}(g(t))$

46. $g(t)\mathbf{v}(t)$

47. $\mathbf{u}(t) \times \mathbf{v}(t)$

48. $\mathbf{u}(t) \cdot \mathbf{v}(t)$

49–52. Derivative rules *Compute the following derivatives.*

49. $\frac{d}{dt}(t^2(\mathbf{i} + 2\,\mathbf{i} - 2t\,\mathbf{k}) \cdot (e^t\,\mathbf{i} + 2e^t\,\mathbf{j} - 3e^{-t}\,\mathbf{k}))$

50. $\frac{d}{dt}((t^3\,\mathbf{i} - 2t\,\mathbf{j} - 2\,\mathbf{k}) \times (t\,\mathbf{i} - t^2\,\mathbf{j} - t^3\,\mathbf{k}))$

51. $\frac{d}{dt}((3t^2\,\mathbf{i} + \sqrt{t}\,\mathbf{j} - 2t^{-1}\,\mathbf{k}) \cdot (\cos t\,\mathbf{i} + \sin 2t\,\mathbf{j} - 3t\,\mathbf{k}))$

52. $\frac{d}{dt}((t^3\,\mathbf{i} + 6\,\mathbf{j} - 2\sqrt{t}\,\mathbf{k}) \times (3t\,\mathbf{i} - 12t^2\,\mathbf{j} - 6t^{-2}\,\mathbf{k}))$

53–58. Higher-order derivatives *Compute* $\mathbf{r}''(t)$ *and* $\mathbf{r}'''(t)$ *for the following functions.*

53. $\mathbf{r}(t) = \langle t^2 + 1, t + 1, 1 \rangle$

54. $\mathbf{r}(t) = \langle 3t^{12} - t^2, t^8 + t^3, t^{-4} - 2 \rangle$

55. $\mathbf{r}(t) = \langle \cos 3t, \sin 4t, \cos 6t \rangle$

56. $\mathbf{r}(t) = \langle e^{4t}, 2e^{-4t} + 1, 2e^{-t} \rangle$

57. $\mathbf{r}(t) = \sqrt{t + 4}\,\mathbf{i} + \frac{t}{t + 1}\,\mathbf{j} - e^{-t^2}\,\mathbf{k}$

58. $\mathbf{r}(t) = \tan t\,\mathbf{i} + \left(t + \frac{1}{t}\right)\mathbf{j} - \ln(t + 1)\,\mathbf{k}$

59–64. Indefinite integrals *Compute the indefinite integral of the following functions.*

59. $\mathbf{r}(t) = \langle t^4 - 3t, 2t - 1, 10 \rangle$

60. $\mathbf{r}(t) = \left\langle 5t^{-4} - t^2, t^6 - 4t^3, \frac{2}{t} \right\rangle$

61. $\mathbf{r}(t) = \langle 2 \cos t, 2 \sin 3t, 4 \cos 8t \rangle$

62. $\mathbf{r}(t) = te^t\,\mathbf{i} + t \sin t^2\,\mathbf{j} - \frac{2t}{\sqrt{t^2 + 4}}\,\mathbf{k}$

63. $\mathbf{r}(t) = e^{3t}\,\mathbf{i} + \frac{1}{1 + t^2}\,\mathbf{j} - \frac{1}{\sqrt{2t}}\,\mathbf{k}$

64. $\mathbf{r}(t) = 2^t\,\mathbf{i} + \frac{1}{1 + 2t}\,\mathbf{j} + \ln t\,\mathbf{k}$

65–70. Finding r from r′ *Find the function* $\mathbf{r}$ *that satisfies the given conditions.*

65. $\mathbf{r}'(t) = \langle e^t, \sin t, \sec^2 t \rangle$; $\quad \mathbf{r}(0) = \langle 2, 2, 2 \rangle$

66. $\mathbf{r}'(t) = \langle 0, 2, 2t \rangle$; $\quad \mathbf{r}(1) = \langle 4, 3, -5 \rangle$

67. $\mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle$; $\quad \mathbf{r}(1) = \langle 4, 3, -5 \rangle$

68. $\mathbf{r}'(t) = \left\langle \sqrt{t}, \cos \pi t, \frac{4}{t} \right\rangle$; $\quad \mathbf{r}(1) = \langle 2, 3, 4 \rangle$

69. $\mathbf{r}'(t) = \langle e^{2t}, 1 - 2e^{-t}, 1 - 2e^t \rangle$; $\quad \mathbf{r}(0) = \langle 1, 1, 1 \rangle$

70. $\mathbf{r}'(t) = \frac{t}{t^2 + 1}\,\mathbf{i} + te^{-t^2}\,\mathbf{j} - \frac{2t}{\sqrt{t^2 + 4}}\,\mathbf{k}$; $\quad \mathbf{r}(0) = \mathbf{i} + \frac{3}{2}\,\mathbf{j} - 3\mathbf{k}$

71–78. Definite integrals *Evaluate the following definite integrals.*

71. $\int_{-1}^{1} (\mathbf{i} + t\,\mathbf{j} + 3t^2\,\mathbf{k})\,dt$

72. $\int_{1}^{4} (6t^2\,\mathbf{i} + 8t^3\,\mathbf{j} + 9t^2\,\mathbf{k})\,dt$

73. $\int_{0}^{\ln 2} (e^t\,\mathbf{i} + e^t \cos(\pi e^t)\,\mathbf{j})\,dt$

74. $\int_{1/2}^{1} \left(\frac{3}{1 + 2t}\,\mathbf{i} - \pi \csc^2\left(\frac{\pi}{2}t\right)\mathbf{k} \right) dt$

75. $\int_{-\pi}^{\pi} (\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + 2t\,\mathbf{k})\,dt$

76. $\int_{0}^{\ln 2} (e^{-t}\,\mathbf{i} + 2e^{2t}\,\mathbf{j} - 4e^t\,\mathbf{k})\,dt$

77. $\int_{0}^{2} te^t(\mathbf{i} + 2\mathbf{j} - \mathbf{k})\,dt$

78. $\int_{0}^{\pi/4} (\sec^2 t\,\mathbf{i} - 2 \cos t\,\mathbf{j} - \mathbf{k})\,dt$

79. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. The vectors $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are parallel for all values of t in the domain.

b. The curve described by the function $\mathbf{r}(t) = \langle t, t^2 - 2t, \cos \pi t \rangle$ is smooth, for $-\infty < t < \infty$.

c. If f, g, and h are odd integrable functions and a is a real number, then

$$\int_{-a}^{a} (f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k})\,dt = \mathbf{0}.$$

80–83. Tangent lines *Suppose the vector-valued function $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ is smooth on an interval containing the point t_0. The line tangent to $\mathbf{r}(t)$ at $t = t_0$ is the line parallel to the tangent vector $\mathbf{r}'(t_0)$ that passes through $(f(t_0), g(t_0), h(t_0))$. For each of the following functions, find an equation of the line tangent to the curve at $t = t_0$. Choose an orientation for the line that is the same as the direction of $\mathbf{r}'$.*

80. $\mathbf{r}(t) = \langle e^t, e^{2t}, e^{3t} \rangle; t_0 = 0$

81. $\mathbf{r}(t) = \langle 2 + \cos t, 3 + \sin 2t, t \rangle; t_0 = \dfrac{\pi}{2}$

82. $\mathbf{r}(t) = \langle \sqrt{2t + 1}, \sin \pi t, 4 \rangle; t_0 = 4$

83. $\mathbf{r}(t) = \langle 3t - 1, 7t + 2, t^2 \rangle; t_0 = 1$

Explorations and Challenges

84–89. Relationship between r and r′

84. Consider the circle $\mathbf{r}(t) = \langle a \cos t, a \sin t \rangle$, for $0 \le t \le 2\pi$, where a is a positive real number. Compute $\mathbf{r}'$ and show that it is orthogonal to $\mathbf{r}$ for all t.

85. Consider the parabola $\mathbf{r}(t) = \langle at^2 + 1, t \rangle$, for $-\infty < t < \infty$, where a is a positive real number. Find all points on the parabola at which $\mathbf{r}$ and $\mathbf{r}'$ are orthogonal.

86. Consider the curve $\mathbf{r}(t) = \langle \sqrt{t}, 1, t \rangle$, for $t > 0$. Find all points on the curve at which $\mathbf{r}$ and $\mathbf{r}'$ are orthogonal.

87. Consider the helix $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$, for $-\infty < t < \infty$. Find all points on the helix at which $\mathbf{r}$ and $\mathbf{r}'$ are orthogonal.

88. Consider the ellipse $\mathbf{r}(t) = \langle 2 \cos t, 8 \sin t, 0 \rangle$, for $0 \le t \le 2\pi$. Find all points on the ellipse at which $\mathbf{r}$ and $\mathbf{r}'$ are orthogonal.

89. Give two families of curves in $\mathbb{R}^3$ for which $\mathbf{r}$ and $\mathbf{r}'$ are parallel for all t in the domain.

90. Motion on a sphere Prove that $\mathbf{r}$ describes a curve that lies on the surface of a sphere centered at the origin $(x^2 + y^2 + z^2 = a^2$ with $a \ge 0)$ if and only if $\mathbf{r}$ and $\mathbf{r}'$ are orthogonal at all points of the curve.

91. Vectors r and r′ for lines

a. If $\mathbf{r}(t) = \langle at, bt, ct \rangle$ with $\langle a, b, c \rangle \ne \langle 0, 0, 0 \rangle$, show that the angle between $\mathbf{r}$ and $\mathbf{r}'$ is constant for all $t > 0$.

b. If $\mathbf{r}(t) = \langle x_0 + at, y_0 + bt, z_0 + ct \rangle$, where x_0, y_0, and z_0 are not all zero, show that the angle between $\mathbf{r}$ and $\mathbf{r}'$ varies with t.

c. Explain the results of parts (a) and (b) geometrically.

92. Proof of Sum Rule By expressing $\mathbf{u}$ and $\mathbf{v}$ in terms of their components, prove that

$$\frac{d}{dt}(\mathbf{u}(t) + \mathbf{v}(t)) = \mathbf{u}'(t) + \mathbf{v}'(t).$$

93. Proof of Product Rule By expressing $\mathbf{u}$ in terms of its components, prove that

$$\frac{d}{dt}(f(t)\mathbf{u}(t)) = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t).$$

94. Proof of Cross Product Rule Prove that

$$\frac{d}{dt}(\mathbf{u}(t) \times \mathbf{v}(t)) = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t).$$

There are two ways to proceed: Either express $\mathbf{u}$ and $\mathbf{v}$ in terms of their three components or use the definition of the derivative.

T **95. Cusps and noncusps**

a. Graph the curve $\mathbf{r}(t) = \langle t^3, t^3 \rangle$. Show that $\mathbf{r}'(0) = \mathbf{0}$ and the curve does not have a cusp at $t = 0$. Explain.

b. Graph the curve $\mathbf{r}(t) = \langle t^3, t^2 \rangle$. Show that $\mathbf{r}'(0) = \mathbf{0}$ and the curve has a cusp at $t = 0$. Explain.

c. The functions $\mathbf{r}(t) = \langle t, t^2 \rangle$ and $\mathbf{p}(t) = \langle t^2, t^4 \rangle$ both satisfy $y = x^2$. Explain how the curves they parameterize are different.

d. Consider the curve $\mathbf{r}(t) = \langle t^m, t^n \rangle$, where $m > 1$ and $n > 1$ are integers with no common factors. Is it true that the curve has a cusp at $t = 0$ if one (not both) of m and n is even? Explain.

QUICK CHECK ANSWERS

1. $\mathbf{r}(t)$ describes a line, so its tangent vector $\mathbf{r}'(t) = \langle 1, 1, 1 \rangle$ has constant direction and magnitude. **2.** Both $\mathbf{r}'$ and $|\mathbf{r}'|$ have units of m/s. In forming $\mathbf{r}'/|\mathbf{r}'|$, the units cancel and $\mathbf{T}(t)$ is without units. **3.** $\dfrac{d}{dt}(\mathbf{u}(t) \cdot \mathbf{v}(t)) = \langle 1, 1, 1 \rangle \cdot \langle 1, 1, 1 \rangle + \langle t, t, t \rangle \cdot \langle 0, 0, 0 \rangle = 3$. $\dfrac{d}{dt}(\langle t, t, t \rangle \cdot \langle 1, 1, 1 \rangle) = \dfrac{d}{dt}(3t) = 3$. **4.** $\langle t, t^2, t^3 \rangle + \mathbf{C}$, where $\mathbf{C} = \langle a, b, c \rangle$, and a, b, and c are real numbers ◄

14.3 Motion in Space

It is a remarkable fact that given the forces acting on an object and its initial position and velocity, the motion of the object in three-dimensional space can be modeled for all future times. To be sure, the accuracy of the results depends on how well the various forces on the object are described. For example, it may be more difficult to predict the trajectory of a spinning soccer ball than the path of a space station orbiting Earth. Nevertheless, as shown in this section, by combining Newton's Second Law of Motion with everything we have learned about vectors, it is possible to solve a variety of moving-body problems.

Position, Velocity, Speed, Acceleration

Until now, we have studied objects that move in one dimension (along a line). The next step is to consider the motion of objects in two dimensions (in a plane) and three

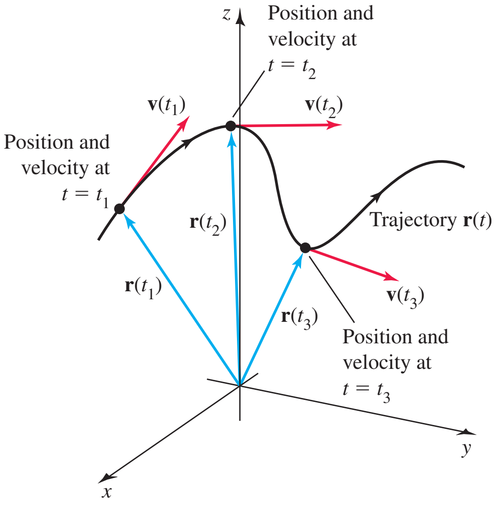


Figure 14.11

dimensions (in space). We work in a three-dimensional coordinate system and let the vector-valued function $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ describe the *position* of a moving object at times $t \geq 0$. The curve described by $\mathbf{r}$ is the *path* or *trajectory* of the object (**Figure 14.11**). Just as with one-dimensional motion, the rate of change of the position function with respect to time is the *instantaneous velocity* of the object—a vector with three components corresponding to the velocity in the x-, y-, and z-directions:

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle.$$

This expression should look familiar. The velocity vectors of a moving object are simply tangent vectors; that is, at any point, the velocity vector is tangent to the trajectory (Figure 14.11).

As with one-dimensional motion, the *speed* of an object moving in three dimensions is the magnitude of its velocity vector:

$$|\mathbf{v}(t)| = |\langle x'(t), y'(t), z'(t) \rangle| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}.$$

The speed is a nonnegative scalar-valued function.

Finally, the *acceleration* of a moving object is the rate of change of the velocity:

$$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t).$$

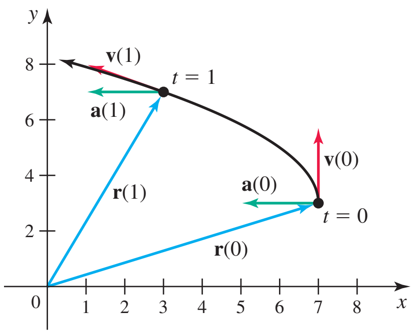


Figure 14.12

Although the position vector gives the path of a moving object and the velocity vector is always tangent to the path, the acceleration vector is more difficult to visualize. **Figure 14.12** shows one particular instance of two-dimensional motion. The trajectory is a segment of a parabola and is traced out by the position vectors (shown at $t = 0$ and $t = 1$). As expected, the velocity vectors are tangent to the trajectory. In this case, the acceleration is $\mathbf{a} = \langle -2, 0 \rangle$; it is constant in magnitude and direction for all times. The relationships among $\mathbf{r}$, $\mathbf{v}$, and $\mathbf{a}$ are explored in the coming examples.

➤ In the case of two-dimensional motion, $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, $\mathbf{v}(t) = \mathbf{r}'(t)$, and $\mathbf{a}(t) = \mathbf{r}''(t)$.

QUICK CHECK 1 Given $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$, find $\mathbf{v}(t)$ and $\mathbf{a}(t)$. ◄

DEFINITION Position, Velocity, Speed, Acceleration

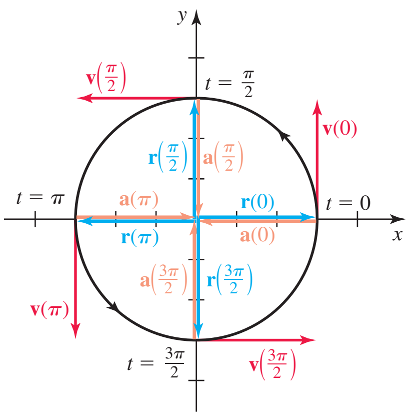
Let the **position** of an object moving in three-dimensional space be given by $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, for $t \geq 0$. The **velocity** of the object is

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle.$$

The **speed** of the object is the scalar function

$$|\mathbf{v}(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}.$$

The **acceleration** of the object is $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$.



Circular motion: At all times $\mathbf{a}(t) = -\mathbf{r}(t)$ and $\mathbf{v}(t)$ is orthogonal to $\mathbf{r}(t)$ and $\mathbf{a}(t)$.

Figure 14.13

EXAMPLE 1 Velocity and acceleration for circular motion Consider the two-dimensional motion given by the position vector

$$\mathbf{r}(t) = \langle x(t), y(t) \rangle = \langle 3 \cos t, 3 \sin t \rangle, \quad \text{for } 0 \leq t \leq 2\pi.$$

a. Sketch the trajectory of the object.

b. Find the velocity and speed of the object.

c. Find the acceleration of the object.

d. Sketch the position, velocity, and acceleration vectors, for $t = 0, \pi/2, \pi$, and $3\pi/2$.

SOLUTION

a. Notice that

$$x(t)^2 + y(t)^2 = 9(\cos^2 t + \sin^2 t) = 9,$$

which is an equation of a circle centered at the origin with radius 3. The object moves on this circle in the counterclockwise direction (**Figure 14.13**).

b. $$\begin{aligned} \mathbf{v}(t) &= \langle x'(t), y'(t) \rangle = \langle -3\sin t, 3\cos t \rangle && \text{Velocity vector} \\ |\mathbf{v}(t)| &= \sqrt{x'(t)^2 + y'(t)^2} && \text{Definition of speed} \\ &= \sqrt{(-3\sin t)^2 + (3\cos t)^2} \\ &= \sqrt{9\underbrace{(\sin^2 t + \cos^2 t)}_{1}} = 3 \end{aligned}$$

The velocity vector has a constant magnitude and a continuously changing direction.

c. Differentiating the velocity, we find that $\mathbf{a}(t) = \mathbf{v}'(t) = \langle -3\cos t, -3\sin t \rangle = -\mathbf{r}(t)$. In this case, the acceleration vector is the negative of the position vector at all times.

d. The relationships among $\mathbf{r}$, $\mathbf{v}$, and $\mathbf{a}$ at four points in time are shown in Figure 14.13. The velocity vector is always tangent to the trajectory and has length 3, while the acceleration vector and position vector each have length 3 and point in opposite directions. At all times, $\mathbf{v}$ is orthogonal to $\mathbf{r}$ and $\mathbf{a}$.

Related Exercise 13 ◄

EXAMPLE 2 **Comparing trajectories** Consider the trajectories described by the position functions

$$\mathbf{r}(t) = \left\langle t, t^2 - 4, \frac{t^3}{4} - 8 \right\rangle, \quad \text{for } t \geq 0, \text{ and}$$

$$\mathbf{R}(t) = \left\langle t^2, t^4 - 4, \frac{t^6}{4} - 8 \right\rangle, \quad \text{for } t \geq 0,$$

where t is measured in the same time units for both functions.

a. Graph and compare the trajectories using a graphing utility.

b. Find the velocity vectors associated with the position functions.

SOLUTION

a. Plotting the position functions at selected values of t results in the trajectories shown in Figure 14.14. Because $\mathbf{r}(0) = \mathbf{R}(0) = \langle 0, -4, -8 \rangle$, both curves have the same initial point. For $t \geq 0$, the two curves consist of the same points, but they are traced out differently. For example, both curves pass through the point $(4, 12, 8)$, but that point corresponds to $\mathbf{r}(4)$ on the first curve and $\mathbf{R}(2)$ on the second curve. In general, $\mathbf{r}(t^2) = \mathbf{R}(t)$, for $t \geq 0$.

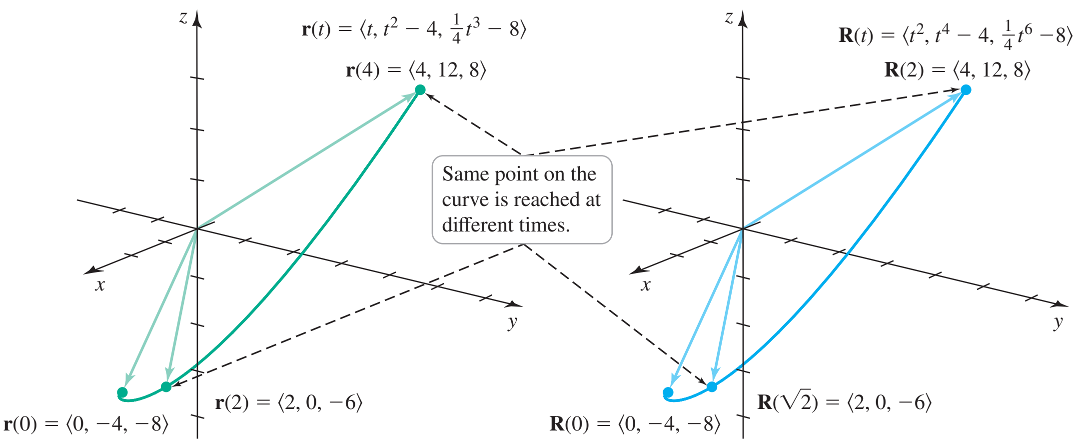


Figure 14.14

b. The velocity vectors are

$$\mathbf{r}'(t) = \left\langle 1, 2t, \frac{3t^2}{4} \right\rangle \quad \text{and} \quad \mathbf{R}'(t) = \left\langle 2t, 4t^3, \frac{3}{2}t^5 \right\rangle.$$

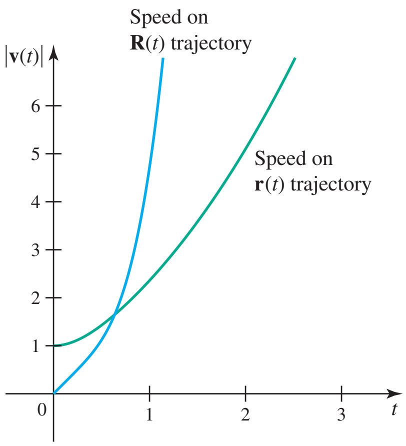


Figure 14.15

The difference in the motion on the two curves is revealed by the graphs of the speeds associated with the trajectories (**Figure 14.15**). The object on the first trajectory reaches the point $(4, 12, 8)$ at $t = 4$, where its speed is $|\mathbf{r}'(4)| = |\langle 1, 8, 12\rangle| \approx 14.5$. The object on the second trajectory reaches the same point $(4, 12, 8)$ at $t = 2$, where its speed is $|\mathbf{R}'(2)| = |\langle 4, 32, 48\rangle| \approx 57.8$.

Related Exercise 21 ◄

QUICK CHECK 2 Find the functions that give the speed of the two objects in Example 2, for $t \geq 0$ (corresponding to the graphs in Figure 14.15). ◄

Straight-Line and Circular Motion

Two types of motion in space arise frequently and deserve to be singled out. First consider a trajectory described by the vector function

$$\mathbf{r}(t) = \langle x_0 + at, y_0 + bt, z_0 + ct\rangle, \quad \text{for } t \geq 0,$$

where x_0, y_0, z_0, a, b, and c are constants. This function describes a straight-line trajectory with an initial position $\langle x_0, y_0, z_0\rangle$ and a direction given by the vector $\langle a, b, c\rangle$ (Section 13.5). The velocity on this trajectory is the constant $\mathbf{v}(t) = \mathbf{r}'(t) = \langle a, b, c\rangle$ in the direction of the trajectory, and the acceleration is $\mathbf{a} = \langle 0, 0, 0\rangle$. The motion associated with this function is **uniform** (constant velocity) **straight-line motion**.

➤ See Exercise 83 for a discussion of nonuniform straight-line motion.

A different situation is **circular motion** (Example 1). Consider the two-dimensional circular path

$$\mathbf{r}(t) = \langle A\cos t, A\sin t\rangle, \quad \text{for } 0 \leq t \leq 2\pi,$$

where A is a nonzero constant (**Figure 14.16**). The velocity and acceleration vectors are

$$\mathbf{v}(t) = \langle -A\sin t, A\cos t\rangle \quad \text{and}$$
$$\mathbf{a}(t) = \langle -A\cos t, -A\sin t\rangle = -\mathbf{r}(t).$$

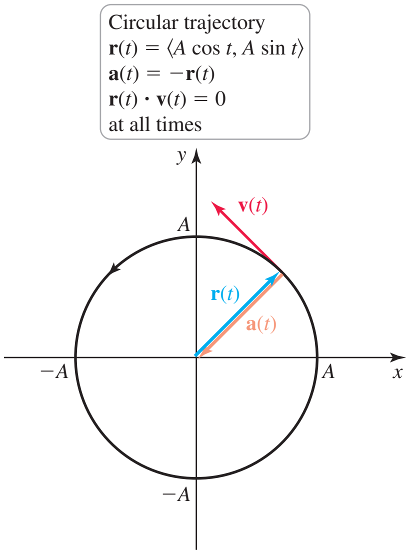


Figure 14.16

Notice that $\mathbf{r}$ and $\mathbf{a}$ are parallel but point in opposite directions. Furthermore, $\mathbf{r}\cdot\mathbf{v} = \mathbf{a}\cdot\mathbf{v} = 0$; therefore, the position and acceleration vectors are both orthogonal to the velocity vectors at any given point (Figure 14.16). Finally, $\mathbf{r}$, $\mathbf{v}$, and $\mathbf{a}$ have constant magnitude A and variable directions. The conclusion that $\mathbf{r}\cdot\mathbf{v} = 0$ applies to any motion for which $|\mathbf{r}|$ is constant; that is, to any motion on a circle or a sphere (**Figure 14.17**).

> **THEOREM 14.2 Motion with Constant $|\mathbf{r}|$**
> Let $\mathbf{r}$ describe a path on which $|\mathbf{r}|$ is constant (motion on a circle or sphere centered at the origin). Then $\mathbf{r}\cdot\mathbf{v} = 0$, which means the position vector and the velocity vector are orthogonal at all times for which the functions are defined.

Proof: If $\mathbf{r}$ has constant magnitude, then $|\mathbf{r}(t)|^2 = \mathbf{r}(t)\cdot\mathbf{r}(t) = c$ for some constant c. Differentiating the equation $\mathbf{r}(t)\cdot\mathbf{r}(t) = c$, we have

$$\begin{aligned}
0 &= \frac{d}{dt}(\mathbf{r}(t)\cdot\mathbf{r}(t)) && \text{Differentiate both sides of } |\mathbf{r}(t)|^2 = c.\\
&= \mathbf{r}'(t)\cdot\mathbf{r}(t) + \mathbf{r}(t)\cdot\mathbf{r}'(t) && \text{Derivative of dot product (Theorem 14.1)}\\
&= 2\mathbf{r}'(t)\cdot\mathbf{r}(t) && \text{Simplify.}\\
&= 2\mathbf{v}(t)\cdot\mathbf{r}(t). && \mathbf{r}'(t) = \mathbf{v}(t)
\end{aligned}$$

Because $\mathbf{r}(t)\cdot\mathbf{v}(t) = 0$ for all t, it follows that $\mathbf{r}$ and $\mathbf{v}$ are orthogonal for all t. ◄

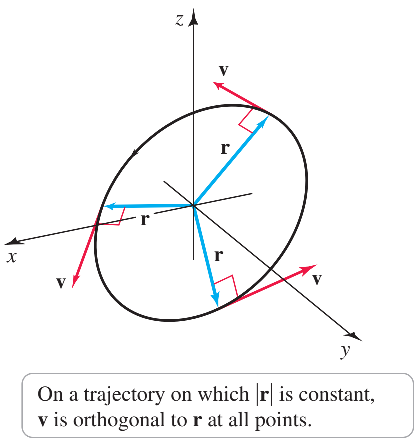


Figure 14.17

EXAMPLE 3 Path on a sphere An object moves on a trajectory described by

$$\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle = \langle 3\cos t, 5\sin t, 4\cos t\rangle, \quad \text{for } 0 \leq t \leq 2\pi.$$

a. Show that the object moves on a sphere and find the radius of the sphere.

b. Find the velocity and speed of the object.

c. Consider the curve $\mathbf{r}(t) = \langle 5\cos t, 5\sin t, 5\sin 2t\rangle$, which is the roller coaster curve from Example 3 of Section 14.1, with different coefficients. Show that this curve does not lie on a sphere. How could $\mathbf{r}$ be modified so that it describes a curve that lies on a sphere of radius 1, centered at the origin?

➤ For generalizations of this example and explorations of trajectories that lie on spheres and ellipses, see Exercises 79 and 82.

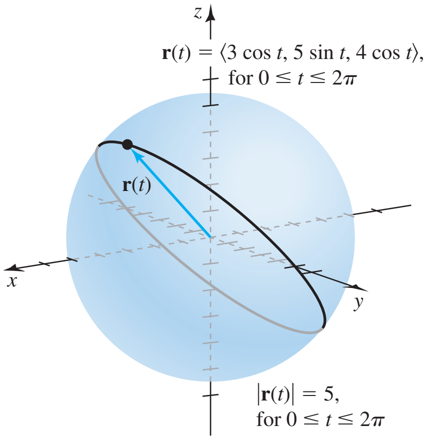


Figure 14.18

SOLUTION

a.
$$\begin{aligned}|\mathbf{r}(t)|^2 &= x(t)^2 + y(t)^2 + z(t)^2 && \text{Square of the distance from the origin}\\ &= (3\cos t)^2 + (5\sin t)^2 + (4\cos t)^2 && \text{Substitute.}\\ &= 25\cos^2 t + 25\sin^2 t && \text{Simplify.}\\ &= 25\underbrace{(\cos^2 t + \sin^2 t)}_{1} = 25 && \text{Factor.}\end{aligned}$$

Therefore, $|\mathbf{r}(t)| = 5$, for $0 \le t \le 2\pi$, and the trajectory lies on a sphere of radius 5 centered at the origin (**Figure 14.18**).

b.
$$\begin{aligned}\mathbf{v}(t) &= \mathbf{r}'(t) = \langle -3\sin t, 5\cos t, -4\sin t\rangle && \text{Velocity vector}\\ |\mathbf{v}(t)| &= \sqrt{\mathbf{v}(t)\cdot\mathbf{v}(t)} && \text{Speed of the object}\\ &= \sqrt{9\sin^2 t + 25\cos^2 t + 16\sin^2 t} && \text{Evaluate the dot product.}\\ &= \sqrt{25\underbrace{(\sin^2 t + \cos^2 t)}_{1}} && \text{Simplify.}\\ &= 5 && \text{Simplify.}\end{aligned}$$

The speed of the object is always 5. You should verify that $\mathbf{r}(t)\cdot\mathbf{v}(t) = 0$, for all t, implying that $\mathbf{r}$ and $\mathbf{v}$ are always orthogonal.

c. We first compute the distance from the origin to the curve:

$$\begin{aligned}|\mathbf{r}(t)| &= \sqrt{(5\cos t)^2 + (5\sin t)^2 + (5\sin 2t)^2} && \text{Distance from origin to curve}\\ &= \sqrt{25(\cos^2 t + \sin^2 t + \sin^2 2t)} && \text{Simplify.}\\ &= 5\sqrt{1 + \sin^2 2t}. && \cos^2 t + \sin^2 t = 1\end{aligned}$$

It is clear that $|\mathbf{r}(t)|$ is not constant, and therefore the curve does not lie on a sphere.

One way to modify the curve so that it does lie on a sphere is to divide each output vector $\mathbf{r}(t)$ by its length. In fact, as long as $|\mathbf{r}(t)| \ne 0$ on the interval of interest, we can force any path onto a sphere (centered at the origin) with this modification. The function

$$\mathbf{u}(t) = \frac{\mathbf{r}(t)}{|\mathbf{r}(t)|} = \left\langle \frac{\cos t}{\sqrt{1 + \sin^2 2t}}, \frac{\sin t}{\sqrt{1 + \sin^2 2t}}, \frac{\sin 2t}{\sqrt{1 + \sin^2 2t}} \right\rangle$$

describes a curve on which $|\mathbf{u}(t)|$ is constant because $\dfrac{\mathbf{r}(t)}{|\mathbf{r}(t)|}$ is a unit vector. We conclude that the new curve, which is reminiscent of the seam on a tennis ball (**Figure 14.19**), lies on the unit sphere centered at the origin.

Related Exercises 32–33 ◀

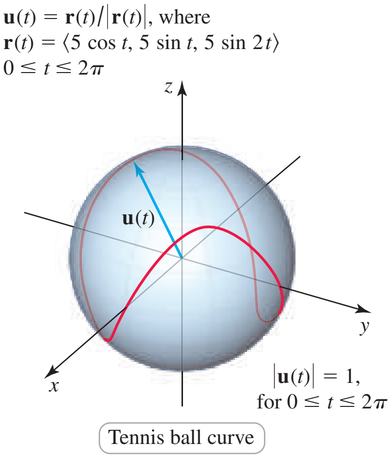


Figure 14.19

QUICK CHECK 3 Explain how to modify the curve $\mathbf{r}(t)$ given in Example 3c so that it lies on a sphere of radius 5 centered at the origin. ◀

Two-Dimensional Motion in a Gravitational Field

Newton's Second Law of Motion, which is used to model the motion of most objects, states that

$$\underbrace{\text{mass}}_{m}\cdot\underbrace{\text{acceleration}}_{\mathbf{a}(t) = \mathbf{r}''(t)} = \underbrace{\text{sum of all forces.}}_{\sum \mathbf{F}_k}$$

The governing law says something about the *acceleration* of an object, and in order to describe the motion fully, we must find the velocity and position from the acceleration.

Finding Velocity and Position from Acceleration We begin with the case of two-dimensional projectile motion in which the only force acting on the object is the gravitational force; for the moment, air resistance and other possible external forces are neglected.

A convenient coordinate system uses a y-axis that points vertically upward and an x-axis that points in the direction of horizontal motion. The gravitational force is in the negative y-direction and is given by $\mathbf{F} = \langle 0, -mg\rangle$, where m is the mass of the object and $g = 9.8\ \text{m/s}^2 = 32\ \text{ft/s}^2$ is the acceleration due to gravity (**Figure 14.20**).

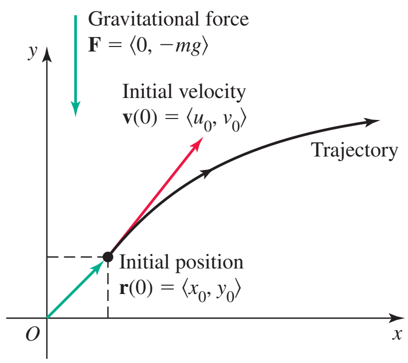


Figure 14.20

With these observations, Newton's Second Law takes the form

$$m\mathbf{a}(t) = \mathbf{F} = \langle 0, -mg \rangle.$$

Significantly, the mass of the object cancels, leaving the vector equation

$$\mathbf{a}(t) = \langle 0, -g \rangle. \tag{1}$$

In order to find the velocity $\mathbf{v}(t) = \langle x'(t), y'(t) \rangle$ and the position $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ from this equation, we must be given the following **initial conditions**:

Initial velocity at $t = 0$: $\mathbf{v}(0) = \langle u_0, v_0 \rangle$ and
Initial position at $t = 0$: $\mathbf{r}(0) = \langle x_0, y_0 \rangle$.

➤ Recall that an antiderivative of 0 is a constant C and an antiderivative of $-g$ is $-gt + C$.

➤ You have a choice. You may do these calculations in vector notation as we have done here, or you may work with individual components.

We proceed in two steps.

1. **Solve for the velocity** The velocity is an antiderivative of the acceleration in equation (1). Integrating the acceleration, we have

$$\mathbf{v}(t) = \int \mathbf{a}(t)\, dt = \int \langle 0, -g \rangle\, dt = \langle 0, -gt \rangle + \mathbf{C},$$

where $\mathbf{C}$ is an arbitrary constant vector. The arbitrary constant is determined by substituting $t = 0$ and using the initial condition $\mathbf{v}(0) = \langle u_0, v_0 \rangle$. We find that $\mathbf{v}(0) = \langle 0, 0 \rangle + \mathbf{C} = \langle u_0, v_0 \rangle$, or $\mathbf{C} = \langle u_0, v_0 \rangle$. Therefore, the velocity is

$$\mathbf{v}(t) = \langle 0, -gt \rangle + \langle u_0, v_0 \rangle = \langle u_0, -gt + v_0 \rangle. \tag{2}$$

Notice that the horizontal component of velocity is simply the initial horizontal velocity u_0 for all time. The vertical component of velocity decreases linearly from its initial value of v_0.

2. **Solve for the position** The position is an antiderivative of the velocity given by equation (2):

$$\mathbf{r}(t) = \int \mathbf{v}(t)\, dt = \int \langle u_0, -gt + v_0 \rangle\, dt = \left\langle u_0\, t, -\frac{1}{2}gt^2 + v_0\, t \right\rangle + \mathbf{C},$$

where $\mathbf{C}$ is an arbitrary constant vector. Substituting $t = 0$, we have $\mathbf{r}(0) = \langle 0, 0 \rangle + \mathbf{C} = \langle x_0, y_0 \rangle$, which implies that $\mathbf{C} = \langle x_0, y_0 \rangle$. Therefore, the position of the object, for $t \geq 0$, is

$$\mathbf{r}(t) = \left\langle u_0\, t, -\frac{1}{2}gt^2 + v_0\, t \right\rangle + \langle x_0, y_0 \rangle = \left\langle \underbrace{u_0\, t + x_0}_{x(t)}, \underbrace{-\frac{1}{2}gt^2 + v_0\, t + y_0}_{y(t)} \right\rangle.$$

SUMMARY Two-Dimensional Motion in a Gravitational Field

Consider an object moving in a plane with a horizontal x-axis and a vertical y-axis, subject only to the force of gravity. Given the initial velocity $\mathbf{v}(0) = \langle u_0, v_0 \rangle$ and the initial position $\mathbf{r}(0) = \langle x_0, y_0 \rangle$, the velocity of the object, for $t \geq 0$, is

$$\mathbf{v}(t) = \langle x'(t), y'(t) \rangle = \langle u_0, -gt + v_0 \rangle$$

and the position is

$$\mathbf{r}(t) = \langle x(t), y(t) \rangle = \left\langle u_0\, t + x_0, -\frac{1}{2}gt^2 + v_0\, t + y_0 \right\rangle.$$

EXAMPLE 4 **Flight of a baseball** A baseball is hit from 3 ft above home plate with an initial velocity in ft/s of $\mathbf{v}(0) = \langle u_0, v_0 \rangle = \langle 80, 80 \rangle$. Neglect all forces other than gravity.

a. Find the position and velocity of the ball between the time it is hit and the time it first hits the ground.

b. Show that the trajectory of the ball is a segment of a parabola.

c. Assuming a flat playing field, how far does the ball travel horizontally? Plot the trajectory of the ball.

d. What is the maximum height of the ball?

e. Does the ball clear a 20-ft fence that is 380 ft from home plate (directly under the path of the ball)?

SOLUTION Assume the origin is located at home plate. Because distances are measured in feet, we use $g = 32\ \text{ft/s}^2$.

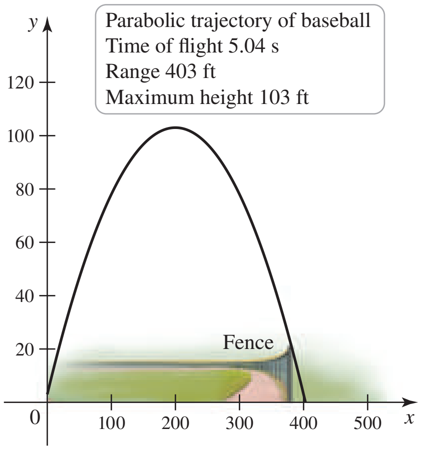


Figure 14.21

➤ The equation in Example 4c can be solved using the quadratic formula or a root-finder on a calculator.

QUICK CHECK 4 Write the functions $x(t)$ and $y(t)$ in Example 4 in the case that $x_0 = 0$, $y_0 = 2$, $u_0 = 100$, and $v_0 = 60$. ◀

a. Substituting $x_0 = 0$ and $y_0 = 3$ into the equation for $\mathbf{r}$, the position of the ball is

$$\mathbf{r}(t) = \langle x(t), y(t) \rangle = \langle 80t, -16t^2 + 80t + 3 \rangle, \quad \text{for } t \geq 0. \tag{3}$$

We then compute $\mathbf{v}(t) = \mathbf{r}'(t) = \langle 80, -32t + 80 \rangle$.

b. Equation (3) says that the horizontal position is $x = 80t$ and the vertical position is $y = -16t^2 + 80t + 3$. Substituting $t = x/80$ into the equation for y gives

$$y = -16\left(\frac{x}{80}\right)^2 + x + 3 = -\frac{x^2}{400} + x + 3,$$

which is an equation of a parabola.

c. The ball lands on the ground at the value of $t > 0$ at which $y = 0$. Solving $y(t) = -16t^2 + 80t + 3 = 0$, we find that $t \approx -0.04$ and $t \approx 5.04$ s. The first root is not relevant for the problem at hand, so we conclude that the ball lands when $t \approx 5.04$ s. The horizontal distance traveled by the ball is $x(5.04) \approx 403$ ft. The path of the ball in the xy-coordinate system on the time interval $[0, 5.04]$ is shown in Figure 14.21.

d. The ball reaches its maximum height at the time its vertical velocity is zero. Solving $y'(t) = -32t + 80 = 0$, we find that $t = 2.5$ s. The height at that time is $y(2.5) = 103$ ft.

e. The ball reaches a horizontal distance of 380 ft (the distance to the fence) when $x(t) = 80t = 380$. Solving for t, we find that $t = 4.75$ s. The height of the ball at that time is $y(4.75) = 22$ ft. So, indeed, the ball clears a 20-ft fence.

Related Exercises 41, 43 ◀

Range, Time of Flight, Maximum Height Having solved one specific motion problem, we can make some general observations about two-dimensional projectile motion in a gravitational field. Assume the motion of an object begins at the origin; that is, $x_0 = y_0 = 0$. Also assume the object is launched at an angle of α $(0 \leq \alpha \leq \pi/2)$ above the horizontal with an initial speed $|\mathbf{v}_0|$ (Figure 14.22). This means that the initial velocity is

$$\langle u_0, v_0 \rangle = \langle |\mathbf{v}_0| \cos \alpha, |\mathbf{v}_0| \sin \alpha \rangle.$$

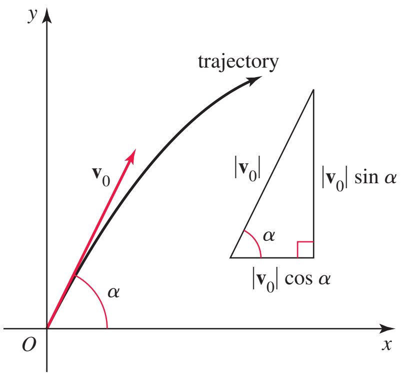


Figure 14.22

Substituting these values into the general expressions for the velocity and position, we find that the velocity of the object is

$$\mathbf{v}(t) = \langle u_0, -gt + v_0 \rangle = \langle |\mathbf{v}_0| \cos \alpha, -gt + |\mathbf{v}_0| \sin \alpha \rangle.$$

The position of the object (with $x_0 = y_0 = 0$) is

$$\mathbf{r}(t) = \langle x(t), y(t) \rangle = \langle (|\mathbf{v}_0| \cos \alpha)t, -gt^2/2 + (|\mathbf{v}_0| \sin \alpha)t \rangle.$$

Notice that the motion is determined entirely by the parameters $|\mathbf{v}_0|$ and α. Several general conclusions now follow.

➤ The other root of the equation $y(t) = 0$ is $t = 0$, the time the object leaves the ground.

1. Assuming the object is launched from the origin over horizontal ground, it returns to the ground when $y(t) = -gt^2/2 + (|\mathbf{v}_0| \sin \alpha)t = 0$. Solving for t, the **time of flight** is $T = 2|\mathbf{v}_0|(\sin \alpha)/g$.

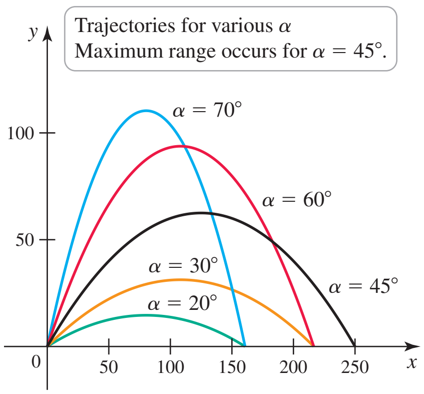


Figure 14.23

QUICK CHECK 5 Show that the range attained with an angle α equals the range attained with the angle $\pi/2 - \alpha$. ◄

2. The **range** of the object, which is the horizontal distance it travels, is the x-coordinate of the trajectory when $t = T$:

$$\begin{aligned} x(T) &= (|\mathbf{v}_0| \cos\alpha)T \\ &= (|\mathbf{v}_0| \cos\alpha)\frac{2|\mathbf{v}_0| \sin\alpha}{g} && \text{Substitute for } T. \\ &= \frac{2|\mathbf{v}_0|^2 \sin\alpha\cos\alpha}{g} && \text{Simplify.} \\ &= \frac{|\mathbf{v}_0|^2 \sin 2\alpha}{g}. && 2\sin\alpha\cos\alpha = \sin 2\alpha \end{aligned}$$

Note that on the interval $0 \le \alpha \le \pi/2$, $\sin 2\alpha$ has a maximum value of 1 when $\alpha = \pi/4$, so the maximum range is $|\mathbf{v}_0|^2/g$. In other words, in an ideal world, firing an object from the ground at an angle of $\pi/4$ (45°) maximizes its range. Notice that the ranges obtained with the angles α and $\pi/2 - \alpha$ are equal (**Figure 14.23**).

3. The maximum height of the object is reached when the vertical velocity is zero, or when $y'(t) = -gt + |\mathbf{v}_0| \sin\alpha = 0$. Solving for t, the maximum height is reached at $t = |\mathbf{v}_0|(\sin\alpha)/g = T/2$, which is half the time of flight. The object spends equal amounts of time ascending and descending. The maximum height is

$$y\left(\frac{T}{2}\right) = \frac{(|\mathbf{v}_0| \sin\alpha)^2}{2g}.$$

4. Finally, by eliminating t from the equations for $x(t)$ and $y(t)$, it can be shown (Exercise 72) that the trajectory of the object is a segment of a parabola.

➤ Use caution with the formulas in the summary box: They are applicable only when the initial position of the object is the origin. Exercise 73 addresses the case where the initial position of the object is $\langle 0, y_0 \rangle$.

SUMMARY Two-Dimensional Motion

Assume an object traveling over horizontal ground, acted on only by the gravitational force, has an initial position $\langle x_0, y_0 \rangle = \langle 0, 0 \rangle$ and initial velocity $\langle u_0, v_0 \rangle = \langle |\mathbf{v}_0| \cos\alpha, |\mathbf{v}_0| \sin\alpha \rangle$. The trajectory, which is a segment of a parabola, has the following properties.

$$\text{time of flight} = T = \frac{2|\mathbf{v}_0| \sin\alpha}{g}$$

$$\text{range} = \frac{|\mathbf{v}_0|^2 \sin 2\alpha}{g}$$

$$\text{maximum height} = y\left(\frac{T}{2}\right) = \frac{(|\mathbf{v}_0| \sin\alpha)^2}{2g}$$

EXAMPLE 5 Flight of a golf ball A golf ball is driven down a horizontal fairway with an initial speed of 55 m/s at an initial angle of 25° (from a tee with negligible height). Neglect all forces except gravity, and assume the ball's trajectory lies in a plane.

a. When the ball first touches the ground, how far has it traveled horizontally and how long has it been in the air?

b. What is the maximum height of the ball?

c. At what angles should the ball be hit to reach a green that is 300 m from the tee?

SOLUTION

a. Using the range formula with $\alpha = 25°$ and $|\mathbf{v}_0| = 55$ m/s, the ball travels

$$\frac{|\mathbf{v}_0|^2 \sin 2\alpha}{g} = \frac{(55\text{ m/s})^2 \sin 50°}{9.8\text{ m/s}^2} \approx 236\text{ m}.$$

The time of the flight is

$$T = \frac{2|\mathbf{v}_0|\sin\alpha}{g} = \frac{2(55\text{ m/s})\sin 25°}{9.8\text{ m/s}^2} \approx 4.7\text{ s}.$$

b. The maximum height of the ball is

$$\frac{(|\mathbf{v}_0|\sin\alpha)^2}{2g} = \frac{((55\text{ m/s})(\sin 25°))^2}{2(9.8\text{ m/s}^2)} \approx 27.6\text{ m}.$$

c. Letting R denote the range and solving the range formula for $\sin 2\alpha$, we find that $\sin 2\alpha = Rg/|\mathbf{v}_0|^2$. For a range of $R = 300$ m and an initial speed of $|\mathbf{v}_0| = 55$ m/s, the required angle satisfies

$$\sin 2\alpha = \frac{Rg}{|\mathbf{v}_0|^2} = \frac{(300\text{ m})(9.8\text{ m/s}^2)}{(55\text{ m/s})^2} \approx 0.972.$$

For the ball to travel a horizontal distance of exactly 300 m, the required angles are $\alpha = \frac{1}{2}\sin^{-1} 0.972 \approx 38.2°$ or $51.8°$.

Related Exercises 42, 45 ◄

Three-Dimensional Motion

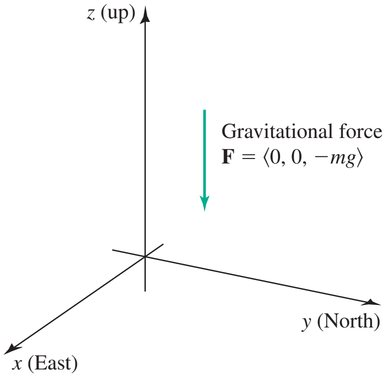


Figure 14.24

To solve three-dimensional motion problems, we adopt a coordinate system in which the x- and y-axes point in two perpendicular horizontal directions (for example, east and north), while the positive z-axis points vertically upward (Figure 14.24). Newton's Second Law now has three components and appears in the form

$$m\mathbf{a}(t) = \langle mx''(t), my''(t), mz''(t)\rangle = \mathbf{F}.$$

If only the gravitational force is present (now in the negative z-direction), then the force vector is $\mathbf{F} = \langle 0, 0, -mg\rangle$; the equation of motion is then $\mathbf{a}(t) = \langle 0, 0, -g\rangle$. Other effects, such as crosswinds, spins, or slices, can be modeled by including other force components.

EXAMPLE 6 **Projectile motion** A small projectile is fired to the east over horizontal ground with an initial speed of $|\mathbf{v}_0| = 300$ m/s at an angle of $\alpha = 30°$ above the horizontal. A crosswind blows from south to north, producing an acceleration of the projectile of 0.36 m/s^2 to the north.

a. Where does the projectile land? How far does it land from its launch site?

b. In order to correct for the crosswind and make the projectile land due east of the launch site, at what angle from due east must the projectile be fired? Assume the initial speed $|\mathbf{v}_0| = 300$ m/s and the angle of elevation $\alpha = 30°$ are the same as in part (a).

SOLUTION

a. Letting $g = 9.8\text{ m/s}^2$, the equations of motion are $\mathbf{a}(t) = \mathbf{v}'(t) = \langle 0, 0.36, -9.8\rangle$. Proceeding as in the two-dimensional case, the indefinite integral of the acceleration is the velocity function

$$\mathbf{v}(t) = \langle 0, 0.36t, -9.8t\rangle + \mathbf{C},$$

where $\mathbf{C}$ is an arbitrary constant vector. With an initial speed $|\mathbf{v}_0| = 300$ m/s and an angle of elevation of $\alpha = 30°$ (Figure 14.25a), the initial velocity is

$$\mathbf{v}(0) = \langle 300\cos 30°, 0, 300\sin 30°\rangle = \langle 150\sqrt{3}, 0, 150\rangle.$$

Substituting $t = 0$ and using the initial condition, we find that $\mathbf{C} = \langle 150\sqrt{3}, 0, 150\rangle$. Therefore, the velocity function is

$$\mathbf{v}(t) = \langle 150\sqrt{3}, 0.36t, -9.8t + 150\rangle.$$

Integrating the velocity function produces the position function

$$\mathbf{r}(t) = \langle 150\sqrt{3}t, 0.18t^2, -4.9t^2 + 150t\rangle + \mathbf{C}.$$

Using the initial condition $\mathbf{r}(0) = \langle 0, 0, 0 \rangle$, we find that $\mathbf{C} = \langle 0, 0, 0 \rangle$, and the position function is

$$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle = \langle 150\sqrt{3}t, 0.18t^2, -4.9t^2 + 150t \rangle.$$

The projectile lands when $z(t) = -4.9t^2 + 150t = 0$. Solving for t, the positive root, which gives the time of flight, is $T = 150/4.9 \approx 30.6$ s. The x- and y-coordinates at that time are

$$x(T) \approx 7953 \text{ m} \quad \text{and} \quad y(T) \approx 169 \text{ m}.$$

Therefore, the projectile lands approximately 7953 m east and 169 m north of the firing site. Because the projectile started at $(0, 0, 0)$, it traveled a horizontal distance of $\sqrt{7953^2 + 169^2} \approx 7955$ m (Figure 14.25a).

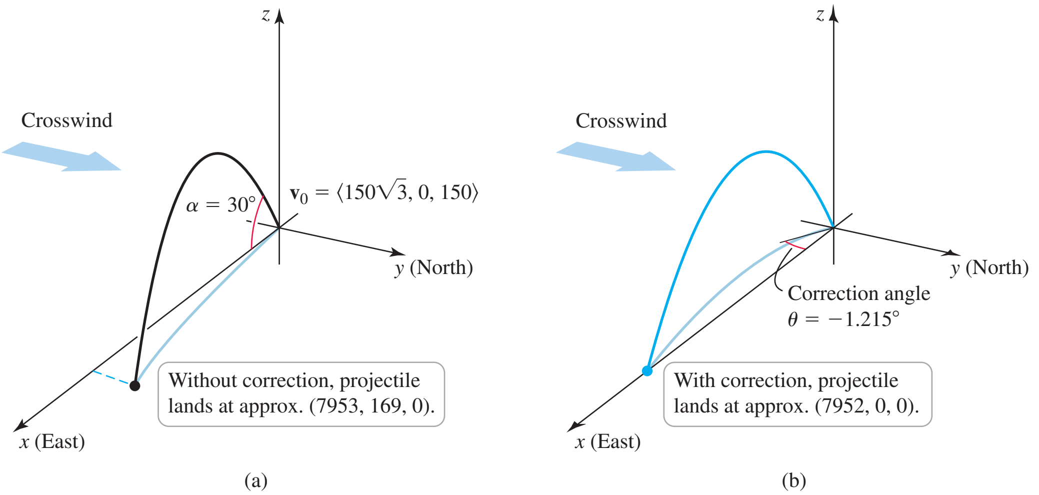


Figure 14.25

b. Keeping the initial speed of the projectile equal to $|\mathbf{v}_0| = 300$ m/s, we decompose the horizontal component of the speed, $150\sqrt{3}$ m/s, into an east component, $u_0 = 150\sqrt{3}\cos\theta$, and a north component, $v_0 = 150\sqrt{3}\sin\theta$, where θ is the angle relative to due east; we must determine the correction angle θ (**Figure 14.25b**). The x- and y-components of the position are

$$x(t) = (150\sqrt{3}\cos\theta)t \quad \text{and} \quad y(t) = 0.18t^2 + (150\sqrt{3}\sin\theta)t.$$

These changes in the initial velocity affect the x- and y-equations, but not the z-equation. Therefore, the time of flight is still $T = 150/4.9 \approx 30.6$ s. The aim is to choose θ so that the projectile lands on the x-axis (due east from the launch site), which means $y(T) = 0$. Solving

$$y(T) = 0.18T^2 + (150\sqrt{3}\sin\theta)T = 0,$$

with $T = 150/4.9$, we find that $\sin\theta \approx -0.0212$; therefore, $\theta \approx -0.0212 \text{ rad} \approx -1.215°$. In other words, the projectile must be fired at a horizontal angle of 1.215° to the *south* of east to correct for the northerly crosswind (Figure 14.25b). The landing location of the projectile is $x(T) \approx 7952$ m and $y(T) = 0$.

Related Exercises 52–53 ◄

SECTION 14.3 EXERCISES

Getting Started

1. Given the position function $\mathbf{r}$ of a moving object, explain how to find the velocity, speed, and acceleration of the object.

2. What is the relationship between the position and velocity vectors for motion on a circle?

3. Write Newton's Second Law of Motion in vector form.

4. Write Newton's Second Law of Motion for three-dimensional motion with only the gravitational force (acting in the z-direction).

5. Given the acceleration of an object and its initial velocity, how do you find the velocity of the object, for $t \geq 0$?

6. Given the velocity of an object and its initial position, how do you find the position of the object, for $t \ge 0$?

7. The velocity of a moving object, for $t \ge 0$, is $\mathbf{r}'(t) = \langle 60, 96 - 32t \rangle$ ft/s.
   a. When is the vertical component of velocity of the object equal to 0?
   b. Find $\mathbf{r}(t)$ if $\mathbf{r}(0) = \langle 0, 3 \rangle$.

8. A baseball is hit 2 feet above home plate, and the position of the ball t seconds later is $\mathbf{r}(t) = \langle 40t, -16t^2 + 31t + 2 \rangle$ ft. Find each of the following values.
   a. The time of flight of the baseball
   b. The range of the baseball

Practice Exercises

9–20. Velocity and acceleration from position *Consider the following position functions.*

a. *Find the velocity and speed of the object.*
b. *Find the acceleration of the object.*

9. $\mathbf{r}(t) = \langle 3t^2 + 1, 4t^2 + 3 \rangle$, for $t \ge 0$

10. $\mathbf{r}(t) = \left\langle \frac{5}{2}t^2 + 3, 6t^2 + 10 \right\rangle$, for $t \ge 0$

11. $\mathbf{r}(t) = \langle 2 + 2t, 1 - 4t \rangle$, for $t \ge 0$

12. $\mathbf{r}(t) = \langle 1 - t^2, 3 + 2t^3 \rangle$, for $t \ge 0$

13. $\mathbf{r}(t) = \langle 8 \sin t, 8 \cos t \rangle$, for $0 \le t \le 2\pi$

14. $\mathbf{r}(t) = \langle 3 \cos t, 4 \sin t \rangle$, for $0 \le t \le 2\pi$

15. $\mathbf{r}(t) = \left\langle t^2 + 3, t^2 + 10, \frac{1}{2}t^2 \right\rangle$, for $t \ge 0$

16. $\mathbf{r}(t) = \langle 2e^{2t} + 1, e^{2t} - 1, 2e^{2t} - 10 \rangle$, for $t \ge 0$

17. $\mathbf{r}(t) = \langle 3 + t, 2 - 4t, 1 + 6t \rangle$, for $t \ge 0$

18. $\mathbf{r}(t) = \langle 3 \sin t, 5 \cos t, 4 \sin t \rangle$, for $0 \le t \le 2\pi$

19. $\mathbf{r}(t) = \langle 1, t^2, e^{-t} \rangle$, for $t \ge 0$

20. $\mathbf{r}(t) = \langle 13 \cos 2t, 12 \text{ sint } 2t, 5 \sin 2t \rangle$, for $0 \le t \le \pi$

T **21–26. Comparing trajectories** *Consider the following position functions* $\mathbf{r}$ *and* $\mathbf{R}$ *for two objects.*

a. *Find the interval $[c, d]$ over which the* $\mathbf{R}$ *trajectory is the same as the* $\mathbf{r}$ *trajectory over $[a, b]$.*
b. *Find the velocity for both objects.*
c. *Graph the speed of the two objects over the intervals $[a, b]$ and $[c, d]$, respectively.*

21. $\mathbf{r}(t) = \langle t, t^2 \rangle$, $[a, b] = [0, 2]$; $\mathbf{R}(t) = \langle 2t, 4t^2 \rangle$ on $[c, d]$

22. $\mathbf{r}(t) = \langle 1 + 3t, 2 + 4t \rangle$, $[a, b] = [0, 6]$;
    $\mathbf{R}(t) = \langle 1 + 9t, 2 + 12t \rangle$ on $[c, d]$

23. $\mathbf{r}(t) = \langle \cos t, 4 \sin t \rangle$, $[a, b] = [0, 2\pi]$;
    $\mathbf{R}(t) = \langle \cos 3t, 4 \sin 3t \rangle$ on $[c, d]$

24. $\mathbf{r}(t) = \langle 2 - e^t, 4 - e^{-t} \rangle$, $[a, b] = [0, \ln 10]$;
    $\mathbf{R}(t) = \langle 2 - t, 4 - 1/t \rangle$ on $[c, d]$

25. $\mathbf{r}(t) = \langle 4 + t^2, 3 - 2t^4, 1 + 3t^6 \rangle$, $[a, b] = [0, 6]$;
    $\mathbf{R}(t) = \langle 4 + \ln t, 3 - 2 \ln^2 t, 1 + 3 \ln^3 t \rangle$ on $[c, d]$
    For graphing, let $c = 1$ and $d = 20$.

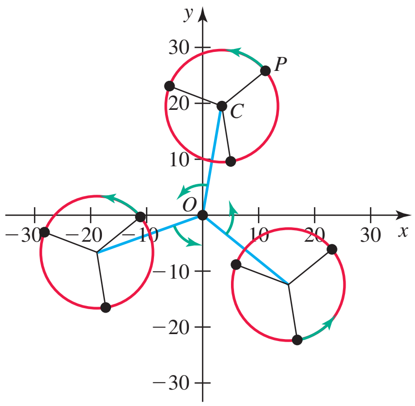
26. $\mathbf{r}(t) = \langle 2 \cos 2t, \sqrt{2} \sin 2t, \sqrt{2} \sin 2t \rangle$, $[a, b] = [0, \pi]$;
    $\mathbf{R}(t) = \langle 2 \cos 4t, \sqrt{2} \sin 4t, \sqrt{2} \sin 4t \rangle$ on $[c, d]$

T **27–28. Carnival rides**

27. Consider a carnival ride where Andrea is at point P that moves counterclockwise around a circle centered at C while the arm, represented by the line segment from the origin O to point C, moves counterclockwise about the origin (see figure). Andrea's position (in feet) at time t (in seconds) is

$$\mathbf{r}(t) = \langle 20 \cos t + 10 \cos 5t, 20 \sin t + 10 \sin 5t \rangle.$$

   a. Plot a graph of $\mathbf{r}(t)$, for $0 \le t \le 2\pi$.
   b. Find the velocity $\mathbf{v}(t)$.
   c. Show that the speed $|\mathbf{v}(t)| = v(t) = 10\sqrt{29 + 20 \cos 4t}$ and plot the speed, for $0 \le t \le 2\pi$. (*Hint:* Use the identity $\sin mx \sin nx + \cos mx \cos nx = \cos((m - n)x)$.)
   d. Determine Andrea's maximum and minimum speeds.



T 28. Suppose the carnival ride in Exercise 27 is modified so that Andrea's position P (in ft) at time t (in s) is

$$\mathbf{r}(t) = \langle 20 \cos t + 10 \cos 5t, 20 \sin t + 10 \sin 5t, 5 \sin 2t \rangle.$$

   a. Describe how this carnival ride differs from the ride in Exercise 27.
   b. Find the speed function $|\mathbf{v}(t)| = v(t)$ and plot its graph.
   c. Find Andrea's maximum and minimum speeds.

29–32. Trajectories on circles and spheres *Determine whether the following trajectories lie on either a circle in $\mathbb{R}^2$ or a sphere in $\mathbb{R}^3$ centered at the origin. If so, find the radius of the circle or sphere, and show that the position vector and the velocity vector are everywhere orthogonal.*

29. $\mathbf{r}(t) = \langle 8 \cos 2t, 8 \sin 2t \rangle$, for $0 \le t \le \pi$

30. $\mathbf{r}(t) = \langle 4 \sin t, 2 \cos t \rangle$, for $0 \le t \le 2\pi$

31. $\mathbf{r}(t) = \langle \sin t + \sqrt{3} \cos t, \sqrt{3} \sin t - \cos t \rangle$, for $0 \le t \le 2\pi$

32. $\mathbf{r}(t) = \langle 3 \sin t, 5 \cos t, 4 \sin t \rangle$, for $0 \le t \le 2\pi$

33–34. Path on a sphere *Show that the following trajectories lie on a sphere centered at the origin, and find the radius of the sphere.*

33. $\mathbf{r}(t) = \left\langle \frac{5 \sin t}{\sqrt{1 + \sin^2 2t}}, \frac{5 \cos t}{\sqrt{1 + \sin^2 2t}}, \frac{5 \sin 2t}{\sqrt{1 + \sin^2 2t}} \right\rangle$,
    for $0 \le t \le 2\pi$

34. $\mathbf{r}(t) = \left\langle \frac{4 \cos t}{\sqrt{4 + t^2}}, \frac{2t}{\sqrt{4 + t^2}}, \frac{4 \sin t}{\sqrt{4 + t^2}} \right\rangle$, for $0 \le t \le 4\pi$

35–40. Solving equations of motion *Given an acceleration vector, initial velocity $\langle u_0, v_0 \rangle$, and initial position $\langle x_0, y_0 \rangle$, find the velocity and position vectors for $t \ge 0$.*

35. $\mathbf{a}(t) = \langle 0, 1 \rangle$, $\langle u_0, v_0 \rangle = \langle 2, 3 \rangle$, $\langle x_0, y_0 \rangle = \langle 0, 0 \rangle$

36. $\mathbf{a}(t) = \langle 1, 2 \rangle$, $\langle u_0, v_0 \rangle = \langle 1, 1 \rangle$, $\langle x_0, y_0 \rangle = \langle 2, 3 \rangle$

37. $\mathbf{a}(t) = \langle 0, 10 \rangle$, $\langle u_0, v_0 \rangle = \langle 0, 5 \rangle$, $\langle x_0, y_0 \rangle = \langle 1, -1 \rangle$

38. $\mathbf{a}(t) = \langle 1, t \rangle$, $\langle u_0, v_0 \rangle = \langle 2, -1 \rangle$, $\langle x_0, y_0 \rangle = \langle 0, 8 \rangle$

39. $\mathbf{a}(t) = \langle \cos t, 2 \sin t \rangle$, $\langle u_0, v_0 \rangle = \langle 0, 1 \rangle$, $\langle x_0, y_0 \rangle = \langle 1, 0 \rangle$

40. $\mathbf{a}(t) = \langle e^{-t}, 1 \rangle$, $\langle u_0, v_0 \rangle = \langle 1, 0 \rangle$, $\langle x_0, y_0 \rangle = \langle 0, 0 \rangle$

T **41–46. Two-dimensional motion** *Consider the motion of the following objects. Assume the x-axis is horizontal, the positive y-axis is vertical, the ground is horizontal, and only the gravitational force acts on the object.*

a. Find the velocity and position vectors, for $t \geq 0$.
b. Graph the trajectory.
c. Determine the time of flight and range of the object.
d. Determine the maximum height of the object.

41. A soccer ball has an initial position (in m) of $\langle x_0, y_0 \rangle = \langle 0, 0 \rangle$ when it is kicked with an initial velocity of $\langle u_0, v_0 \rangle = \langle 30, 6 \rangle$ m/s.

42. A golf ball has an initial position (in ft) of $\langle x_0, y_0 \rangle = \langle 0, 0 \rangle$ when it is hit at an angle of 30° with an initial speed of 150 ft/s.

43. A baseball has an initial position (in ft) of $\langle x_0, y_0 \rangle = \langle 0, 6 \rangle$ when it is thrown with an initial velocity of $\langle u_0, v_0 \rangle = \langle 80, 10 \rangle$ ft/s.

44. A baseball is thrown horizontally from a height of 10 ft above the ground with a speed of 132 ft/s.

45. A projectile is launched from a platform 20 ft above the ground at an angle of 60° above the horizontal with a speed of 250 ft/s. Assume the origin is at the base of the platform.

46. A rock is thrown from the edge of a vertical cliff 40 m above the ground at an angle of 45° above the horizontal with a speed of $10\sqrt{2}$ m/s. Assume the origin is at the foot of the cliff.

47–50. Solving equations of motion *Given an acceleration vector, initial velocity $\langle u_0, v_0, w_0 \rangle$, and initial position $\langle x_0, y_0, z_0 \rangle$, find the velocity and position vectors, for $t \geq 0$.*

47. $\mathbf{a}(t) = \langle 0, 0, 10 \rangle$, $\langle u_0, v_0, w_0 \rangle = \langle 1, 5, 0 \rangle$, $\langle x_0, y_0, z_0 \rangle = \langle 0, 5, 0 \rangle$

48. $\mathbf{a}(t) = \langle 1, t, 4t \rangle$, $\langle u_0, v_0, w_0 \rangle = \langle 20, 0, 0 \rangle$, $\langle x_0, y_0, z_0 \rangle = \langle 0, 0, 0 \rangle$

49. $\mathbf{a}(t) = \langle \sin t, \cos t, 1 \rangle$, $\langle u_0, v_0, w_0 \rangle = \langle 0, 2, 0 \rangle$, $\langle x_0, y_0, z_0 \rangle = \langle 0, 0, 0 \rangle$

50. $\mathbf{a}(t) = \langle t, e^{-t}, 1 \rangle$, $\langle u_0, v_0, w_0 \rangle = \langle 0, 0, 1 \rangle$, $\langle x_0, y_0, z_0 \rangle = \langle 4, 0, 0 \rangle$

T **51–56. Three-dimensional motion** *Consider the motion of the following objects. Assume the x-axis points east, the y-axis points north, the positive z-axis is vertical and opposite g, the ground is horizontal, and only the gravitational force acts on the object unless otherwise stated.*

a. Find the velocity and position vectors, for $t \geq 0$.
b. Make a sketch of the trajectory.
c. Determine the time of flight and range of the object.
d. Determine the maximum height of the object.

51. A bullet is fired from a rifle 1 m above the ground in a northeast direction. The initial velocity of the bullet is $\langle 200, 200, 0 \rangle$ m/s.

52. A golf ball is hit east down a fairway with an initial velocity of $\langle 50, 0, 30 \rangle$ m/s. A crosswind blowing to the south produces an acceleration of the ball of -0.8 m/s^2.

53. A baseball is hit 3 ft above home plate with an initial velocity of $\langle 60, 80, 80 \rangle$ ft/s. The spin on the baseball produces a horizontal acceleration of the ball of 10 ft/s^2 in the eastward direction.

54. A baseball is hit 3 ft above home plate with an initial velocity of $\langle 30, 30, 80 \rangle$ ft/s. The spin on the baseball produces a horizontal acceleration of the ball of 5 ft/s^2 in the northward direction.

55. A small rocket is fired from a launch pad 10 m above the ground with an initial velocity, in m/s, of $\langle 300, 400, 500 \rangle$. A crosswind blowing to the north produces an acceleration of the rocket of 2.5 m/s^2.

56. A soccer ball is kicked from the point $\langle 0, 0, 0 \rangle$ with an initial velocity of $\langle 0, 80, 80 \rangle$ ft/s. The spin on the ball produces an acceleration of $\langle 1.2, 0, 0 \rangle$ ft/s^2.

57. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. If the speed of an object is constant, then its velocity components are constant.
b. The functions $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$ and $\mathbf{R}(t) = \langle \sin t^2, \cos t^2 \rangle$ generate the same set of points, for $t \geq 0$.
c. A velocity vector of variable magnitude cannot have a constant direction.
d. If the acceleration of an object is $\mathbf{a}(t) = \mathbf{0}$, for all $t \geq 0$, then the velocity of the object is constant.
e. If you double the initial speed of a projectile, its range also doubles (assume no forces other than gravity act on the projectile).
f. If you double the initial speed of a projectile, its time of flight also doubles (assume no forces other than gravity).
g. A trajectory with $\mathbf{v}(t) = \mathbf{a}(t) \neq \mathbf{0}$, for all t, is possible.

T **58–61. Trajectory properties** *Find the time of flight, range, and maximum height of the following two-dimensional trajectories, assuming no forces other than gravity. In each case, the initial position is $\langle 0, 0 \rangle$ and the initial velocity is $\mathbf{v}_0 = \langle u_0, v_0 \rangle$.*

58. $\langle u_0, v_0 \rangle = \langle 10, 20 \rangle$ ft/s

59. Initial speed $|\mathbf{v}_0| = 150$ m/s, launch angle $\alpha = 30°$

60. $\langle u_0, v_0 \rangle = \langle 40, 80 \rangle$ m/s

61. Initial speed $|\mathbf{v}_0| = 400$ ft/s, launch angle $\alpha = 60°$

62. Motion on the moon The acceleration due to gravity on the moon is approximately $g/6$ (one-sixth its value on Earth). Compare the time of flight, range, and maximum height of a projectile on the moon with the corresponding values on Earth.

63. Firing angles A projectile is fired over horizontal ground from the origin with an initial speed of 60 m/s. What firing angles produce a range of 300 m?

T **64. Firing strategies** Suppose you wish to fire a projectile over horizontal ground from the origin and attain a range of 1000 m.

a. Sketch a graph of the initial speed required for all firing angles $0 < \alpha < \pi/2$.
b. What firing angle requires the least initial speed?

65. Speed on an ellipse An object moves along an ellipse given by the function $\mathbf{r}(t) = \langle a \cos t, b \sin t \rangle$, for $0 \leq t \leq 2\pi$, where $a > 0$ and $b > 0$.

a. Find the velocity and speed of the object in terms of a and b, for $0 \leq t \leq 2\pi$.

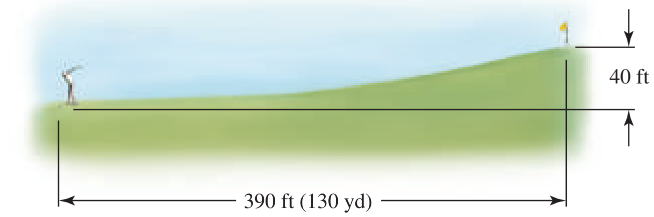
b. With $a = 1$ and $b = 6$, graph the speed function, for $0 \le t \le 2\pi$. Mark the points on the trajectory at which the speed is a minimum and a maximum.

c. Is it true that the object speeds up along the flattest (straightest) parts of the trajectory and slows down where the curves are sharpest?

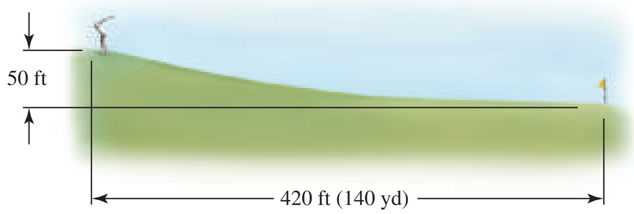
d. For general a and b, find the ratio of the maximum speed to the minimum speed on the ellipse (in terms of a and b).

Explorations and Challenges

66. Golf shot A golfer stands 390 ft (130 yd) horizontally from the hole and 40 ft below the hole (see figure). Assuming the ball is hit with an initial speed of 150 ft/s, at what angle(s) should it be hit to land in the hole? Assume the path of the ball lies in a plane.



67. Another golf shot A golfer stands 420 ft (140 yd) horizontally from the hole and 50 ft above the hole (see figure). Assuming the ball is hit with an initial speed of 120 ft/s, at what angle(s) should it be hit to land in the hole? Assume the path of the ball lies in a plane.



T 68. Initial speed of a golf shot A golfer stands 390 ft horizontally from the hole and 40 ft below the hole (see figure for Exercise 66). If the ball leaves the ground at an initial angle of 45° with the horizontal, with what initial speed should it be hit to land in the hole?

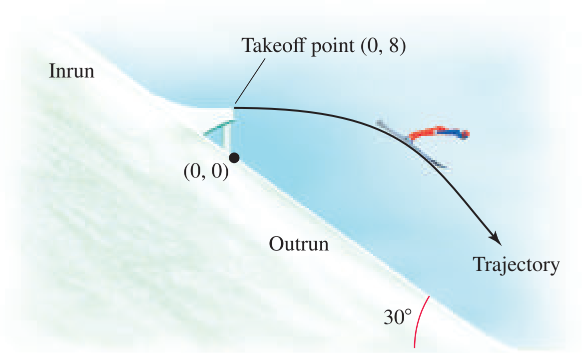
T 69. Initial speed of a golf shot A golfer stands 420 ft horizontally from the hole and 50 ft above the hole (see figure for Exercise 67). If the ball leaves the ground at an initial angle of 30° with the horizontal, with what initial speed should it be hit to land in the hole?

T 70. Ski jump The lip of a ski jump is 8 m above the outrun that is sloped at an angle of 30° to the horizontal (see figure).

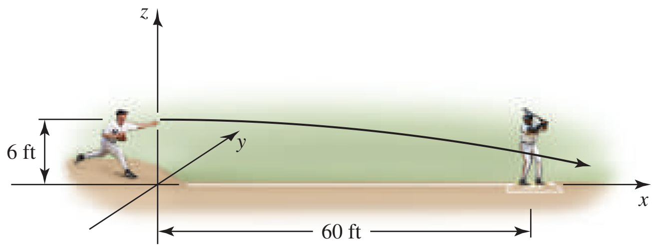
a. If the initial velocity of a ski jumper at the lip of the jump is $\langle 40, 0 \rangle$ m/s, what is the length of the jump (distance from the origin to the landing point)? Assume only gravity affects the motion.

b. Assume air resistance produces a constant horizontal acceleration of 0.15 m/s^2 opposing the motion. What is the length of the jump?

c. Suppose the takeoff ramp is tilted upward at an angle of $\theta°$, so that the skier's initial velocity is $40\langle \cos\theta, \sin\theta \rangle$ m/s. What value of θ maximizes the length of the jump? Express your answer in degrees and neglect air resistance.



71. Designing a baseball pitch A baseball leaves the hand of a pitcher 6 vertical feet above and 60 horizontal feet from home plate. Assume the coordinate axes are oriented as shown in the figure.



a. Suppose a pitch is thrown with an initial velocity of $\langle 130, 0, -3 \rangle$ ft/s (about 90 mi/hr). In the absence of all forces except gravity, how far above the ground is the ball when it crosses home plate and how long does it take the pitch to arrive?

b. What vertical velocity component should the pitcher use so that the pitch crosses home plate exactly 3 ft above the ground?

c. A simple model to describe the curve of a baseball assumes the spin of the ball produces a constant sideways acceleration (in the y-direction) of c ft/s^2. Suppose a pitcher throws a curve ball with $c = 8$ ft/s^2 (one fourth the acceleration of gravity). How far does the ball move in the y-direction by the time it reaches home plate, assuming an initial velocity of $\langle 130, 0, -3 \rangle$ ft/s?

d. In part (c), does the ball curve more in the first half of its trip to the plate or in the second half? How does this fact affect the batter?

e. Suppose the pitcher releases the ball from an initial position of $\langle 0, -3, 6 \rangle$ with initial velocity $\langle 130, 0, -3 \rangle$. What value of the spin parameter c is needed to put the ball over home plate passing through the point $(60, 0, 3)$?

72. Parabolic trajectories Show that the two-dimensional trajectory

$$x(t) = u_0 t + x_0 \text{ and } y(t) = -\frac{gt^2}{2} + v_0 t + y_0, \text{ for } 0 \le t \le T,$$

of an object moving in a gravitational field is a segment of a parabola for some value of $T > 0$. Find T such that $y(T) = 0$.

73. Time of flight, range, height Derive the formulas for time of flight, range, and maximum height in the case that an object is launched from the initial position $\langle 0, y_0 \rangle$ above the horizontal ground with initial velocity $|\mathbf{v}_0| \langle \cos\alpha, \sin\alpha \rangle$.

74. A race Two people travel from $P(4, 0)$ to $Q(-4, 0)$ along the paths given by

$$\mathbf{r}(t) = \left\langle 4\cos\frac{\pi t}{8}, 4\sin\frac{\pi t}{8}\right\rangle \quad \text{and}$$
$$\mathbf{R}(t) = \langle 4 - t, (4 - t)^2 - 16\rangle.$$

a. Graph both paths between P and Q.
b. Graph the speeds of both people between P and Q.
c. Who arrives at Q first?

75. Circular motion Consider an object moving along the circular trajectory $\mathbf{r}(t) = \langle A\cos\omega t, A\sin\omega t\rangle$, where A and ω are constants.

a. Over what time interval $[0, T]$ does the object traverse the circle once?
b. Find the velocity and speed of the object. Is the velocity constant in either direction or magnitude? Is the speed constant?
c. Find the acceleration of the object.
d. How are the position and velocity related? How are the position and acceleration related?
e. Sketch the position, velocity, and acceleration vectors at four different points on the trajectory with $A = \omega = 1$.

76. A linear trajectory An object moves along a straight line from the point $P(1, 2, 4)$ to the point $Q(-6, 8, 10)$.

a. Find a position function $\mathbf{r}$ that describes the motion if it occurs with a constant speed over the time interval $[0, 5]$.
b. Find a position function $\mathbf{r}$ that describes the motion if it occurs with speed e^t.

77. A circular trajectory An object moves clockwise around a circle centered at the origin with radius 5 m beginning at the point $(0, 5)$.

a. Find a position function $\mathbf{r}$ that describes the motion if the object moves with a constant speed, completing 1 lap every 12 s.
b. Find a position function $\mathbf{r}$ that describes the motion if it occurs with speed e^{-t}.

78. A helical trajectory An object moves on the helix $\langle \cos t, \sin t, t\rangle$, for $t \geq 0$.

a. Find a position function $\mathbf{r}$ that describes the motion if it occurs with a constant speed of 10.
b. Find a position function $\mathbf{r}$ that describes the motion if it occurs with speed t.

79. Tilted ellipse Consider the curve $\mathbf{r}(t) = \langle \cos t, \sin t, c\sin t\rangle$, for $0 \leq t \leq 2\pi$, where c is a real number. Assuming the curve lies in a plane, prove that the curve is an ellipse in that plane.

80. Equal area property Consider the ellipse $\mathbf{r}(t) = \langle a\cos t, b\sin t\rangle$, for $0 \leq t \leq 2\pi$, where a and b are real numbers. Let θ be the angle between the position vector and the x-axis.

a. Show that $\tan\theta = \frac{b}{a}\tan t$.
b. Find $\theta'(t)$.
c. Recall that the area bounded by the polar curve $r = f(\theta)$ on the interval $[0, \theta]$ is $A(\theta) = \frac{1}{2}\int_0^\theta (f(u))^2\,du$. Letting $f(\theta(t)) = |\mathbf{r}(\theta(t))|$, show that $A'(t) = \frac{1}{2}ab$.
d. Conclude that as an object moves around the ellipse, it sweeps out equal areas in equal times.

81. Another property of constant $|\mathbf{r}|$ motion Suppose an object moves on the surface of a sphere with $|\mathbf{r}(t)|$ constant for all t. Show that $\mathbf{r}(t)$ and $\mathbf{a}(t) = \mathbf{r}''(t)$ satisfy $\mathbf{r}(t)\cdot\mathbf{a}(t) = -|\mathbf{v}(t)|^2$.

82. Conditions for a circular/elliptical trajectory in the plane An object moves along a path given by

$$\mathbf{r}(t) = \langle a\cos t + b\sin t, c\cos t + d\sin t\rangle, \text{ for } 0 \leq t \leq 2\pi.$$

a. What conditions on a, b, c, and d guarantee that the path is a circle?
b. What conditions on a, b, c, and d guarantee that the path is an ellipse?

83. Nonuniform straight-line motion Consider the motion of an object given by the position function

$$\mathbf{r}(t) = f(t)\langle a, b, c\rangle + \langle x_0, y_0, z_0\rangle, \quad \text{for } t \geq 0,$$

where a, b, c, x_0, y_0, and z_0 are constants, and f is a differentiable scalar function, for $t \geq 0$.

a. Explain why $\mathbf{r}$ describes motion along a line.
b. Find the velocity function. In general, is the velocity constant in magnitude or direction along the path?

QUICK CHECK ANSWERS

1. $\mathbf{v}(t) = \langle 1, 2t, 3t^2\rangle$, $\mathbf{a}(t) = \langle 0, 2, 6t\rangle$
2. $|\mathbf{r}'(t)| = \sqrt{1 + 4t^2 + 9t^4/16}$; $|\mathbf{R}'(t)| = \sqrt{4t^2 + 16t^6 + 9t^{10}/4}$ **3.** The curve $\mathbf{R}(t) = 5\mathbf{r}(t)/|\mathbf{r}(t)|$ lies on a sphere of radius 5.
4. $x(t) = 100t$, $y(t) = -16t^2 + 60t + 2$
5. $\sin(2(\pi/2 - \alpha)) = \sin(\pi - 2\alpha) = \sin 2\alpha$ ◄

14.4 Length of Curves

We return now to a recurring theme: determining the arc length of a curve. In Section 6.5, we learned how to find the arc length of curves of the form $y = f(x)$, and in Sections 12.1 and 12.3, we discovered formulas for the arc length of a plane curve described parametrically or described in polar coordinates. In this section, we extend these ideas to handle the arc length of a three-dimensional curve described by a vector function. We also discover how to formulate a parametric description of a curve using arc length as a parameter.

Arc Length

Suppose a curve is described by the vector-valued function $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, for $a \le t \le b$, where f', g', and h' are continuous on $[a, b]$. In Section 12.1, we showed that the arc length L of the two-dimensional curve $\mathbf{r}(t) = \langle f(t), g(t) \rangle$, for $a \le t \le b$ is given by

$$L = \int_a^b \sqrt{f'(t)^2 + g'(t)^2}\, dt.$$

An analogous arc length formula for three-dimensional curves follows using a similar argument. The length of the curve $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ on the interval $[a, b]$ is

$$L = \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2}\, dt.$$

Noting that $\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$, we state the following definition.

➤ An important fact is that the arc length of a smooth parameterized curve is independent of the choice of parameter (Exercise 54).

➤ For curves in the xy-plane, we set $h(t) = 0$ in the definition of arc length.

DEFINITION Arc Length for Vector Functions

Consider the parameterized curve $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, where f', g', and h' are continuous, and the curve is traversed once for $a \le t \le b$. The **arc length** of the curve between $(f(a), g(a), h(a))$ and $(f(b), g(b), h(b))$ is

$$L = \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2}\, dt = \int_a^b |\mathbf{r}'(t)|\, dt.$$

QUICK CHECK 1 What does the arc length formula give for the length of the line $\mathbf{r}(t) = \langle 2t, t, -2t \rangle$, for $0 \le t \le 3$? ◄

The following application of arc length leads to an integral that is difficult to evaluate exactly.

EXAMPLE 1 Lengths of planetary orbits According to Kepler's first law, the planets revolve about the sun in elliptical orbits. A vector function that describes an ellipse in the xy-plane is

$$\mathbf{r}(t) = \langle a \cos t, b \sin t \rangle, \qquad \text{where } 0 \le t \le 2\pi.$$

If $a > b > 0$, then $2a$ is the length of the major axis and $2b$ is the length of the minor axis (Figure 14.26). Verify the lengths of the planetary orbits given in Table 14.1. Distances are given in terms of the astronomical unit (AU), which is the length of the semimajor axis of Earth's orbit, or about 93 million miles.

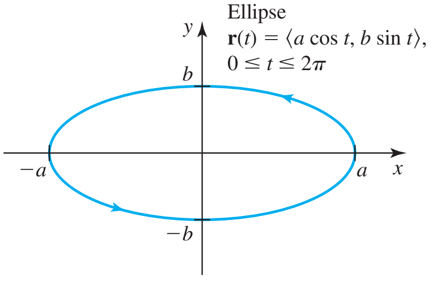


Figure 14.26

➤ The German astronomer and mathematician Johannes Kepler (1571–1630) worked with the meticulously gathered data of Tycho Brahe to formulate three empirical laws obeyed by planets and comets orbiting the sun. The work of Kepler formed the foundation for Newton's laws of gravitation developed 50 years later.

➤ In September 2006, Pluto joined the ranks of Ceres, Haumea, Makemake, and Eris as one of five dwarf planets in our solar system.

Table 14.1

Planet	Semimajor axis, a (AU)	Semiminor axis, b (AU)	$\alpha = b/a$	Orbit length (AU)
Mercury	0.387	0.379	0.979	2.407
Venus	0.723	0.723	1.000	4.543
Earth	1.000	0.999	0.999	6.280
Mars	1.524	1.517	0.995	9.554
Jupiter	5.203	5.179	0.995	32.616
Saturn	9.539	9.524	0.998	59.888
Uranus	19.182	19.161	0.999	120.458
Neptune	30.058	30.057	1.000	188.857

SOLUTION Using the arc length formula, the length of a general elliptical orbit is

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{x'(t)^2 + y'(t)^2}\, dt \\ &= \int_0^{2\pi} \sqrt{(-a \sin t)^2 + (b \cos t)^2}\, dt && \text{Substitute for } x'(t) \text{ and } y'(t). \\ &= \int_0^{2\pi} \sqrt{a^2 \sin^2 t + b^2 \cos^2 t}\, dt. && \text{Simplify.} \end{aligned}$$

Factoring a^2 out of the square root and letting $\alpha = b/a$, we have

$$L = \int_0^{2\pi} \sqrt{a^2(\sin^2 t + (b/a)^2 \cos^2 t)}\, dt \quad \text{Factor out } a^2.$$
$$= a\int_0^{2\pi} \sqrt{\sin^2 t + \alpha^2 \cos^2 t}\, dt \quad \text{Let } \alpha = b/a.$$
$$= 4a\int_0^{\pi/2} \sqrt{\sin^2 t + \alpha^2 \cos^2 t}\, dt. \quad \text{Use symmetry; quarter orbit on } [0, \pi/2].$$

➤ The integral that gives the length of an ellipse is a *complete elliptic integral of the second kind*. Many reference books and software packages provide approximate values of this integral.

Unfortunately, an antiderivative for this integrand cannot be found in terms of elementary functions, so we have two options: This integral is well known and values have been tabulated for various values of α. Alternatively, we may use a calculator to approximate the integral numerically (see Section 8.8). Using numerical integration, the orbit lengths in Table 14.1 are obtained. For example, the length of Mercury's orbit with $a = 0.387$ and $\alpha = 0.979$ is

➤ Although rounded values for α appear in Table 14.1, the calculations in Example 1 were done in full precision and were rounded to three decimal places only in the final step.

$$L = 4a\int_0^{\pi/2} \sqrt{\sin^2 t + \alpha^2 \cos^2 t}\, dt$$
$$= 1.548\int_0^{\pi/2} \sqrt{\sin^2 t + 0.959 \cos^2 t}\, dt \quad \text{Simplify.}$$
$$\approx 2.407. \quad \text{Approximate using calculator.}$$

The fact that α is close to 1 for all the planets means that their orbits are nearly circular. For this reason, the lengths of the orbits shown in the table are nearly equal to $2\pi a$, which is the length of a circular orbit with radius a.

Related Exercises 27–28 ◄

➤ Recall from Chapter 6 that the distance traveled by an object in one dimension is $\int_a^b |v(t)|\, dt$. The arc length formula generalizes this formula to three dimensions.

If the function $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ is the position function for a moving object, then the arc length formula has a natural interpretation. Recall that $\mathbf{v}(t) = \mathbf{r}'(t)$ is the velocity of the object and $|\mathbf{v}(t)| = |\mathbf{r}'(t)|$ is the speed of the object. Therefore, the arc length formula becomes

$$L = \int_a^b |\mathbf{r}'(t)|\, dt = \int_a^b |\mathbf{v}(t)|\, dt.$$

This formula is an analog of the familiar *distance* = *speed* × *elapsed time* formula.

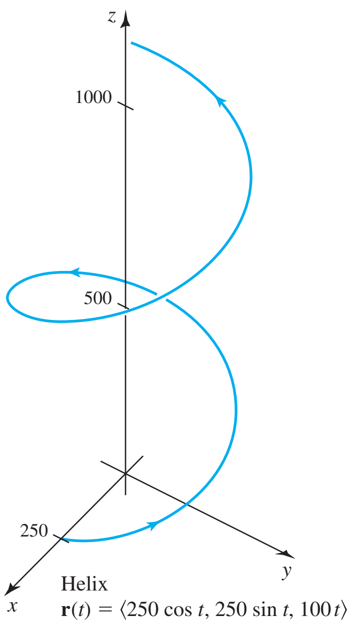


Figure 14.27

EXAMPLE 2 **Flight of an eagle** An eagle rises at a rate of 100 vertical ft/min on a helical path given by

$$\mathbf{r}(t) = \langle 250 \cos t, 250 \sin t, 100t \rangle$$

(**Figure 14.27**), where $\mathbf{r}$ is measured in feet and t is measured in minutes. How far does it travel in 10 min?

SOLUTION The speed of the eagle is

$$|\mathbf{v}(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$$
$$= \sqrt{(-250 \sin t)^2 + (250 \cos t)^2 + 100^2} \quad \text{Substitute derivatives.}$$
$$= \sqrt{250^2(\sin^2 t + \cos^2 t) + 100^2} \quad \text{Combine terms.}$$
$$= \sqrt{250^2 + 100^2} \approx 269. \quad \sin^2 t + \cos^2 t = 1$$

The constant speed makes the arc length integral easy to evaluate:

$$L = \int_0^{10} |\mathbf{v}(t)|\, dt \approx \int_0^{10} 269\, dt = 2690.$$

The eagle travels approximately 2690 ft in 10 min.

Related Exercise 25 ◄

➤ The standard parameterization for a helix winding counterclockwise around the z-axis is $\mathbf{r}(t) = \langle a \cos t, a \sin t, bt \rangle$. A helix has the property that its tangent vector makes a constant angle with the axis around which it winds.

Arc Length as a Parameter

Until now, the parameter t used to describe a curve $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ has been chosen either for convenience or because it represents time in some specified unit. We now introduce the most natural parameter for describing curves; that parameter is *arc length*. Let's see what it means for a curve to be *parameterized by arc length*.

Consider the following two characterizations of the unit circle centered at the origin:

$$\langle \cos t, \sin t \rangle, \text{ for } 0 \leq t \leq 2\pi \quad \text{and} \quad \langle \cos 2t, \sin 2t \rangle, \text{ for } 0 \leq t \leq \pi.$$

In the first description, as the parameter t increases from $t = 0$ to $t = 2\pi$, the full circle is generated and the arc length s of the curve also increases from $s = 0$ to $s = 2\pi$. In other words, as the parameter t increases, it measures the arc length of the curve that is generated (**Figure 14.28a**).

In the second description, as t varies from $t = 0$ to $t = \pi$, the full circle is generated and the arc length increases from $s = 0$ to $s = 2\pi$. In this case, the length of the interval in t does not equal the length of the curve generated; therefore, the parameter t does not correspond to arc length (**Figure 14.28b**). In general, there are infinitely many ways to parameterize a given curve; however, for a given initial point and orientation, arc length is the parameter for only one of them.

QUICK CHECK 2 Consider the portion of a circle $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$, for $a \leq t \leq b$. Show that the arc length of the curve is $b - a$. ◂

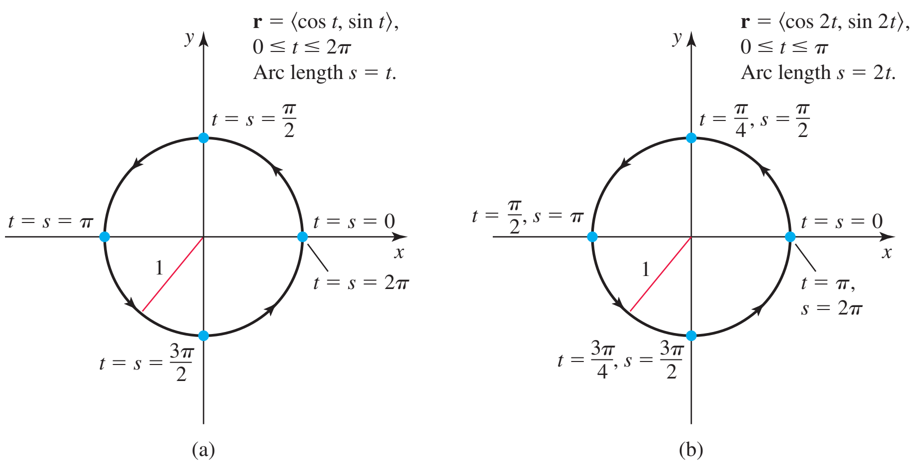


Figure 14.28

The Arc Length Function Suppose a smooth curve is represented by the function $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, for $t \geq a$, where t is a parameter. Notice that as t increases, the length of the curve also increases. Using the arc length formula, the length of the curve from $\mathbf{r}(a)$ to $\mathbf{r}(t)$ is

➤ Notice that t is the independent variable of the function $s(t)$, so a different symbol u is used for the variable of integration. It is common to use s as the arc length function.

$$s(t) = \int_a^t \sqrt{f'(u)^2 + g'(u)^2 + h'(u)^2}\, du = \int_a^t |\mathbf{v}(u)|\, du.$$

This equation gives the relationship between the arc length of a curve and any parameter t used to describe the curve.

An important consequence of this relationship arises if we differentiate both sides with respect to t using the Fundamental Theorem of Calculus:

$$\frac{ds}{dt} = \frac{d}{dt}\left(\int_a^t |\mathbf{v}(u)|\, du \right) = |\mathbf{v}(t)|.$$

Specifically, if t represents time and $\mathbf{r}$ is the position of an object moving on the curve, then the rate of change of the arc length with respect to time is the speed of the object. Notice that if $\mathbf{r}(t)$ describes a smooth curve, then $|\mathbf{v}(t)| \neq 0$; hence $ds/dt > 0$, and s is an increasing function of t—as t increases, the arc length also increases. If $\mathbf{r}(t)$ is a curve on which $|\mathbf{v}(t)| = 1$, then

$$s(t) = \int_a^t |\mathbf{v}(u)|\, du = \int_a^t 1\, du = t - a,$$

which means the parameter t corresponds to arc length. These observations are summarized in the following theorem.

THEOREM 14.3 Arc Length as a Function of a Parameter
Let $\mathbf{r}(t)$ describe a smooth curve, for $t \geq a$. The arc length is given by

$$s(t) = \int_a^t |\mathbf{v}(u)|\, du,$$

where $|\mathbf{v}| = |\mathbf{r}'|$. Equivalently, $\dfrac{ds}{dt} = |\mathbf{v}(t)|$. If $|\mathbf{v}(t)| = 1$, for all $t \geq a$, then the parameter t corresponds to arc length.

EXAMPLE 3 Arc length parameterization Consider the helix $\mathbf{r}(t) = \langle 2\cos t, 2\sin t, 4t\rangle$, for $t \geq 0$.

a. Find the arc length function $s(t)$.

b. Find another description of the helix that uses arc length as the parameter.

SOLUTION

a. Note that $\mathbf{r}'(t) = \langle -2\sin t, 2\cos t, 4\rangle$ and

$$\begin{aligned} |\mathbf{v}(t)| = |\mathbf{r}'(t)| &= \sqrt{(-2\sin t)^2 + (2\cos t)^2 + 4^2} \\ &= \sqrt{4(\sin^2 t + \cos^2 t) + 4^2} && \text{Simplify.} \\ &= \sqrt{4 + 4^2} && \sin^2 t + \cos^2 t = 1 \\ &= \sqrt{20} = 2\sqrt{5}. && \text{Simplify.} \end{aligned}$$

Therefore, the relationship between the arc length s and the parameter t is

$$s(t) = \int_a^t |\mathbf{v}(u)|\, du = \int_0^t 2\sqrt{5}\, du = 2\sqrt{5}\, t.$$

An increase of $1/(2\sqrt{5})$ in the parameter t corresponds to an increase of 1 in the arc length. Therefore, the curve is not parameterized by arc length.

b. Substituting $t = s/(2\sqrt{5})$ into the original parametric description of the helix, we find that the description with arc length as a parameter is (using a different function name)

$$\mathbf{r}_1(s) = \left\langle 2\cos\left(\frac{s}{2\sqrt{5}}\right), 2\sin\left(\frac{s}{2\sqrt{5}}\right), \frac{2s}{\sqrt{5}}\right\rangle, \quad \text{for } s \geq 0.$$

This description has the property that an increment of Δs in the parameter corresponds to an increment of exactly Δs in the arc length.

Related Exercises 37–39 ◄

QUICK CHECK 3 Does the line $\mathbf{r}(t) = \langle t, t, t\rangle$ have arc length as a parameter? Explain. ◄

As you will see in Section 14.5, using arc length as a parameter—when it can be done—generally leads to simplified calculations.

SECTION 14.4 EXERCISES

Getting Started

1. Find the length of the line given by $\mathbf{r}(t) = \langle t, 2t\rangle$, for $a \leq t \leq b$.

2. Explain how to find the length of the curve
$$\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle, \text{ for } a \leq t \leq b.$$

3. Express the arc length of a curve in terms of the speed of an object moving along the curve.

4. Suppose an object moves in space with the position function $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$. Write the integral that gives the distance it travels between $t = a$ and $t = b$.

5. An object moves on a trajectory given by
$$\mathbf{r}(t) = \langle 10\cos 2t, 10\sin 2t\rangle, \text{ for } 0 \leq t \leq \pi.$$
How far does it travel?

6. Use calculus to find the length of the line segment $\mathbf{r}(t) = \langle t, -8t, 4t\rangle$, for $0 \leq t \leq 2$. Verify your answer without using calculus.

7. Explain what it means for a curve to be parameterized by its arc length.

8. Is the curve $\mathbf{r}(t) = \langle \cos t, \sin t\rangle$ parameterized by its arc length? Explain.

Practice Exercises

9–22. Arc length calculations *Find the length of the following two- and three-dimensional curves.*

9. $\mathbf{r}(t) = \langle 3t^2 - 1, 4t^2 + 5 \rangle$, for $0 \le t \le 1$

10. $\mathbf{r}(t) = \langle 3t - 1, 4t + 5, t \rangle$, for $0 \le t \le 1$

11. $\mathbf{r}(t) = \langle 3\cos t, 3\sin t \rangle$, for $0 \le t \le \pi$

12. $\mathbf{r}(t) = \langle 4\cos 3t, 4\sin 3t \rangle$, for $0 \le t \le 2\pi/3$

13. $\mathbf{r}(t) = \langle \cos t + t\sin t, \sin t - t\cos t \rangle$, for $0 \le t \le \pi/2$

14. $\mathbf{r}(t) = \langle \cos t + \sin t, \cos t - \sin t \rangle$, for $0 \le t \le 2\pi$

15. $\mathbf{r}(t) = \langle 2 + 3t, 1 - 4t, -4 + 3t \rangle$, for $1 \le t \le 6$

16. $\mathbf{r}(t) = \langle 4\cos t, 4\sin t, 3t \rangle$, for $0 \le t \le 6\pi$

17. $\mathbf{r}(t) = \langle t, 8\sin t, 8\cos t \rangle$, for $0 \le t \le 4\pi$

18. $\mathbf{r}(t) = \left\langle \dfrac{t^2}{2}, \dfrac{(2t+1)^{3/2}}{3} \right\rangle$, for $0 \le t \le 2$

19. $\mathbf{r}(t) = \langle e^{2t}, 2e^{2t} + 5, 2e^{2t} - 20 \rangle$, for $0 \le t \le \ln 2$

20. $\mathbf{r}(t) = \langle t^2, t^3 \rangle$, for $0 \le t \le 4$

21. $\mathbf{r}(t) = \langle \cos^3 t, \sin^3 t \rangle$, for $0 \le t \le \pi/2$

22. $\mathbf{r}(t) = \langle 2\cos t, 2\sqrt{3}\cos t, 4\sin t \rangle$, for $0 \le t \le 2\pi$

23–26. Speed and arc length *For the following trajectories, find the speed associated with the trajectory, and then find the length of the trajectory on the given interval.*

23. $\mathbf{r}(t) = \langle 2t^3, -t^3, 5t^3 \rangle$, for $0 \le t \le 4$

24. $\mathbf{r}(t) = \langle 5\cos t^2, 5\sin t^2, 12t^2 \rangle$, for $0 \le t \le 2$

25. $\mathbf{r}(t) = \langle 13\sin 2t, 12\cos 2t, 5\cos 2t \rangle$, for $0 \le t \le \pi$

26. $\mathbf{r}(t) = \langle e^t \sin t, e^t \cos t, e^t \rangle$, for $0 \le t \le \ln 2$

T **27. Speed of Earth** Verify that the length of one orbit of Earth is approximately 6.280 AU (see Table 14.1). Then determine the average speed of Earth relative to the sun in miles per hour. (*Hint:* It takes Earth 365.25 days to orbit the sun.)

T **28. Speed of Jupiter** Verify that the length of one orbit of Jupiter is approximately 32.616 AU (see Table 14.1). Then determine the average speed of Jupiter relative to the sun in miles per hour. (*Hint:* It takes Jupiter 11.8618 Earth years to orbit the sun.)

T **29–32. Arc length approximations** *Use a calculator to approximate the length of the following curves. In each case, simplify the arc length integral as much as possible before finding an approximation.*

29. $\mathbf{r}(t) = \langle 2\cos t, 4\sin t \rangle$, for $0 \le t \le 2\pi$

30. $\mathbf{r}(t) = \langle 2\cos t, 4\sin t, 6\cos t \rangle$, for $0 \le t \le 2\pi$

31. $\mathbf{r}(t) = \langle t, 4t^2, 10 \rangle$, for $-2 \le t \le 2$

32. $\mathbf{r}(t) = \langle e^t, 2e^{-t}, t \rangle$, for $0 \le t \le \ln 3$

33–42. Arc length parametrization *Determine whether the following curves use arc length as a parameter. If not, find a description that uses arc length as a parameter.*

33. $\mathbf{r}(t) = \langle 1, \sin t, \cos t \rangle$, for $t \ge 1$

34. $\mathbf{r}(t) = \left\langle \dfrac{t}{\sqrt{3}}, \dfrac{t}{\sqrt{3}}, \dfrac{t}{\sqrt{3}} \right\rangle$, for $0 \le t \le 10$

35. $\mathbf{r}(t) = \langle t, 2t \rangle$, for $0 \le t \le 3$

36. $\mathbf{r}(t) = \langle t + 1, 2t - 3, 6t \rangle$, for $0 \le t \le 10$

37. $\mathbf{r}(t) = \langle 2\cos t, 2\sin t \rangle$, for $0 \le t \le 2\pi$

38. $\mathbf{r}(t) = \langle 17\cos t, 15\sin t, 8\sin t \rangle$, for $0 \le t \le \pi$

39. $\mathbf{r}(t) = \langle \cos t^2, \sin t^2 \rangle$, for $0 \le t \le \sqrt{\pi}$

40. $\mathbf{r}(t) = \langle t^2, 2t^2, 4t^2 \rangle$, for $1 \le t \le 4$

41. $\mathbf{r}(t) = \langle e^t, e^t, e^t \rangle$, for $t \ge 0$

42. $\mathbf{r}(t) = \left\langle \dfrac{\cos t}{\sqrt{2}}, \dfrac{\cos t}{\sqrt{2}}, \sin t \right\rangle$, for $0 \le t \le 10$

43. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. If an object moves on a trajectory with constant speed S over a time interval $a \le t \le b$, then the length of the trajectory is $S(b - a)$.

b. The curves defined by

$$\mathbf{r}(t) = \langle f(t), g(t) \rangle \quad \text{and} \quad \mathbf{R}(t) = \langle g(t), f(t) \rangle$$

have the same length over the interval $[a, b]$.

c. The curve $\mathbf{r}(t) = \langle f(t), g(t) \rangle$, for $0 \le a \le t \le b$, and the curve $\mathbf{R}(t) = \langle f(t^2), g(t^2) \rangle$, for $\sqrt{a} \le t \le \sqrt{b}$, have the same length.

d. The curve $\mathbf{r}(t) = \langle t, t^2, 3t^2 \rangle$, for $1 \le t \le 4$, is parameterized by arc length.

44. Length of a line segment Consider the line segment joining the points $P(x_0, y_0, z_0)$ and $Q(x_1, y_1, z_1)$.

a. Find a function $\mathbf{r}(t)$ for the segment PQ.

b. Use the arc length formula to find the length of PQ.

c. Use geometry (distance formula) to verify the result of part (b).

45. Tilted circles Let the curve C be described by

$$\mathbf{r}(t) = \langle a\cos t, b\sin t, c\sin t \rangle,$$

where a, b, and c are real positive numbers.

a. Assume C lies in a plane. Show that C is a circle centered at the origin, provided $a^2 = b^2 + c^2$.

b. Find the arc length of the circle in part (a).

c. Assuming C lies in a plane, find the conditions for which $\mathbf{r}(t) = \langle a\cos t + b\sin t, c\cos t + d\sin t, e\cos t + f\sin t \rangle$ describes a circle. Then find its arc length.

46. A family of arc length integrals Find the length of the curve $\mathbf{r}(t) = \langle t^m, t^m, t^{3m/2} \rangle$, for $0 \le a \le t \le b$, where m is a real number. Express the result in terms of m, a, and b.

47. A special case Suppose a curve is described by

$$\mathbf{r}(t) = \langle Ah(t), Bh(t) \rangle, \text{ for } a \le t \le b,$$

where A and B are constants and h has a continuous derivative.

a. Show that the length of the curve is

$$\sqrt{A^2 + B^2} \int_a^b |h'(t)|\, dt.$$

b. Use part (a) to find the length of the curve $x = 2t^3$, $y = 5t^3$, for $0 \le t \le 4$.

c. Use part (a) to find the length of the curve $x = 4/t$, $y = 10/t$, for $1 \le t \le 8$.

Explorations and Challenges

T 48. Toroidal magnetic field A circle of radius a that is centered at $(A, 0)$ is revolved about the y-axis to create a torus (assume $a < A$). When current flows through a copper wire that is wrapped around this torus, a magnetic field is created and the strength of this field depends on the amount of copper wire used. If the wire is wrapped evenly around the torus a total of k times, the shape of the wire is modeled by the function

$$\mathbf{r}(t) = \langle (A + a\cos kt)\cos t, (A + a\cos kt)\sin t, a\sin kt \rangle,$$

for $0 \le t \le 2\pi$. Determine the amount of copper required if $A = 4$ in, $a = 1$ in, and $k = 35$.

T 49. Projectile trajectories A projectile (such as a baseball or a cannonball) launched from the origin with an initial horizontal velocity u_0 and an initial vertical velocity v_0 moves in a parabolic trajectory given by

$$\mathbf{r}(t) = \left\langle u_0 t, -\frac{1}{2}gt^2 + v_0 t \right\rangle, \quad \text{for } t \ge 0,$$

where air resistance is neglected and $g = 9.8\text{ m/s}^2$ is the acceleration due to gravity (see Section 14.3).

a. Let $u_0 = 20$ m/s and $v_0 = 25$ m/s. Assuming the projectile is launched over horizontal ground, at what time does it return to Earth?

b. Find the integral that gives the length of the trajectory from launch to landing.

c. Evaluate the integral in part (b) by first making the change of variables $u = -gt + v_0$. The resulting integral is evaluated either by making a second change of variables or by using a calculator. What is the length of the trajectory?

d. How far does the projectile land from its launch site?

50. Variable speed on a circle Consider a particle that moves in a plane according to the function $\mathbf{r}(t) = \langle \sin t^2, \cos t^2 \rangle$ with an initial position $(0, 1)$ at $t = 0$.

a. Describe the path of the particle, including the time required to return to the initial position.

b. What is the length of the path in part (a)?

c. Describe how the motion of this particle differs from the motion described by the equations $x = \sin t$ and $y = \cos t$.

d. Consider the motion described by $x = \sin t^n$ and $y = \cos t^n$, where n is a positive integer. Describe the path of the particle, including the time required to return to the initial position.

e. What is the length of the path in part (d) for any positive integer n?

f. If you were watching a race on a circular path between two runners, one moving according to $x = \sin t$ and $y = \cos t$ and one according to $x = \sin t^2$ and $y = \cos t^2$, who would win and when would one runner pass the other?

51. Arc length parameterization Prove that the line $\mathbf{r}(t) = \langle x_0 + at, y_0 + bt, z_0 + ct \rangle$ is parameterized by arc length, provided $a^2 + b^2 + c^2 = 1$.

52. Arc length parameterization Prove that the curve $\mathbf{r}(t) = \langle a\cos t, b\sin t, c\sin t \rangle$ is parameterized by arc length, provided $a^2 = b^2 + c^2 = 1$.

53. Lengths of related curves Suppose a curve is given by $\mathbf{r}(t) = \langle f(t), g(t) \rangle$, where f' and g' are continuous, for $a \le t \le b$. Assume the curve is traversed once, for $a \le t \le b$, and the length of the curve between $(f(a), g(a))$ and $(f(b), g(b))$ is L. Prove that for any nonzero constant c, the length of the curve defined by $\mathbf{r}(t) = \langle cf(t), cg(t) \rangle$, for $a \le t \le b$, is $|c|L$.

54. Change of variables Consider the parameterized curves $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ and $\mathbf{R}(t) = \langle f(u(t)), g(u(t)), h(u(t)) \rangle$, where f, g, h, and u are continuously differentiable functions and u has an inverse on $[a, b]$.

a. Show that the curve generated by $\mathbf{r}$ on the interval $a \le t \le b$ is the same as the curve generated by $\mathbf{R}$ on $u^{-1}(a) \le t \le u^{-1}(b)$ (or $u^{-1}(b) \le t \le u^{-1}(a)$).

b. Show that the lengths of the two curves are equal. (*Hint:* Use the Chain Rule and a change of variables in the arc length integral for the curve generated by $\mathbf{R}$.)

QUICK CHECK ANSWERS

1. 9 **2.** For $a \le t \le b$, the curve C generated is $(b - a)/2\pi$ of a full circle. Because the full circle has a length of 2π, the curve C has a length of $b - a$. **3.** No. If t increases by 1 unit, the length of the curve increases by $\sqrt{3}$ units. ➤

14.5 Curvature and Normal Vectors

We know how to find tangent vectors and lengths of curves in space, but much more can be said about the shape of such curves. In this section, we introduce several new concepts. *Curvature* measures how *fast* a curve turns at a point, the *normal vector* gives the *direction* in which a curve turns, and the *binormal vector* and the *torsion* describe the twisting of a curve.

Curvature

Imagine driving a car along a winding mountain road. There are two ways to change the velocity of the car (that is, to accelerate). You can change the *speed* of the car or you can change the *direction* of the car. A change of speed is relatively easy to describe, so we postpone that discussion and focus on the change of direction. The rate at which the car changes direction is related to the notion of *curvature*.

Unit Tangent Vector Recall from Section 14.2 that if $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ is a smooth oriented curve, then the unit tangent vector at a point is the unit vector that points in the direction of the tangent vector $\mathbf{r}'(t)$; that is,

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|}.$$

Because $\mathbf{T}$ is a unit vector, its length does not change along the curve. The only way $\mathbf{T}$ can change is through a change in direction.

How quickly does $\mathbf{T}$ change (in direction) as we move along the curve? If a small increment in arc length Δs along the curve results in a large change in the direction of $\mathbf{T}$, the curve is turning quickly over that interval and we say it has a large *curvature* (**Figure 14.29a**). If a small increment Δs in arc length results in a small change in the direction of $\mathbf{T}$, the curve is turning slowly over that interval and it has a small curvature (**Figure 14.29b**). The magnitude of the rate at which the direction of $\mathbf{T}$ changes with respect to arc length is the curvature of the curve.

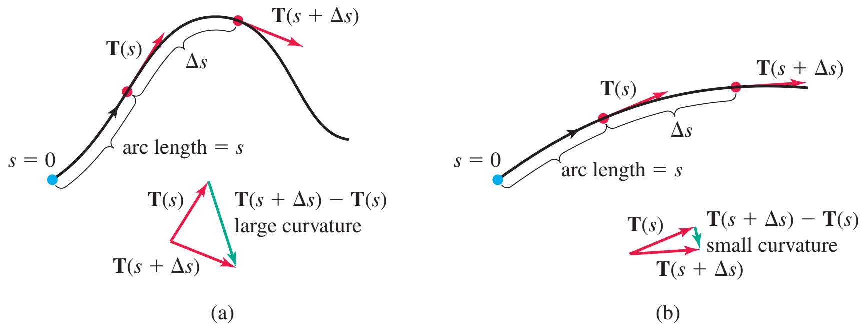


Figure 14.29

➤ Recall that the unit tangent vector at a point depends on the orientation of the curve. The curvature does not depend on the orientation of the curve, but it does depend on the shape of the curve. The Greek letter κ (kappa) is used to denote curvature.

DEFINITION Curvature

Let $\mathbf{r}$ describe a smooth parameterized curve. If s denotes arc length and $\mathbf{T} = \mathbf{r}'/|\mathbf{r}'|$ is the unit tangent vector, the **curvature** is $\kappa(s) = \left|\dfrac{d\mathbf{T}}{ds}\right|$.

Note that κ is a nonnegative scalar-valued function. A large value of κ at a point indicates a tight curve that changes direction quickly. If κ is small, then the curve is relatively flat and its direction changes slowly. The minimum curvature (zero) occurs on a straight line, where the tangent vector never changes direction along the curve.

In order to evaluate $d\mathbf{T}/ds$, a description of the curve in terms of the arc length appears to be needed, but it may be difficult to obtain. A short calculation leads to the first of two practical curvature formulas.

Letting t be an arbitrary parameter, we begin with the Chain Rule and write $\dfrac{d\mathbf{T}}{dt} = \dfrac{d\mathbf{T}}{ds} \cdot \dfrac{ds}{dt}$. After dividing both sides of this equation by $ds/dt = |\mathbf{v}|$, we take absolute values and arrive at

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{|d\mathbf{T}/dt|}{|ds/dt|} = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right|.$$

This derivation is a proof of the following theorem.

THEOREM 14.4 Curvature Formula

Let $\mathbf{r}(t)$ describe a smooth parameterized curve, where t is any parameter. If $\mathbf{v} = \mathbf{r}'$ is the velocity and $\mathbf{T}$ is the unit tangent vector, then the curvature is

$$\kappa(t) = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}.$$

EXAMPLE 1 **Lines have zero curvature** Consider the line

$$\mathbf{r}(t) = \langle x_0 + at, y_0 + bt, z_0 + ct \rangle, \text{ for } -\infty < t < \infty.$$

Show that $\kappa = 0$ at all points on the line.

SOLUTION Note that $\mathbf{r}'(t) = \langle a, b, c \rangle$ and $|\mathbf{r}'(t)| = |\mathbf{v}(t)| = \sqrt{a^2 + b^2 + c^2}$. Therefore,

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\langle a, b, c \rangle}{\sqrt{a^2 + b^2 + c^2}}.$$

Because $\mathbf{T}$ is a constant, $\dfrac{d\mathbf{T}}{dt} = \mathbf{0}$; therefore, $\kappa = 0$ at all points of the line.

Related Exercise 11 ◀

EXAMPLE 2 **Circles have constant curvature** Consider the circle $\mathbf{r}(t) = \langle R\cos t, R\sin t \rangle$, for $0 \le t \le 2\pi$, where $R > 0$. Show that $\kappa = 1/R$.

SOLUTION We compute $\mathbf{r}'(t) = \langle -R\sin t, R\cos t \rangle$ and

$$\begin{aligned} |\mathbf{v}(t)| = |\mathbf{r}'(t)| &= \sqrt{(-R\sin t)^2 + (R\cos t)^2} \\ &= \sqrt{R^2(\sin^2 t + \cos^2 t)} && \text{Simplify.} \\ &= R. && \sin^2 t + \cos^2 t = 1, R > 0 \end{aligned}$$

Therefore,

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\langle -R\sin t, R\cos t \rangle}{R} = \langle -\sin t, \cos t \rangle, \text{ and}$$

$$\frac{d\mathbf{T}}{dt} = \langle -\cos t, -\sin t \rangle.$$

Combining these observations, the curvature is

$$\kappa = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{R}|\langle -\cos t, -\sin t \rangle| = \frac{1}{R}\sqrt{\underbrace{\cos^2 t + \sin^2 t}_{1}} = \frac{1}{R}.$$

The curvature of a circle is constant; a circle with a small radius has a large curvature, and vice versa.

Related Exercise 12 ◀

➤ The curvature of a curve at a point can also be visualized in terms of a **circle of curvature**, which is a circle of radius R that is tangent to the curve at that point. The curvature at the point is $\kappa = 1/R$. See Exercises 70–73.

QUICK CHECK 1 What is the curvature of the circle $\mathbf{r}(t) = \langle 3\sin t, 3\cos t \rangle$? ◀

An Alternative Curvature Formula A second curvature formula, which pertains specifically to trajectories of moving objects, is easier to use in some cases. The calculation is instructive because it relies on many properties of vector functions. In the end, a remarkably simple formula emerges.

Again consider a smooth curve $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, where $\mathbf{v}(t) = \mathbf{r}'(t)$ and $\mathbf{a}(t) = \mathbf{v}'(t)$ are the velocity and acceleration of an object moving along that curve, respectively. We assume $\mathbf{v}(t) \neq \mathbf{0}$ and $\mathbf{a}(t) \neq \mathbf{0}$. Because $\mathbf{T} = \mathbf{v}/|\mathbf{v}|$, we begin by writing $\mathbf{v} = |\mathbf{v}|\,\mathbf{T}$ and differentiating both sides with respect to t:

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}(|\mathbf{v}(t)|\,\mathbf{T}(t)) = \frac{d}{dt}(|\mathbf{v}(t)|)\mathbf{T}(t) + |\mathbf{v}(t)|\frac{d\mathbf{T}}{dt}. \qquad \text{Product Rule}$$

We now form $\mathbf{v} \times \mathbf{a}$:

➤ Distributive law for cross products:
$\mathbf{w} \times (\mathbf{u} + \mathbf{v}) = (\mathbf{w} \times \mathbf{u}) + (\mathbf{w} \times \mathbf{v})$
$(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$

$$\begin{aligned} \mathbf{v} \times \mathbf{a} &= \underbrace{|\mathbf{v}|\mathbf{T}}_{\mathbf{v}} \times \underbrace{\left(\frac{d}{dt}(|\mathbf{v}|)\mathbf{T} + |\mathbf{v}|\frac{d\mathbf{T}}{dt}\right)}_{\mathbf{a}} \\ &= \underbrace{|\mathbf{v}|\mathbf{T} \times \left(\frac{d}{dt}(|\mathbf{v}|)\right)\mathbf{T}}_{\mathbf{0}} + |\mathbf{v}|\mathbf{T} \times |\mathbf{v}|\frac{d\mathbf{T}}{dt} && \text{Distributive law for cross products} \end{aligned}$$

The first term in this expression has the form $a\mathbf{T} \times b\mathbf{T}$, where a and b are scalars. Therefore, $a\mathbf{T}$ and $b\mathbf{T}$ are parallel vectors and $a\mathbf{T} \times b\mathbf{T} = \mathbf{0}$. To simplify the second term, recall that a vector $\mathbf{u}(t)$ of constant length has the property that $\mathbf{u}$ and $d\mathbf{u}/dt$ are orthogonal (Section 14.3). Because $\mathbf{T}$ is a unit vector, it has constant length, and $\mathbf{T}$ and $d\mathbf{T}/dt$ are orthogonal. Furthermore, scalar multiples of $\mathbf{T}$ and $d\mathbf{T}/dt$ are also orthogonal. Therefore, the magnitude of the second term simplifies as follows:

$$\begin{aligned}\left|\,|\mathbf{v}|\mathbf{T} \times |\mathbf{v}|\frac{d\mathbf{T}}{dt}\right| &= |\mathbf{v}|\,|\mathbf{T}|\,\left||\mathbf{v}|\frac{d\mathbf{T}}{dt}\right| \underbrace{\sin\theta}_{1} && |\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}|\sin\theta \\ &= |\mathbf{v}|^2\left|\frac{d\mathbf{T}}{dt}\right|\underbrace{|\mathbf{T}|}_{1} && \text{Simplify, } \theta = \pi/2. \\ &= |\mathbf{v}|^2\left|\frac{d\mathbf{T}}{dt}\right|. && |\mathbf{T}| = 1\end{aligned}$$

➤ Recall that the magnitude of the cross product of nonzero vectors is $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}|\sin\theta$, where θ is the angle between the vectors. If the vectors are orthogonal, $\sin\theta = 1$ and $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}|$.

The final step is to use Theorem 14.4 and substitute $\left|\frac{d\mathbf{T}}{dt}\right| = \kappa|\mathbf{v}|$. Putting these results together, we find that

$$|\mathbf{v} \times \mathbf{a}| = |\mathbf{v}|^2\left|\frac{d\mathbf{T}}{dt}\right| = |\mathbf{v}|^2\,\kappa|\mathbf{v}| = \kappa|\mathbf{v}|^3.$$

Solving for the curvature gives $\kappa = \dfrac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}$.

➤ Note that $\mathbf{a}(t) = \mathbf{0}$ corresponds to straight-line motion and $\kappa = 0$. If $\mathbf{v}(t) = \mathbf{0}$, the object is at rest and κ is undefined.

THEOREM 14.5 Alternative Curvature Formula
Let $\mathbf{r}$ be the position of an object moving on a smooth curve. The **curvature** at a point on the curve is

$$\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3},$$

where $\mathbf{v} = \mathbf{r}'$ is the velocity and $\mathbf{a} = \mathbf{v}'$ is the acceleration.

QUICK CHECK 2 Use the alternative curvature formula to compute the curvature of the curve $\mathbf{r}(t) = \langle t^2, 10, -10\rangle$. ◄

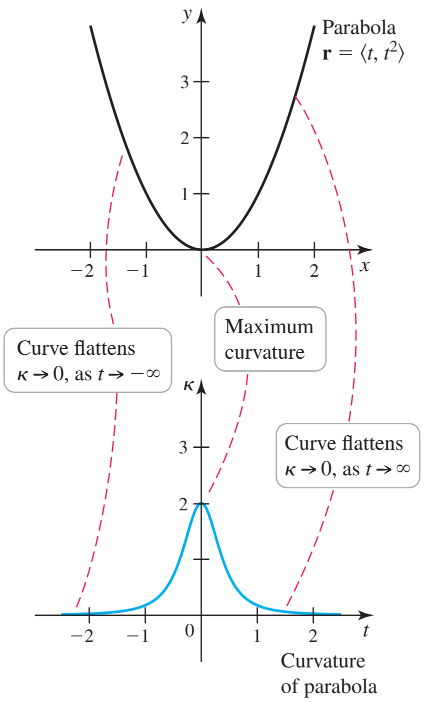


Figure 14.30

EXAMPLE 3 Curvature of a parabola Find the curvature of the parabola $\mathbf{r}(t) = \langle t, at^2\rangle$, for $-\infty < t < \infty$, where $a > 0$ is a real number.

SOLUTION The alternative formula works well in this case. We find that $\mathbf{v}(t) = \mathbf{r}'(t) = \langle 1, 2at\rangle$ and $\mathbf{a}(t) = \mathbf{v}'(t) = \langle 0, 2a\rangle$. To compute the cross product $\mathbf{v} \times \mathbf{a}$, we append a third component of 0 to each vector:

$$\mathbf{v} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2at & 0 \\ 0 & 2a & 0 \end{vmatrix} = 2a\,\mathbf{k}.$$

Therefore, the curvature is

$$\kappa(t) = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} = \frac{|2a\,\mathbf{k}|}{|\langle 1, 2at\rangle|^3} = \frac{2a}{(1 + 4a^2t^2)^{3/2}}.$$

The curvature is a maximum at the vertex of the parabola where $t = 0$ and $\kappa = 2a$. The curvature decreases as one moves along the curve away from the vertex, as shown in Figure 14.30 with $a = 1$.

Related Exercise 23 ◄

EXAMPLE 4 Curvature of a helix Find the curvature of the helix $\mathbf{r}(t) = \langle a\cos t, a\sin t, bt\rangle$, for $-\infty < t < \infty$, where $a > 0$ and $b > 0$ are real numbers.

SOLUTION We use the alternative curvature formula, with

$$\begin{aligned}\mathbf{v}(t) &= \mathbf{r}'(t) = \langle -a\sin t, a\cos t, b\rangle \quad \text{and} \\ \mathbf{a}(t) &= \mathbf{v}'(t) = \langle -a\cos t, -a\sin t, 0\rangle.\end{aligned}$$

The cross product $\mathbf{v} \times \mathbf{a}$ is

$$\mathbf{v} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a \sin t & a \cos t & b \\ -a \cos t & -a \sin t & 0 \end{vmatrix} = ab \sin t\,\mathbf{i} - ab \cos t\,\mathbf{j} + a^2\,\mathbf{k}.$$

Therefore,

$$\begin{aligned} |\mathbf{v} \times \mathbf{a}| &= |ab \sin t\,\mathbf{i} - ab \cos t\,\mathbf{j} + a^2\,\mathbf{k}| \\ &= \sqrt{a^2 b^2 \underbrace{(\sin^2 t + \cos^2 t)}_{1} + a^4} \\ &= a\sqrt{a^2 + b^2}. \end{aligned}$$

By a familiar calculation, $|\mathbf{v}| = |\langle -a \sin t, a \cos t, b \rangle| = \sqrt{a^2 + b^2}$. Therefore,

$$\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} = \frac{a\sqrt{a^2 + b^2}}{(\sqrt{a^2 + b^2})^3} = \frac{a}{a^2 + b^2}.$$

A similar calculation shows that all helices of this form have constant curvature.

Related Exercise 22 ◀

➤ In the curvature formula for the helix, if $b = 0$, the helix becomes a circle of radius a with $\kappa = \frac{1}{a}$. At the other extreme, holding a fixed and letting $b \to \infty$ stretches and straightens the helix so that $\kappa \to 0$.

Principal Unit Normal Vector

The curvature answers the question of how *fast* a curve turns. The principal unit normal vector determines the *direction* in which a curve turns. Specifically, the magnitude of $d\mathbf{T}/ds$ is the curvature: $\kappa = |d\mathbf{T}/ds|$. What about the direction of $d\mathbf{T}/ds$? If only the direction, but not the magnitude, of a vector is of interest, it is convenient to work with a unit vector that has the same direction as the original vector. We apply this idea to $d\mathbf{T}/ds$. The unit vector that points in the direction of $d\mathbf{T}/ds$ is the *principal unit normal vector*.

➤ The principal unit normal vector depends on the shape of the curve but not on the orientation of the curve.

DEFINITION Principal Unit Normal Vector

Let $\mathbf{r}$ describe a smooth curve parameterized by arc length. The **principal unit normal vector** at a point P on the curve at which $\kappa \neq 0$ is

$$\mathbf{N}(s) = \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds}.$$

For other parameters, we use the equivalent formula

$$\mathbf{N}(t) = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|},$$

evaluated at the value of t corresponding to P.

The practical formula $\mathbf{N} = \dfrac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$ follows from the definition by using the Chain Rule to write $\dfrac{d\mathbf{T}}{ds} = \dfrac{d\mathbf{T}}{dt} \cdot \dfrac{dt}{ds}$ (Exercise 80). Two important properties of the principal unit normal vector follow from the definition.

THEOREM 14.6 Properties of the Principal Unit Normal Vector

Let $\mathbf{r}$ describe a smooth parameterized curve with unit tangent vector $\mathbf{T}$ and principal unit normal vector $\mathbf{N}$.

1. $\mathbf{T}$ and $\mathbf{N}$ are orthogonal at all points of the curve; that is, $\mathbf{T} \cdot \mathbf{N} = 0$ at all points where $\mathbf{N}$ is defined.
2. The principal unit normal vector points to the inside of the curve—in the direction that the curve is turning.

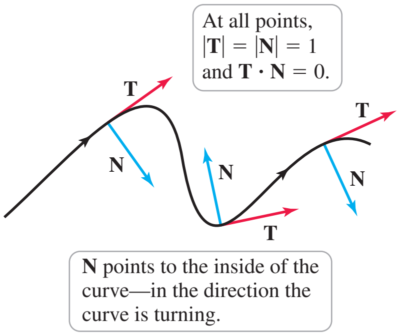


Figure 14.31

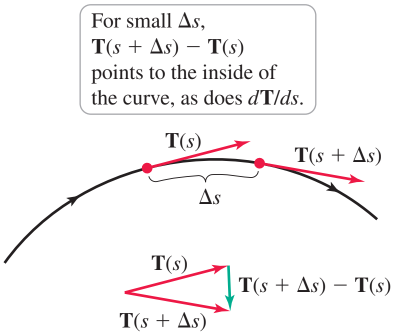


Figure 14.32

QUICK CHECK 3 Consider the parabola $\mathbf{r}(t) = \langle t, -t^2 \rangle$. Does the principal unit normal vector point in the positive y-direction or the negative y-direction along the curve? ◄

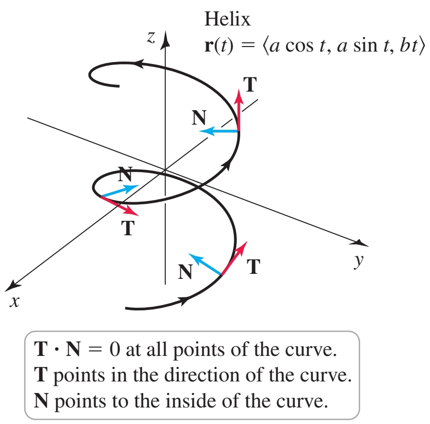


Figure 14.33

QUICK CHECK 4 Why is the principal unit normal vector for a straight line undefined? ◄

Proof:

1. As a unit vector, $\mathbf{T}$ has constant length. Therefore, by Theorem 14.2, $\mathbf{T}$ and $d\mathbf{T}/dt$ (or $\mathbf{T}$ and $d\mathbf{T}/ds$) are orthogonal. Because $\mathbf{N}$ is a scalar multiple of $d\mathbf{T}/ds$, $\mathbf{T}$ and $\mathbf{N}$ are orthogonal (Figure 14.31).
2. We motivate—but do not prove—this fact by recalling that

$$\frac{d\mathbf{T}}{ds} = \lim_{\Delta s \to 0} \frac{\mathbf{T}(s + \Delta s) - \mathbf{T}(s)}{\Delta s}.$$

Therefore, $d\mathbf{T}/ds$ points in the approximate direction of $\mathbf{T}(s + \Delta s) - \mathbf{T}(s)$ when Δs is small. As shown in Figure 14.32, this difference points in the direction in which the curve is turning. Because $\mathbf{N}$ is a positive scalar multiple of $d\mathbf{T}/ds$, it points in the same direction. ◄

EXAMPLE 5 Principal unit normal vector for a helix Find the principal unit normal vector for the helix $\mathbf{r}(t) = \langle a \cos t, a \sin t, bt \rangle$, for $-\infty < t < \infty$, where $a > 0$ and $b > 0$ are real numbers.

SOLUTION Several preliminary calculations are needed. First, we have $\mathbf{v}(t) = \mathbf{r}'(t) = \langle -a \sin t, a \cos t, b \rangle$. Therefore,

$$\begin{aligned} |\mathbf{v}(t)| = |\mathbf{r}'(t)| &= \sqrt{(-a \sin t)^2 + (a \cos t)^2 + b^2} \\ &= \sqrt{a^2 (\sin^2 t + \cos^2 t) + b^2} \qquad \text{Simplify.} \\ &= \sqrt{a^2 + b^2}. \qquad \sin^2 t + \cos^2 t = 1 \end{aligned}$$

The unit tangent vector is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\langle -a \sin t, a \cos t, b \rangle}{\sqrt{a^2 + b^2}}.$$

Notice that $\mathbf{T}$ points along the curve in an upward direction (at an angle to the horizontal that satisfies the equation $\tan \theta = b/a$; Figure 14.33). We can now calculate the principal unit normal vector. First, we determine that

$$\frac{d\mathbf{T}}{dt} = \frac{d}{dt}\left(\frac{\langle -a \sin t, a \cos t, b \rangle}{\sqrt{a^2 + b^2}} \right) = \frac{\langle -a \cos t, -a \sin t, 0 \rangle}{\sqrt{a^2 + b^2}}$$

and

$$\left| \frac{d\mathbf{T}}{dt} \right| = \frac{a}{\sqrt{a^2 + b^2}}.$$

The principal unit normal vector now follows:

$$\mathbf{N} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|} = \frac{\dfrac{\langle -a \cos t, -a \sin t, 0 \rangle}{\sqrt{a^2 + b^2}}}{\dfrac{a}{\sqrt{a^2 + b^2}}} = \langle -\cos t, -\sin t, 0 \rangle.$$

Several important checks should be made. First note that $\mathbf{N}$ is a unit vector; that is, $|\mathbf{N}| = 1$. It should also be confirmed that $\mathbf{T} \cdot \mathbf{N} = 0$; that is, the unit tangent vector and the principal unit normal vector are everywhere orthogonal. Finally, $\mathbf{N}$ is parallel to the xy-plane and points inward toward the z-axis, in the direction the curve turns (Figure 14.33). Notice that in the special case $b = 0$, the trajectory is a circle, but the normal vector is still $\mathbf{N} = \langle -\cos t, -\sin t, 0 \rangle$.

Related Exercise 28 ◄

Components of the Acceleration

The vectors $\mathbf{T}$ and $\mathbf{N}$ may be used to gain insight into how moving objects accelerate. Recall the observation made earlier that the two ways to change the velocity of an object (accelerate) are to change its *speed* and to change its *direction* of motion. We show that changing the speed produces acceleration in the direction of **T** and changing the direction produces acceleration in the direction of **N**.

We begin with the fact that $\mathbf{T} = \dfrac{\mathbf{v}}{|\mathbf{v}|}$ or $\mathbf{v} = \mathbf{T}|\mathbf{v}| = \mathbf{T}\dfrac{ds}{dt}$. Differentiating both sides of $\mathbf{v} = \mathbf{T}\dfrac{ds}{dt}$ with respect to t gives

➤ Recall that the speed is $|\mathbf{v}| = ds/dt$, where s is arc length.

$$\begin{aligned}
\mathbf{a} = \frac{d\mathbf{v}}{dt} &= \frac{d}{dt}\left(\mathbf{T}\frac{ds}{dt}\right) \\
&= \frac{d\mathbf{T}}{dt}\frac{ds}{dt} + \mathbf{T}\frac{d^2s}{dt^2} && \text{Product Rule} \\
&= \underbrace{\frac{d\mathbf{T}}{ds}}_{\kappa\mathbf{N}}\underbrace{\frac{ds}{dt}}_{|\mathbf{v}|}\frac{ds}{dt} + \mathbf{T}\frac{d^2s}{dt^2} && \text{Chain Rule: } \frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds}\frac{ds}{dt} \\
&= \kappa\mathbf{N}|\mathbf{v}|^2 + \mathbf{T}\frac{d^2s}{dt^2}. && \text{Substitute.}
\end{aligned}$$

We now identify the normal and tangential components of the acceleration.

➤ Note that a_N and a_T are defined even at points where $\kappa = 0$ and **N** is undefined.

THEOREM 14.7 Tangential and Normal Components of the Acceleration
The acceleration vector of an object moving in space along a smooth curve has the following representation in terms of its **tangential component** a_T (in the direction of **T**) and its **normal component** a_N (in the direction of **N**):

$$\mathbf{a} = a_N\mathbf{N} + a_T\mathbf{T},$$

where $a_N = \kappa|\mathbf{v}|^2 = \dfrac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|}$ and $a_T = \dfrac{d^2s}{dt^2}$.

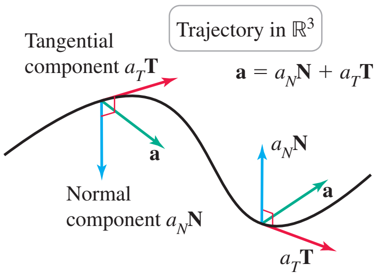


Figure 14.34

The tangential component of the acceleration, in the direction of **T**, is the usual acceleration $a_T = d^2s/dt^2$ of an object moving along a straight line (Figure 14.34). The normal component, in the direction of **N**, increases with the speed $|\mathbf{v}|$ and with the curvature. Higher speeds on tighter curves produce greater normal accelerations.

EXAMPLE 6 Acceleration on a circular path Find the components of the acceleration on the circular trajectory

$$\mathbf{r}(t) = \langle R\cos\omega t, R\sin\omega t\rangle,$$

where R and ω are positive real numbers.

SOLUTION We find that $\mathbf{r}'(t) = \langle -R\omega\sin\omega t, R\omega\cos\omega t\rangle$, $|\mathbf{v}(t)| = |\mathbf{r}'(t)| = R\omega$, and, by Example 2, $\kappa = 1/R$. Recall that $ds/dt = |\mathbf{v}(t)|$, which is constant; therefore, $d^2s/dt^2 = 0$ and the tangential component of the acceleration is zero. The acceleration is

$$\mathbf{a} = \kappa|\mathbf{v}|^2\,\mathbf{N} + \underbrace{\frac{d^2s}{dt^2}}_{0}\mathbf{T} = \frac{1}{R}(R\omega)^2\,\mathbf{N} = R\omega^2\,\mathbf{N}.$$

On a circular path (traversed at constant speed), the acceleration is entirely in the normal direction, orthogonal to the tangent vectors. The acceleration increases with the radius of the circle R and with the frequency of the motion ω.

Related Exercise 36 ➤

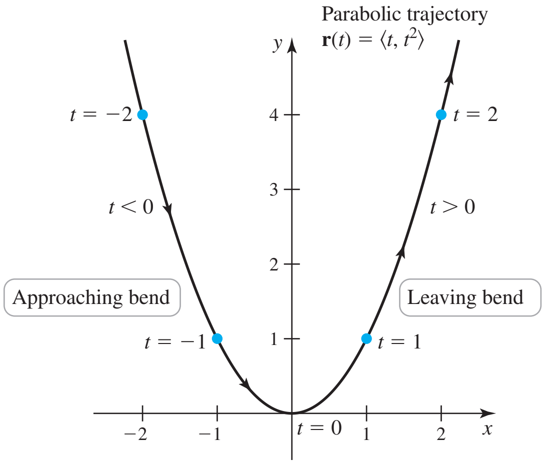


Figure 14.35

EXAMPLE 7 A bend in the road The driver of a car follows the parabolic trajectory $\mathbf{r}(t) = \langle t, t^2\rangle$, for $-2 \le t \le 2$, through a sharp bend (Figure 14.35). Find the tangential and normal components of the acceleration of the car.

SOLUTION The velocity and acceleration vectors are easily computed: $\mathbf{v}(t) = \mathbf{r}'(t) = \langle 1, 2t \rangle$ and $\mathbf{a}(t) = \mathbf{r}''(t) = \langle 0, 2 \rangle$. The goal is to express $\mathbf{a} = \langle 0, 2 \rangle$ in terms of $\mathbf{T}$ and $\mathbf{N}$. A short calculation reveals that

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\langle 1, 2t \rangle}{\sqrt{1 + 4t^2}} \quad \text{and} \quad \mathbf{N} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|} = \frac{\langle -2t, 1 \rangle}{\sqrt{1 + 4t^2}}.$$

We now have two ways to proceed. One is to compute the normal and tangential components of the acceleration directly using the definitions. More efficient is to note that $\mathbf{T}$ and $\mathbf{N}$ are orthogonal unit vectors, and then to compute the scalar projections of $\mathbf{a} = \langle 0, 2 \rangle$ in the directions of $\mathbf{T}$ and $\mathbf{N}$. We find that

➤ Using the fact that $|\mathbf{T}| = |\mathbf{N}| = 1$, we have, from Section 13.3,

$$a_N = \text{scal}_{\mathbf{N}}\mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{N}}{|\mathbf{N}|} = \mathbf{a} \cdot \mathbf{N}$$

and

$$a_T = \text{scal}_{\mathbf{T}}\mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{T}}{|\mathbf{T}|} = \mathbf{a} \cdot \mathbf{T} = \frac{\mathbf{v} \cdot \mathbf{a}}{|\mathbf{v}|}.$$

$$a_N = \mathbf{a} \cdot \mathbf{N} = \langle 0, 2 \rangle \cdot \frac{\langle -2t, 1 \rangle}{\sqrt{1 + 4t^2}} = \frac{2}{\sqrt{1 + 4t^2}}$$

and

$$a_T = \mathbf{a} \cdot \mathbf{T} = \langle 0, 2 \rangle \cdot \frac{\langle 1, 2t \rangle}{\sqrt{1 + 4t^2}} = \frac{4t}{\sqrt{1 + 4t^2}}.$$

You should verify that at all times (Exercise 76),

$$\mathbf{a} = a_N\mathbf{N} + a_T\mathbf{T} = \frac{2}{\sqrt{1 + 4t^2}}(\mathbf{N} + 2t\,\mathbf{T}) = \langle 0, 2 \rangle.$$

Let's interpret these results. First notice that the driver negotiates the curve in a sensible way: The speed $|\mathbf{v}| = \sqrt{1 + 4t^2}$ decreases as the car approaches the origin (the tightest part of the curve) and increases as it moves away from the origin (Figure 14.36). As the car approaches the origin ($t < 0$), $\mathbf{T}$ points in the direction of the trajectory and $\mathbf{N}$ points to the inside of the curve. However, $a_T = \frac{d^2s}{dt^2} < 0$ when $t < 0$, so $a_T\mathbf{T}$ points in the direction opposite that of $\mathbf{T}$ (corresponding to a deceleration). As the car leaves the origin ($t > 0$), $a_T > 0$ (corresponding to an acceleration) and $a_T\mathbf{T}$ and $\mathbf{T}$ point in the direction of the trajectory. At all times, $\mathbf{N}$ points to the inside of the curve (Figure 14.36; Exercise 78).

Related Exercise 38 ➤

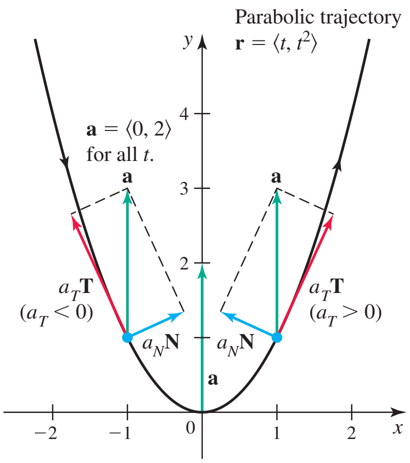


Figure 14.36

QUICK CHECK 5 Verify that $\mathbf{T}$ and $\mathbf{N}$ given in Example 7 satisfy $|\mathbf{T}| = |\mathbf{N}| = 1$ and that $\mathbf{T} \cdot \mathbf{N} = 0$. ➤

The Binormal Vector and Torsion

We have seen that the curvature function and the principal unit normal vector tell us how quickly and in what direction a curve turns. For curves in two dimensions, these quantities give a fairly complete description of motion along the curve. However, in three dimensions, a curve has more "room" in which to change its course, and another descriptive function is often useful. Figure 14.37 shows a smooth parameterized curve C with its unit tangent vector $\mathbf{T}$ and its principal unit normal vector $\mathbf{N}$ at two different points. These two vectors determine a plane called the *osculating plane* (Figure 14.37b). The question we now ask is, How quickly does the curve C move out of the plane determined by $\mathbf{T}$ and $\mathbf{N}$?

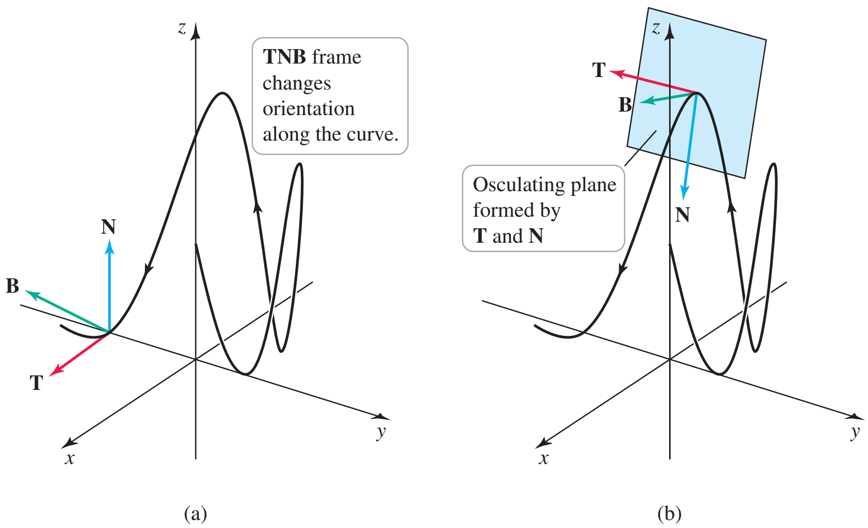


Figure 14.37

To answer this question, we begin by defining the *unit binormal vector* $\mathbf{B} = \mathbf{T} \times \mathbf{N}$. By the definition of the cross product, $\mathbf{B}$ is orthogonal to $\mathbf{T}$ and $\mathbf{N}$. Because $\mathbf{T}$ and $\mathbf{N}$ are unit vectors, $\mathbf{B}$ is also a unit vector. Notice that $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ form a right-handed coordinate system (like the *xyz*-coordinate system) that changes its orientation as we move along the curve. This coordinate system is often called the **TNB frame** (Figure 14.37).

➤ The **TNB** frame is also called the Frenet-Serret frame, after two 19th-century French mathematicians, Jean Frenet and Joseph Serret.

The rate at which the curve C twists out of the plane determined by $\mathbf{T}$ and $\mathbf{N}$ is the rate at which $\mathbf{B}$ changes as we move along C, which is $\dfrac{d\mathbf{B}}{ds}$. A short calculation leads to a practical formula for the twisting of the curve. Differentiating the cross product $\mathbf{T} \times \mathbf{N}$, we find that

$$\begin{aligned}\frac{d\mathbf{B}}{ds} &= \frac{d}{ds}(\mathbf{T} \times \mathbf{N}) \\ &= \underbrace{\frac{d\mathbf{T}}{ds} \times \mathbf{N}}_{\text{parallel vectors}} + \mathbf{T} \times \frac{d\mathbf{N}}{ds} && \text{Product Rule for cross products} \\ &= \mathbf{T} \times \frac{d\mathbf{N}}{ds}. && \frac{d\mathbf{T}}{ds} \text{ and } \mathbf{N} \text{ are parallel; } \frac{d\mathbf{T}}{ds} \times \mathbf{N} = \mathbf{0}.\end{aligned}$$

QUICK CHECK 6 Explain why $\mathbf{B} = \mathbf{T} \times \mathbf{N}$ is a unit vector. ◂

Notice that by definition, $\mathbf{N} = \dfrac{1}{\kappa}\dfrac{d\mathbf{T}}{ds}$, which implies that $\mathbf{N}$ and $\dfrac{d\mathbf{T}}{ds}$ are scalar multiples of each other. Therefore, their cross product is the zero vector.

The properties of $\dfrac{d\mathbf{B}}{ds}$ become clear with the following observations.

- $\dfrac{d\mathbf{B}}{ds}$ is orthogonal to both $\mathbf{T}$ and $\dfrac{d\mathbf{N}}{ds}$, because it is the cross product of $\mathbf{T}$ and $\dfrac{d\mathbf{N}}{ds}$.
- Applying Theorem 14.2 to the unit vector $\mathbf{B}$, it follows that $\dfrac{d\mathbf{B}}{ds}$ is also orthogonal to $\mathbf{B}$.
- By the previous two observations, $\dfrac{d\mathbf{B}}{ds}$ is orthogonal to both $\mathbf{B}$ and $\mathbf{T}$, so it must be parallel to $\mathbf{N}$.

➤ Note that $\mathbf{B}$ is a unit vector (of constant length). Therefore, by Theorem 14.2, $\mathbf{B}$ and $\mathbf{B}'(t)$ are orthogonal. Because $\mathbf{B}'(t)$ and $\mathbf{B}'(s)$ are parallel, it follows that $\mathbf{B}$ and $\mathbf{B}'(s)$ are orthogonal.

Because $\dfrac{d\mathbf{B}}{ds}$ is parallel to (a scalar multiple of) $\mathbf{N}$, we write

$$\frac{d\mathbf{B}}{ds} = -\tau\mathbf{N},$$

where the scalar τ is the *torsion*. Notice that $\left|\dfrac{d\mathbf{B}}{ds}\right| = |-\tau\mathbf{N}| = |-\tau|$, so the magnitude of the torsion equals the magnitude of $\dfrac{d\mathbf{B}}{ds}$, which is the rate at which the curve twists out of the **TN**-plane.

➤ The negative sign in the definition of the torsion is conventional. However, τ may be positive or negative (or zero), and in general, it varies along the curve.

A short calculation gives a method for computing the torsion. We take the dot product of both sides of the equation defining the torsion with $\mathbf{N}$:

$$\begin{aligned}\frac{d\mathbf{B}}{ds}\cdot\mathbf{N} &= -\tau\underbrace{\mathbf{N}\cdot\mathbf{N}}_{1} \\ \frac{d\mathbf{B}}{ds}\cdot\mathbf{N} &= -\tau. && \mathbf{N} \text{ is a unit vector.}\end{aligned}$$

➤ Notice that $\mathbf{B}$ and τ depend on the orientation of the curve.

QUICK CHECK 7 Explain why $\mathbf{N}\cdot\mathbf{N} = 1$. ◂

DEFINITION Unit Binormal Vector and Torsion

Let C be a smooth parameterized curve with unit tangent and principal unit normal vectors $\mathbf{T}$ and $\mathbf{N}$, respectively. Then at each point of the curve at which the curvature is nonzero, the **unit binormal vector** is

$$\mathbf{B} = \mathbf{T} \times \mathbf{N},$$

and the **torsion** is

$$\tau = -\frac{d\mathbf{B}}{ds}\cdot\mathbf{N}.$$

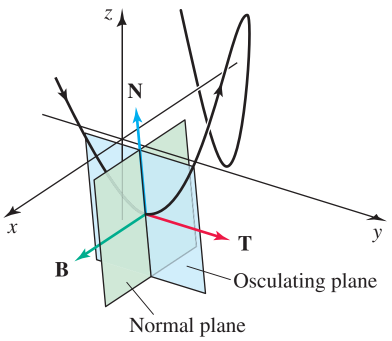


Figure 14.38

➤ The third plane formed by the vectors **T** and **B** is called the *rectifying plane.*

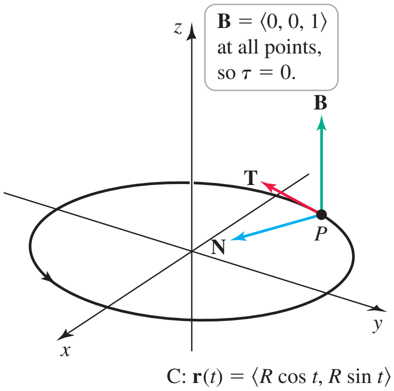


Figure 14.39

Figure 14.38 provides some interpretation of the curvature and the torsion. First, we see a smooth curve C passing through a point where the mutually orthogonal vectors **T**, **N**, and **B** are defined. The **osculating plane** is defined by the vectors **T** and **N**. The plane orthogonal to the osculating plane containing **N** is called the **normal plane**. Because **N** and $\frac{d\mathbf{B}}{ds}$ are parallel, $\frac{d\mathbf{B}}{ds}$ also lies in the normal plane. The torsion, which is equal in magnitude to $\left|\frac{d\mathbf{B}}{ds}\right|$, gives the rate at which the curve moves *out of* the osculating plane. In a complementary way, the curvature, which is equal to $\left|\frac{d\mathbf{T}}{ds}\right|$, gives the rate at which the curve turns *within* the osculating plane. Two examples will clarify these concepts.

EXAMPLE 8 **Unit binormal vector** Consider the circle C defined by

$$\mathbf{r}(t) = \langle R\cos t, R\sin t\rangle, \text{ for } 0 \le t \le 2\pi, \text{ with } R > 0.$$

a. Without doing any calculations, find the unit binormal vector **B** and determine the torsion.

b. Use the definition of **B** to calculate **B** and confirm your answer in part (a).

SOLUTION

a. The circle C lies in the xy-plane, so at all points on the circle, **T** and **N** are in the xy-plane. Therefore, at all points of the circle, $\mathbf{B} = \mathbf{T} \times \mathbf{N}$ is the unit vector in the positive z-direction (by the right-hand rule); that is, $\mathbf{B} = \mathbf{k}$. Because **B** changes in neither length nor direction, $\frac{d\mathbf{B}}{ds} = \mathbf{0}$ and $\tau = 0$ (Figure 14.39).

b. Building on the calculations of Example 2, we find that

$$\mathbf{T} = \langle -\sin t, \cos t\rangle \quad \text{and} \quad \mathbf{N} = \langle -\cos t, -\sin t\rangle.$$

Therefore, the unit binormal vector is

$$\mathbf{B} = \mathbf{T} \times \mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin t & \cos t & 0 \\ -\cos t & -\sin t & 0 \end{vmatrix} = 0\cdot\mathbf{i} - 0\cdot\mathbf{j} + 1\cdot\mathbf{k} = \mathbf{k}.$$

As in part (a), it follows that the torsion is zero.

Related Exercise 41 ◀

Generalizing Example 8, it can be shown that the binormal vector of any curve that lies in the xy-plane is always parallel to the z-axis; therefore, the torsion of the curve is everywhere zero.

EXAMPLE 9 **Torsion of a helix** Compute the torsion of the helix $\mathbf{r}(t) = \langle a\cos t, a\sin t, bt\rangle$, for $t \ge 0$, $a > 0$, and $b > 0$.

SOLUTION In Example 5, we found that

$$\mathbf{T} = \frac{\langle -a\sin t, a\cos t, b\rangle}{\sqrt{a^2 + b^2}} \quad \text{and} \quad \mathbf{N} = \langle -\cos t, -\sin t, 0\rangle.$$

Therefore,

$$\mathbf{B} = \mathbf{T} \times \mathbf{N} = \frac{1}{\sqrt{a^2+b^2}} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin t & a\cos t & b \\ -\cos t & -\sin t & 0 \end{vmatrix} = \frac{\langle b\sin t, -b\cos t, a\rangle}{\sqrt{a^2+b^2}}.$$

The next step is to determine $\frac{d\mathbf{B}}{ds}$, which we do in the same way we computed $\frac{d\mathbf{T}}{ds}$, by writing

$$\frac{d\mathbf{B}}{dt} = \frac{d\mathbf{B}}{ds}\cdot\frac{ds}{dt} \quad \text{or} \quad \frac{d\mathbf{B}}{ds} = \frac{d\mathbf{B}/dt}{ds/dt}.$$

In this case,

$$\frac{ds}{dt} = |\mathbf{r}'(t)| = \sqrt{a^2 \sin^2 t + a^2 \cos^2 t + b^2} = \sqrt{a^2 + b^2}.$$

Computing $\frac{d\mathbf{B}}{dt}$, we have

$$\frac{d\mathbf{B}}{ds} = \frac{d\mathbf{B}/dt}{ds/dt} = \frac{\langle b \cos t, b \sin t, 0 \rangle}{a^2 + b^2}.$$

The final step is to compute the torsion:

$$\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N} = -\frac{\langle b \cos t, b \sin t, 0 \rangle}{a^2 + b^2} \cdot \langle -\cos t, -\sin t, 0 \rangle = \frac{b}{a^2 + b^2}.$$

We see that the torsion is constant over the helix. In Example 4, we found that the curvature of a helix is also constant. This special property of circular helices means that the curve turns about its axis at a constant rate and rises vertically at a constant rate (**Figure 14.40**).

Related Exercise 47 ◄

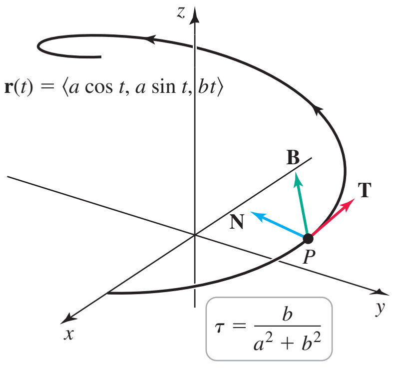


Figure 14.40

Example 9 suggests that the computation of the binormal vector and the torsion can be involved. We close by stating some alternative formulas for $\mathbf{B}$ and τ that may simplify calculations in some cases. Letting $\mathbf{v} = \mathbf{r}'(t)$ and $\mathbf{a} = \mathbf{v}'(t) = \mathbf{r}''(t)$, the binormal vector can be written compactly as (Exercise 83)

$$\mathbf{B} = \mathbf{T} \times \mathbf{N} = \frac{\mathbf{v} \times \mathbf{a}}{|\mathbf{v} \times \mathbf{a}|}.$$

We also state without proof that the torsion may be expressed in either of the forms

$$\tau = \frac{(\mathbf{v} \times \mathbf{a}) \cdot \mathbf{a}'}{|\mathbf{v} \times \mathbf{a}|^2} \quad \text{or} \quad \tau = \frac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2}.$$

SUMMARY Formulas for Curves in Space

Position function: $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$

Velocity: $\mathbf{v} = \mathbf{r}'$

Acceleration: $\mathbf{a} = \mathbf{v}'$

Unit tangent vector: $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$

Principal unit normal vector: $\mathbf{N} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$ (provided $d\mathbf{T}/dt \neq \mathbf{0}$)

Curvature: $\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}$

Components of acceleration: $\mathbf{a} = a_N \mathbf{N} + a_T \mathbf{T}$, where $a_N = \kappa |\mathbf{v}|^2 = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|}$ and $a_T = \frac{d^2 s}{dt^2} = \frac{\mathbf{v} \cdot \mathbf{a}}{|\mathbf{v}|}$

Unit binormal vector: $\mathbf{B} = \mathbf{T} \times \mathbf{N} = \frac{\mathbf{v} \times \mathbf{a}}{|\mathbf{v} \times \mathbf{a}|}$

Torsion: $\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N} = \frac{(\mathbf{v} \times \mathbf{a}) \cdot \mathbf{a}'}{|\mathbf{v} \times \mathbf{a}|^2} = \frac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2}$

SECTION 14.5 EXERCISES

Getting Started

1. What is the curvature of a straight line?
2. Explain in words the meaning of *the curvature of a curve*. Is it a scalar function or a vector function?
3. Give a practical formula for computing curvature.
4. Interpret *the principal unit normal vector of a curve*. Is it a scalar function or a vector function?
5. Give a practical formula for computing the principal unit normal vector.
6. Explain how to decompose the acceleration vector of a moving object into its tangential and normal components.
7. Explain how the vectors **T**, **N**, and **B** are related geometrically.
8. How do you compute **B**?
9. Give a geometrical interpretation of torsion.
10. How do you compute torsion?

Practice Exercises

11–20. Curvature *Find the unit tangent vector* **T** *and the curvature* κ *for the following parameterized curves.*

11. $\mathbf{r}(t) = \langle 2t + 1, 4t - 5, 6t + 12 \rangle$
12. $\mathbf{r}(t) = \langle 2\cos t, -2\sin t \rangle$
13. $\mathbf{r}(t) = \langle 2t, 4\sin t, 4\cos t \rangle$
14. $\mathbf{r}(t) = \langle \cos t^2, \sin t^2 \rangle$
15. $\mathbf{r}(t) = \langle \sqrt{3}\sin t, \sin t, 2\cos t \rangle$
16. $\mathbf{r}(t) = \langle t, \ln \cos t \rangle$
17. $\mathbf{r}(t) = \langle t, 2t^2 \rangle$
18. $\mathbf{r}(t) = \langle \cos^3 t, \sin^3 t \rangle$
19. $\mathbf{r}(t) = \left\langle \int_0^t \cos \frac{\pi u^2}{2}\,du, \int_0^t \sin \frac{\pi u^2}{2}\,du \right\rangle, t > 0$
20. $\mathbf{r}(t) = \left\langle \int_0^t \cos u^2\,du, \int_0^t \sin u^2\,du \right\rangle, t > 0$

21–26. Alternative curvature formula *Use the alternative curvature formula* $\kappa = \dfrac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}$ *to find the curvature of the following parameterized curves.*

21. $\mathbf{r}(t) = \langle -3\cos t, 3\sin t, 0 \rangle$
22. $\mathbf{r}(t) = \langle 4t, 3\sin t, 3\cos t \rangle$
23. $\mathbf{r}(t) = \langle 4 + t^2, t, 0 \rangle$
24. $\mathbf{r}(t) = \langle \sqrt{3}\sin t, \sin t, 2\cos t \rangle$
25. $\mathbf{r}(t) = \langle 4\cos t, \sin t, 2\cos t \rangle$
26. $\mathbf{r}(t) = \langle e^t \cos t, e^t \sin t, e^t \rangle$

27–34. Principal unit normal vector *Find the unit tangent vector* **T** *and the principal unit normal vector* **N** *for the following parameterized curves. In each case, verify that* $|\mathbf{T}| = |\mathbf{N}| = 1$ *and* $\mathbf{T} \cdot \mathbf{N} = 0$.

27. $\mathbf{r}(t) = \langle 2\sin t, 2\cos t \rangle$
28. $\mathbf{r}(t) = \langle 4\sin t, 4\cos t, 10t \rangle$
29. $\mathbf{r}(t) = \left\langle \frac{t^2}{2}, 4 - 3t, 1 \right\rangle$
30. $\mathbf{r}(t) = \left\langle \frac{t^2}{2}, \frac{t^3}{3} \right\rangle, t > 0$
31. $\mathbf{r}(t) = \langle \cos t^2, \sin t^2 \rangle$
32. $\mathbf{r}(t) = \langle \cos^3 t, \sin^3 t \rangle$
33. $\mathbf{r}(t) = \langle t^2, t \rangle$
34. $\mathbf{r}(t) = \langle t, \ln \cos t \rangle$

35–40. Components of the acceleration *Consider the following trajectories of moving objects. Find the tangential and normal components of the acceleration.*

35. $\mathbf{r}(t) = \langle t, 1 + 4t, 2 - 6t \rangle$
36. $\mathbf{r}(t) = \langle 10\cos t, -10\sin t \rangle$
37. $\mathbf{r}(t) = \langle e^t \cos t, e^t \sin t, e^t \rangle$
38. $\mathbf{r}(t) = \langle t, t^2 + 1 \rangle$
39. $\mathbf{r}(t) = \langle t^3, t^2 \rangle$
40. $\mathbf{r}(t) = \langle 20\cos t, 20\sin t, 30t \rangle$

41–44. Computing the binormal vector and torsion *In Exercises 27–30, the unit tangent vector* **T** *and the principal unit normal vector* **N** *were computed for the following parameterized curves. Use the definitions to compute their unit binormal vector and torsion.*

41. $\mathbf{r}(t) = \langle 2\sin t, 2\cos t \rangle$
42. $\mathbf{r}(t) = \langle 4\sin t, 4\cos t, 10t \rangle$
43. $\mathbf{r}(t) = \left\langle \frac{t^2}{2}, 4 - 3t, 1 \right\rangle$
44. $\mathbf{r}(t) = \left\langle \frac{t^2}{2}, \frac{t^3}{3} \right\rangle, t > 0$

45–48. Computing the binormal vector and torsion *Use the definitions to compute the unit binormal vector and torsion of the following curves.*

45. $\mathbf{r}(t) = \langle 2\cos t, 2\sin t, -t \rangle$
46. $\mathbf{r}(t) = \langle t, \cosh t, -\sinh t \rangle$
47. $\mathbf{r}(t) = \langle 12t, 5\cos t, 5\sin t \rangle$
48. $\mathbf{r}(t) = \langle \sin t - t\cos t, \cos t + t\sin t, t \rangle$

49. **Explain why or why not** Determine whether the following statements are true and give an explanation or counterexample.
    a. The position, unit tangent, and principal unit normal vectors (**r**, **T**, and **N**) at a point lie in the same plane.
    b. The vectors **T** and **N** at a point depend on the orientation of a curve.
    c. The curvature at a point depends on the orientation of a curve.
    d. An object with unit speed ($|\mathbf{v}| = 1$) on a circle of radius R has an acceleration of $\mathbf{a} = \mathbf{N}/R$.
    e. If the speedometer of a car reads a constant 60 mi/hr, the car is not accelerating.
    f. A curve in the xy-plane that is concave up at all points has positive torsion.
    g. A curve with large curvature also has large torsion.

50. **Special formula: Curvature for $y = f(x)$** Assume f is twice differentiable. Prove that the curve $y = f(x)$ has curvature

$$\kappa(x) = \frac{|f''(x)|}{(1 + f'(x)^2)^{3/2}}.$$

(*Hint:* Use the parametric description $x = t$, $y = f(t)$.)

51–54. Curvature for $y = f(x)$ *Use the result of Exercise 50 to find the curvature function of the following curves.*

51. $f(x) = x^2$
52. $f(x) = \sqrt{a^2 - x^2}$
53. $f(x) = \ln x$
54. $f(x) = \ln \cos x$

55. Special formula: Curvature for plane curves Show that the parametric curve $\mathbf{r}(t) = \langle f(t), g(t) \rangle$, where f and g are twice differentiable, has curvature

$$\kappa(t) = \frac{|f'g'' - f''g'|}{((f')^2 + (g')^2)^{3/2}},$$

where all derivatives are taken with respect to t.

56–59. Curvature for plane curves *Use the result of Exercise 55 to find the curvature function of the following curves.*

56. $\mathbf{r}(t) = \langle a \sin t, a \cos t \rangle$ (circle)

57. $\mathbf{r}(t) = \langle a \sin t, b \cos t \rangle$ (ellipse)

58. $\mathbf{r}(t) = \langle a \cos^3 t, a \sin^3 t \rangle$ (astroid)

59. $\mathbf{r}(t) = \langle t, at^2 \rangle$ (parabola)

When appropriate, consider using the special formulas derived in Exercises 50 and 55 in the remaining exercises.

60–63. Same paths, different velocity *The position functions of objects A and B describe different motion along the same path for $t \geq 0$.*

- **a.** *Sketch the path followed by both A and B.*
- **b.** *Find the velocity and acceleration of A and B and discuss the differences.*
- **c.** *Express the acceleration of A and B in terms of the tangential and normal components and discuss the differences.*

60. A: $\mathbf{r}(t) = \langle 1 + 2t, 2 - 3t, 4t \rangle$, B: $\mathbf{r}(t) = \langle 1 + 6t, 2 - 9t, 12t \rangle$

61. A: $\mathbf{r}(t) = \langle t, 2t, 3t \rangle$, B: $\mathbf{r}(t) = \langle t^2, 2t^2, 3t^2 \rangle$

62. A: $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$, B: $\mathbf{r}(t) = \langle \cos 3t, \sin 3t \rangle$

63. A: $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$, B: $\mathbf{r}(t) = \langle \cos t^2, \sin t^2 \rangle$

T 64–67. Graphs of the curvature *Consider the following curves.*

- **a.** *Graph the curve.*
- **b.** *Compute the curvature.*
- **c.** *Graph the curvature as a function of the parameter.*
- **d.** *Identify the points (if any) at which the curve has a maximum or minimum curvature.*
- **e.** *Verify that the graph of the curvature is consistent with the graph of the curve.*

64. $\mathbf{r}(t) = \langle t, t^2 \rangle$, for $-2 \leq t \leq 2$ (parabola)

65. $\mathbf{r}(t) = \langle t - \sin t, 1 - \cos t \rangle$, for $0 \leq t \leq 2\pi$ (cycloid)

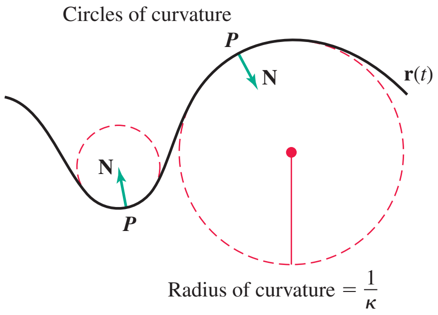
66. $\mathbf{r}(t) = \langle t, \sin t \rangle$, for $0 \leq t \leq \pi$ (sine curve)

67. $\mathbf{r}(t) = \left\langle \frac{t^2}{2}, \frac{t^3}{3} \right\rangle$, for $t > 0$

68. Curvature of $\ln x$ Find the curvature of $f(x) = \ln x$, for $x > 0$, and find the point at which it is a maximum. What is the value of the maximum curvature?

69. Curvature of e^x Find the curvature of $f(x) = e^x$ and find the point at which it is a maximum. What is the value of the maximum curvature?

70. Circle and radius of curvature Choose a point P on a smooth curve C in the plane. The **circle of curvature** (or **osculating circle**) at P is the circle that (a) is tangent to C at P, (b) has the same curvature as C at P, and (c) lies on the same side of C as the principal unit normal $\mathbf{N}$ (see figure). The **radius of curvature** is the radius of the circle of curvature. Show that the radius of curvature is $1/\kappa$, where κ is the curvature of C at P.



71–73. Finding the radius of curvature *Find the radius of curvature (see Exercise 70) of the following curves at the given point. Then write an equation of the circle of curvature at the point.*

71. $\mathbf{r}(t) = \langle t, t^2 \rangle$ (parabola) at $t = 0$

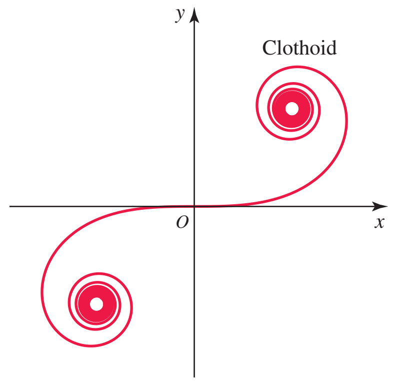
72. $y = \ln x$ at $x = 1$

73. $\mathbf{r}(t) = \langle t - \sin t, 1 - \cos t \rangle$ (cycloid) at $t = \pi$

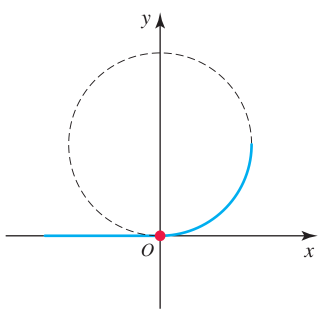
74. Designing a highway curve The function

$$\mathbf{r}(t) = \left\langle \int_0^t \cos \frac{u^2}{2} du, \int_0^t \sin \frac{u^2}{2} du \right\rangle,$$

whose graph is called a **clothoid** or **Euler spiral** (Figure A), has applications in the design of railroad tracks, roller coasters, and highways.



(A)



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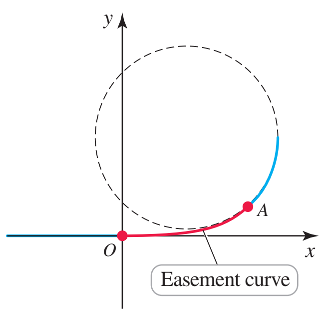


(C)

a. A car moves from left to right on a straight highway, approaching a curve at the origin (Figure B). Sudden changes in curvature at the start of the curve may cause the driver to jerk the steering wheel. Suppose the curve starting at the origin is a segment of a circle of radius a. Explain why there is a sudden change in the curvature of the road at the origin. (*Hint:* See Exercise 70.)

b. A better approach is to use a segment of a clothoid as an easement curve, in between the straight highway and a circle, to avoid sudden changes in curvature (Figure C). Assume the easement curve corresponds to the clothoid $\mathbf{r}(t)$, for $0 \le t \le 1.2$. Find the curvature of the easement curve as a function of t, and explain why this curve eliminates the sudden change in curvature at the origin.

c. Find the radius of a circle connected to the easement curve at point A (that corresponds to $t = 1.2$ on the curve $\mathbf{r}(t)$) so that the curvature of the circle matches the curvature of the easement curve at point A.

75. Curvature of the sine curve The function $f(x) = \sin nx$, where n is a positive real number, has a local maximum at $x = \frac{\pi}{2n}$. Compute the curvature κ of f at this point. How does κ vary (if at all) as n varies?

76. Parabolic trajectory In Example 7 it was shown that for the parabolic trajectory $\mathbf{r}(t) = \langle t, t^2 \rangle$, $\mathbf{a} = \langle 0, 2 \rangle$ and $\mathbf{a} = \frac{2}{\sqrt{1 + 4t^2}}(\mathbf{N} + 2t\,\mathbf{T})$. Show that the second expression for $\mathbf{a}$ reduces to the first expression.

T 77. Parabolic trajectory Consider the parabolic trajectory

$$x = (V_0 \cos \alpha)t,\ y = (V_0 \sin \alpha)t - \frac{1}{2}gt^2,$$

where V_0 is the initial speed, α is the angle of launch, and g is the acceleration due to gravity. Consider all times $[0, T]$ for which $y \ge 0$.

a. Find and graph the speed, for $0 \le t \le T$.
b. Find and graph the curvature, for $0 \le t \le T$.
c. At what times (if any) do the speed and curvature have maximum and minimum values?

Explorations and Challenges

78. Relationship between T, N, and a Show that if an object accelerates in the sense that $\frac{d^2s}{dt^2} > 0$ and $\kappa \ne 0$, then the acceleration vector lies between $\mathbf{T}$ and $\mathbf{N}$ in the plane of $\mathbf{T}$ and $\mathbf{N}$. Show that if an object decelerates in the sense that $\frac{d^2s}{dt^2} < 0$, then the acceleration vector lies in the plane of $\mathbf{T}$ and $\mathbf{N}$, but not between $\mathbf{T}$ and $\mathbf{N}$.

79. Zero curvature Prove that the curve

$$\mathbf{r}(t) = \langle a + bt^p, c + dt^p, e + ft^p \rangle,$$

where a, b, c, d, e, and f are real numbers and p is a positive integer, has zero curvature. Give an explanation.

80. Practical formula for N Show that the definition of the principal unit normal vector $\mathbf{N} = \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|}$ implies the practical formula $\mathbf{N} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$. Use the Chain Rule and recall that $|\mathbf{v}| = \frac{ds}{dt} > 0$.

T 81. Maximum curvature Consider the "superparabolas" $f_n(x) = x^{2n}$, where n is a positive integer.

a. Find the curvature function of f_n, for $n = 1, 2$, and 3.
b. Plot f_n and their curvature functions, for $n = 1, 2$, and 3, and check for consistency.
c. At what points does the maximum curvature occur, for $n = 1, 2$, and 3?
d. Let the maximum curvature for f_n occur at $x = \pm z_n$. Using either analytical methods or a calculator, determine $\lim_{n \to \infty} z_n$. Interpret your result.

82. Alternative derivation of curvature Derive the computational formula for curvature using the following steps.

a. Use the tangential and normal components of the acceleration to show that $\mathbf{v} \times \mathbf{a} = \kappa |\mathbf{v}|^3 \mathbf{B}$. (Note that $\mathbf{T} \times \mathbf{T} = \mathbf{0}$.)
b. Solve the equation in part (a) for κ and conclude that $\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}$, as shown in the text.

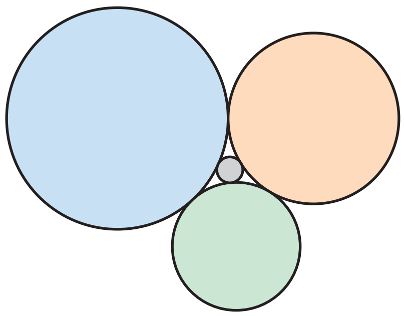
83. Computational formula for B Use the result of part (a) of Exercise 82 and the formula for κ to show that

$$\mathbf{B} = \frac{\mathbf{v} \times \mathbf{a}}{|\mathbf{v} \times \mathbf{a}|}.$$

84. Torsion formula Show that the formula defining torsion, $\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N}$, is equivalent to $\tau = -\frac{1}{|\mathbf{v}|}\frac{d\mathbf{B}}{dt} \cdot \mathbf{N}$. The second formula is generally easier to use.

85. Descartes' four-circle solution Consider the four mutually tangent circles shown in the figure that have radii a, b, c, and d, and curvatures $A = \frac{1}{a}$, $B = \frac{1}{b}$, $C = \frac{1}{c}$, and $D = \frac{1}{d}$. Prove Descartes' result (1643) that

$$(A + B + C + D)^2 = 2(A^2 + B^2 + C^2 + D^2).$$



QUICK CHECK ANSWERS

1. $\kappa = \frac{1}{3}$ **2.** $\kappa = 0$ **3.** Negative y-direction **4.** $\kappa = 0$, so $\mathbf{N}$ is undefined. **6.** $|\mathbf{T}| = |\mathbf{N}| = 1$, so $|\mathbf{B}| = 1$ **7.** For any vector, $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$. Because $|\mathbf{N}| = 1$, $\mathbf{N} \cdot \mathbf{N} = 1$. ◄

CHAPTER 14 REVIEW EXERCISES

1. **Explain why or why not** Determine whether the following statements are true and give an explanation or counterexample.
   a. If $\mathbf{r}(t) = \langle \cos t, e^t, t \rangle + \mathbf{C}$ and $\mathbf{r}(0) = \langle 0, 0, 0 \rangle$, then $\mathbf{C} = \langle 0, 0, 0 \rangle$.
   b. The curvature of a circle of radius 5 is $\kappa = 1/5$.
   c. The graph of $\mathbf{r}(t) = \langle 3 \cos t, 0, 6 \sin t \rangle$ is an ellipse in the xz-plane.
   d. If $\mathbf{r}'(t) = \mathbf{0}$, then $\mathbf{r}(t) = \langle a, b, c \rangle$, where a, b, and c are real numbers.
   e. The parameterized curve $\mathbf{r}(t) = \langle 5 \cos t, 12 \cos t, 13 \sin t \rangle$ has arc length as a parameter.
   f. The position vector and the principal unit normal are always parallel on a smooth curve.

2. **Sets of points** Describe the set of points satisfying the equations $x^2 + z^2 = 1$ and $y = 2$.

T **3–6. Graphing curves** *Sketch the curves described by the following functions, indicating the orientation of the curve. Use analysis and describe the shape of the curve before using a graphing utility.*

3. $\mathbf{r}(t) = (2t + 1)\mathbf{i} + t\,\mathbf{j}$

4. $\mathbf{r}(t) = \langle \cos t, 1 + \cos^2 t \rangle$, for $0 \le t \le \pi/2$

5. $\mathbf{r}(t) = 4 \cos t\,\mathbf{i} + \mathbf{j} + 4 \sin t\,\mathbf{k}$, for $0 \le t \le 2\pi$

6. $\mathbf{r}(t) = e^t\,\mathbf{i} + 2e^t\,\mathbf{j} + \mathbf{k}$, for $t \ge 0$

7. **Intersection curve** A sphere S and a plane P intersect along the curve $\mathbf{r}(t) = \sin t\,\mathbf{i} + \sqrt{2} \cos t\,\mathbf{j} + \sin t\,\mathbf{k}$, for $0 \le t \le 2\pi$. Find equations for S and P and describe the curve $\mathbf{r}$.

8–13. Vector-valued functions *Find a function $\mathbf{r}(t)$ that describes each of the following curves.*

8. The line segment from $P(2, -3, 0)$ to $Q(1, 4, 9)$

9. The line passing through the point $P(4, -2, 3)$ that is orthogonal to the lines $\mathbf{R}(t) = \langle t, 5t, 2t \rangle$ and $\mathbf{S}(t) = \langle t + 1, -1, 3t - 1 \rangle$

10. A circle of radius 3 centered at $(2, 1, 0)$ that lies in the plane $y = 1$

11. An ellipse in the plane $x = 2$ satisfying the equation $\dfrac{y^2}{9} + \dfrac{z^2}{16} = 1$

12. The projection of the curve onto the xy-plane is the parabola $y = x^2$, and the projection of the curve onto the xz-plane is the line $z = x$.

13. The projection of the curve onto the xy-plane is the unit circle $x^2 + y^2 = 1$, and the projection of the curve onto the yz-plane is the line segment $z = y$, for $-1 \le y \le 1$.

14–15. Intersection curve *Find the curve $\mathbf{r}(t)$ where the following surfaces intersect.*

14. $z = x^2 - 5y^2$; $z = 10x^2 + 4y^2 - 36$

15. $x^2 + 7y^2 + 2z^2 = 9$; $z = y$

16–19. Working with vector-valued functions *For each vector-valued function $\mathbf{r}$, carry out the following steps.*

a. Evaluate $\lim_{t \to 0} \mathbf{r}(t)$ and $\lim_{t \to \infty} \mathbf{r}(t)$, if each exists.
b. Find $\mathbf{r}'(t)$ and evaluate $\mathbf{r}'(0)$.
c. Find $\mathbf{r}''(t)$.
d. Evaluate $\int \mathbf{r}(t)\,dt$.

16. $\mathbf{r}(t) = \langle t + 1, t^2 - 3 \rangle$

17. $\mathbf{r}(t) = \left\langle \dfrac{1}{2t + 1}, \dfrac{t}{t + 1} \right\rangle$

18. $\mathbf{r}(t) = \langle e^{-2t}, te^{-t}, \tan^{-1} t \rangle$

19. $\mathbf{r}(t) = \langle \sin 2t, 3 \cos 4t, t \rangle$

20–21. Definite integrals *Evaluate the following definite integrals.*

20. $\displaystyle\int_1^3 \left(6t^2\,\mathbf{i} + 4t\,\mathbf{j} + \frac{1}{t}\,\mathbf{k} \right) dt$

21. $\displaystyle\int_{-1}^1 \left(\sin \pi t\,\mathbf{i} + \mathbf{j} + \frac{2}{1 + t^2}\,\mathbf{k} \right) dt$

22–24. Derivative rules *Suppose $\mathbf{u}$ and $\mathbf{v}$ are differentiable functions at $t = 0$ with $\mathbf{u}(0) = \langle 2, 7, 0 \rangle$, $\mathbf{u}'(0) = \langle 3, 1, 2 \rangle$, $\mathbf{v}(0) = \langle 3, -1, 0 \rangle$, and $\mathbf{v}'(0) = \langle 5, 0, 3 \rangle$. Evaluate the following expressions.*

22. $\dfrac{d}{dt}(\mathbf{u} \cdot \mathbf{v})\Big|_{t=0}$

23. $\dfrac{d}{dt}(\mathbf{u} \times \mathbf{v})\Big|_{t=0}$

24. $\dfrac{d}{dt}(\mathbf{u}(e^{5t} - 1))\Big|_{t=0}$

25–27. Finding r from r′ *Find the function $\mathbf{r}$ that satisfies the given conditions.*

25. $\mathbf{r}'(t) = \langle 1, \sin 2t, \sec^2 t \rangle$; $\mathbf{r}(0) = \langle 2, 2, 2 \rangle$

26. $\mathbf{r}'(t) = \langle e^t, 2e^{2t}, 6e^{3t} \rangle$; $\mathbf{r}(0) = \langle 1, 3, -1 \rangle$

27. $\mathbf{r}'(t) = \left\langle \dfrac{4}{1 + t^2}, 2t + 1, 3t^2 \right\rangle$; $\mathbf{r}(1) = \langle 0, 0, 0 \rangle$

28–29. Unit tangent vectors *Find the unit tangent vector $\mathbf{T}(t)$ for the following parameterized curves. Then determine the unit tangent vector at the given value of t.*

28. $\mathbf{r}(t) = \langle 8, 3 \sin 2t, 3 \cos 2t \rangle$, for $0 \le t \le \pi$; $t = \pi/4$

29. $\mathbf{r}(t) = \langle 2e^t, e^{2t}, t \rangle$, for $0 \le t \le 2\pi$; $t = 0$

30–31. Velocity and acceleration from position *Consider the following position functions.*

a. Find the velocity and speed of the object.
b. Find the acceleration of the object.

30. $\mathbf{r}(t) = \left\langle \dfrac{5}{3} t^3 + 1, t^2 + 10t \right\rangle$, for $t \ge 0$

31. $\mathbf{r}(t) = \left\langle e^{4t} + 1, e^{4t}, \dfrac{1}{2} e^{4t} + 1 \right\rangle$, for $t \ge 0$

32–33. Solving equations of motion *Given an acceleration vector, initial velocity $\langle u_0, v_0 \rangle$, and initial position $\langle x_0, y_0 \rangle$, find the velocity and position vectors for $t \ge 0$.*

32. $\mathbf{a}(t) = \langle 1, 4 \rangle$, $\langle u_0, v_0 \rangle = \langle 4, 3 \rangle$, $\langle x_0, y_0 \rangle = \langle 0, 2 \rangle$

33. $\mathbf{a}(t) = \langle \cos t, 2 \sin t \rangle$, $\langle u_0, v_0 \rangle = \langle 2, 1 \rangle$, $\langle x_0, y_0 \rangle = \langle 1, 2 \rangle$

T 34. **Orthogonal r and r′** Find all points on the ellipse $\mathbf{r}(t) = \langle 1, 8 \sin t, \cos t \rangle$, for $0 \le t \le 2\pi$, at which $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are orthogonal. Sketch the curve and the tangent vectors to verify your conclusion.

T 35–36. Modeling motion *Consider the motion of the following objects. Assume the x-axis is horizontal, the positive y-axis is vertical, the ground is horizontal, and only the gravitational force acts on the object.*

a. Find the velocity and position vectors, for $t \geq 0$.
b. Determine the time of flight and range of the object.
c. Determine the maximum height of the object.

35. A baseball has an initial position $\langle x_0, y_0 \rangle = \langle 0, 3 \rangle$ ft when it is hit at an angle of 60° with an initial speed of 80 ft/s.

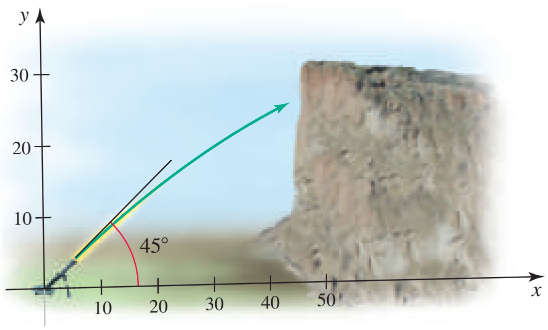
36. A rock is thrown from 2 m above horizontal ground at an angle of 30° above the horizontal with a speed of 6 m/s. Assume the initial position of the rock is $\langle x_0, y_0 \rangle = \langle 0, 2 \rangle$.

T 37. A baseball is hit 2 ft above home plate with an initial velocity of $\langle 40, 20, 40 \rangle$ ft/s. The spin on the baseball produces a horizontal acceleration of the ball of 4 ft/s^2 in the eastward direction. Assume the positive x-axis points east and the positive y-axis points north.

a. Find the velocity and position vectors, for $t \geq 0$. Assume the origin is at home plate.
b. When does the ball hit the ground? Round your answer to three digits to the right of the decimal place.
c. How far does it land from home plate? Round your answer to the nearest whole number.

38. Firing angles A projectile is fired over horizontal ground from the ground with an initial speed of 40 m/s. What firing angles produce a range of 100 m?

T 39. Projectile motion A projectile is launched from the origin, which is a point 50 ft from a 30-ft vertical cliff (see figure). It is launched at a speed of $50\sqrt{2}$ ft/s at an angle of 45° to the horizontal. Assume the ground is horizontal on top of the cliff and that only the gravitational force affects the motion of the object.

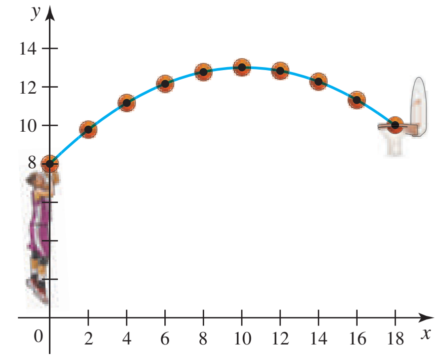


a. Give the coordinates of the landing spot of the projectile on the top of the cliff.
b. What is the maximum height reached by the projectile?
c. What is the time of flight?
d. Write the integral that gives the length of the trajectory.
e. Approximate the length of the trajectory.
f. What is the range of launch angles needed to clear the edge of the cliff?

40. Baseball motion A toddler on level ground throws a baseball into the air at an angle of 30° with the ground from a height of 2 ft. If the ball lands 10 ft from the child, determine the initial speed of the ball.

T 41. Closest point Find the approximate location of the point on the curve $\mathbf{r}(t) = \langle t, t^2 + 1, 3t \rangle$ that lies closest to the point $P(3, 1, 6)$.

T 42. Basketball shot A basketball is shot at an angle of 45° to the horizontal. The center of the basketball is at the point $A(0, 8)$ at the moment it is released, and it passes through the center of the basketball hoop that is located at the point $B(18, 10)$. Assume the basketball does not hit the front of the hoop (otherwise it might not pass through the basket). The validity of this assumption is investigated in parts (d), (e), and (f).



a. Determine the initial speed of the basketball.
b. Find the initial velocity $\mathbf{v}(0)$ at the moment it is released.
c. Find the position function $\mathbf{r}(t)$ of the center of the basketball t seconds after the ball is released. Assume $\mathbf{r}(0) = \langle 0, 8 \rangle$.
d. Find the distance $s(t)$ between the center of the basketball and the front of the basketball hoop t seconds after the ball is released. Assume the diameter of the basketball hoop is 18 inches.
e. Determine the closest distance (in inches) between the center of the basketball and the front of the basketball hoop.
f. Is the assumption that the basketball does not hit the front of the hoop valid? Use the fact that the diameter of a women's basketball is about 9.23 inches. (*Hint:* The ball hits the front of the hoop if, during its flight, the distance from the center of the ball to the front of the hoop is less than the radius of the basketball.)

43–46. Arc length *Find the arc length of the following curves.*

43. $\mathbf{r}(t) = \left\langle t^2, \frac{4\sqrt{2}}{3} t^{3/2}, 2t \right\rangle$, for $1 \leq t \leq 3$

44. $\mathbf{r}(t) = \langle 2t^{9/2}, t^3 \rangle$, for $0 \leq t \leq 2$

T 45. $\mathbf{r}(t) = \langle \sin t, t + \cos t, 4t \rangle$, for $0 \leq t \leq \frac{\pi}{2}$

46. $\mathbf{r}(t) = \langle t, \ln \sec t, \ln (\sec t + \tan t) \rangle$, for $0 \leq t \leq \frac{\pi}{4}$

47. Velocity and trajectory length The acceleration of a wayward firework is given by $\mathbf{a}(t) = \sqrt{2}\,\mathbf{j} + 2t\mathbf{k}$, for $0 \leq t \leq 3$. Suppose the initial velocity of the firework is $\mathbf{v}(0) = \mathbf{i}$.

a. Find the velocity of the firework, for $0 \leq t \leq 3$.
b. Find the length of the trajectory of the firework over the interval $0 \leq t \leq 3$.

48–49. Arc length parameterization *Find a description of the following curves that uses arc length as a parameter.*

48. $\mathbf{r}(t) = (1 + 4t)\mathbf{i} - 3t\mathbf{j}$, for $t \geq 1$

49. $\mathbf{r}(t) = \left\langle t^2, \frac{4\sqrt{2}}{3} t^{3/2}, 2t \right\rangle$, for $t \geq 0$

50. Tangents and normals for an ellipse Consider the ellipse $\mathbf{r}(t) = \langle 3\cos t, 4\sin t\rangle$, for $0 \le t \le 2\pi$.

a. Find the tangent vector $\mathbf{r}'$, the unit tangent vector $\mathbf{T}$, and the principal unit normal vector $\mathbf{N}$ at all points on the curve.
b. At what points does $|\mathbf{r}'|$ have maximum and minimum values?
c. At what points does the curvature have maximum and minimum values? Interpret this result in light of part (b).
d. Find the points (if any) at which $\mathbf{r}$ and $\mathbf{N}$ are parallel.

T 51–54. Properties of space curves *Do the following calculations.*

a. *Find the tangent vector and the unit tangent vector.*
b. *Find the curvature.*
c. *Find the principal unit normal vector.*
d. *Verify that* $|\mathbf{N}| = 1$ *and* $\mathbf{T}\cdot\mathbf{N} = 0$.
e. *Graph the curve and sketch* $\mathbf{T}$ *and* $\mathbf{N}$ *at two points.*

51. $\mathbf{r}(t) = \langle 6\cos t, 3\sin t\rangle$, for $0 \le t \le 2\pi$

52. $\mathbf{r}(t) = \cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + \mathbf{k}$, for $0 \le t \le 2\pi$

53. $\mathbf{r}(t) = \cos t\,\mathbf{i} + 2\cos t\,\mathbf{j} + \sqrt{5}\sin t\,\mathbf{k}$, for $0 \le t \le 2\pi$

54. $\mathbf{r}(t) = t\,\mathbf{i} + 2\cos t\,\mathbf{j} + 2\sin t\,\mathbf{k}$, for $0 \le t \le 2\pi$

55–58. Analyzing motion *Consider the position vector of the following moving objects.*

a. *Find the normal and tangential components of the acceleration.*
b. *Graph the trajectory and sketch the normal and tangential components of the acceleration at two points on the trajectory. Show that their sum gives the total acceleration.*

55. $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}$, for $0 \le t \le 2\pi$

56. $\mathbf{r}(t) = 3t\,\mathbf{i} + (4 - t)\mathbf{j} + t\,\mathbf{k}$, for $t \ge 0$

57. $\mathbf{r}(t) = (t^2 + 1)\mathbf{i} + 2t\,\mathbf{j}$, for $t \ge 0$

58. $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + 10t\,\mathbf{k}$, for $0 \le t \le 2\pi$

59. Computing the binormal vector and torsion Compute the unit binormal vector $\mathbf{B}$ and the torsion of the curve $\mathbf{r}(t) = \langle t, t^2, t^3\rangle$ at $t = 1$.

60–61. Curve analysis *Carry out the following steps for the given curves C.*

a. *Find* $\mathbf{T}(t)$ *at all points of C.*
b. *Find* $\mathbf{N}(t)$ *and the curvature at all points of C.*
c. *Sketch the curve and show* $\mathbf{T}(t)$ *and* $\mathbf{N}(t)$ *at the points of C corresponding to* $t = 0$ *and* $t = \pi/2$.
d. *Are the results of parts (a) and (b) consistent with the graph?*
e. *Find* $\mathbf{B}(t)$ *at all points of C.*
f. *On the graph of part (c), plot* $\mathbf{B}(t)$ *at the points of C corresponding to* $t = 0$ *and* $t = \pi/2$.
g. *Describe three calculations that serve to check the accuracy of your results in parts (a)–(f).*
h. *Compute the torsion at all points of C. Interpret this result.*

60. $C: \mathbf{r}(t) = \langle 3\sin t, 4\sin t, 5\cos t\rangle$, for $0 \le t \le 2\pi$

61. $C: \mathbf{r}(t) = \langle 3\sin t, 3\cos t, 4t\rangle$, for $0 \le t \le 2\pi$

62. Torsion of a plane curve Suppose $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$, where f, g, and h are the quadratic functions $f(t) = a_1t^2 + b_1t + c_1$, $g(t) = a_2t^2 + b_2t + c_2$, and $h(t) = a_3t^2 + b_3t + c_3$, and where at least one of the leading coefficients a_1, a_2, and a_3 is nonzero. Apart from a set of degenerate cases (for example, $\mathbf{r}(t) = \langle t^2, t^2, t^2\rangle$, whose graph is a line), it can be shown that the graph of $\mathbf{r}(t)$ is a parabola that lies in a plane (Exercise 63).

a. Show by direct computation that $\mathbf{v} \times \mathbf{a}$ is constant. Then explain why the unit binormal vector is constant at all points on the curve. What does this result say about the torsion of the curve?
b. Compute $\mathbf{a}'(t)$ and explain why the torsion is zero at all points on the curve for which the torsion is defined.

63. Families of plane curves Let f and g be continuous on an interval I. Consider the curve

$$C: \mathbf{r}(t) = \langle a_1f(t) + a_2g(t) + a_3, b_1f(t) + b_2g(t) + b_3, c_1f(t) + c_2g(t) + c_3\rangle,$$

for t in I, and where a_i, b_i, and c_i, for $i = 1, 2$, and 3, are real numbers.

a. Show that, in general, C lies in a plane.
b. Explain why the torsion is zero at all points of C for which the torsion is defined.

64. Length of a DVD groove The capacity of a single-sided, single-layer digital versatile disc (DVD) is approximately 4.7 billion bytes—enough to store a two-hour movie. (Newer double-sided, double-layer DVDs have about four times that capacity, and Blu-ray discs are in the range of 50 gigabytes.) A DVD consists of a single "groove" that spirals outward from the inner edge to the outer edge of the storage region.

a. First consider the spiral given in polar coordinates by $r = \dfrac{t\theta}{2\pi}$, where $0 \le \theta \le 2\pi N$ and successive loops of the spiral are t units apart. Explain why this spiral has N loops and why the entire spiral has a radius of $R = Nt$ units. Sketch three loops of the spiral.
b. Write an integral for the length L of the spiral with N loops.
c. The integral in part (b) can be evaluated exactly, but a good approximation can also be made. Assuming N is large, explain why $\theta^2 + 1 \approx \theta^2$. Use this approximation to simplify the integral in part (b) and show that $L \approx t\pi N^2 = \dfrac{\pi R^2}{t}$.
d. Now consider a DVD with an inner radius of $r = 2.5$ cm and an outer radius of $R = 5.9$ cm. Model the groove by a spiral with a thickness of $t = 1.5$ microns $= 1.5 \times 10^{-6}$ m. Because of the hole in the DVD, the lower limit in the arc length integral is not $\theta = 0$. What are the limits of integration?
e. Use the approximation in part (c) to find the length of the DVD groove. Express your answer in centimeters and miles.

Chapter 14 Guided Projects

Applications of the material in this chapter and related topics can be found in the following Guided Projects. For additional information, see the Preface.

- Designing a trajectory
- Bezier curves for graphic design
- Kepler's laws

15

Functions of Several Variables

Chapter Preview The vectors of Chapter 13 and the vector-valued functions of Chapter 14 took us into three-dimensional space for the first time. In this chapter, we step into three-dimensional space along a different path by considering functions with several independent variables and one dependent variable. All the familiar properties of single-variable functions—domains, graphs, limits, continuity, and derivatives—have generalizations for multivariable functions, although you will also see subtle differences and new features. With functions of several independent variables, we work with *partial derivatives*, which, in turn, give rise to directional derivatives and the *gradient*, a fundamental concept in calculus. Partial derivatives allow us to find maximum and minimum values of multivariable functions. We define tangent planes, rather than tangent lines, that enable us to make linear approximations. The chapter ends with a survey of optimization problems in several variables.

15.1 Graphs and Level Curves

In Chapter 14, we discussed vector-valued functions with one independent variable and several dependent variables. We now reverse the situation and consider functions with several independent variables and one dependent variable. Such functions are aptly called *functions of several variables* or *multivariable functions*.

To set the stage, consider the following practical questions that illustrate a few of the many applications of functions of several variables.

- What is the probability that one man selected randomly from a large group of men weighs more than 200 pounds and is over 6 feet tall? (The answer depends on two variables, weight and height.)
- Where on the wing of an airliner flying at a speed of 550 mi/hr is the pressure greatest? (Pressure depends on the x-, y-, and z-coordinates of various points on the wing.)
- A physician knows the optimal blood concentration of an antibiotic needed by a patient. What dose of antibiotic is needed and how often should it be given to reach this optimal level? (The concentration depends (at least) on the amount of the dose, the frequency with which it is administered, and the weight of the patient.)

Although we don't answer these questions immediately, they clearly suggest the scope and importance of the topic. First, we must introduce the idea of a function of several variables.

Functions of Two Variables

The key concepts related to functions of several variables are most easily presented in the case of two independent variables; the extension to three or more variables is then straightforward. In general, functions of two variables are written *explicitly* in the form

$$z = f(x, y)$$

or *implicitly* in the form

$$F(x, y, z) = 0.$$

Both forms are important, but for now, we consider explicitly defined functions.

The concepts of domain and range carry over directly from functions of a single variable.

> **DEFINITION Function, Domain, and Range with Two Independent Variables**
>
> A **function** $z = f(x, y)$ assigns to each point (x, y) in a set D in $\mathbb{R}^2$ a unique real number z in a subset of $\mathbb{R}$. The set D is the **domain** of f. The **range** of f is the set of real numbers z that are assumed as the points (x, y) vary over the domain (Figure 15.1).

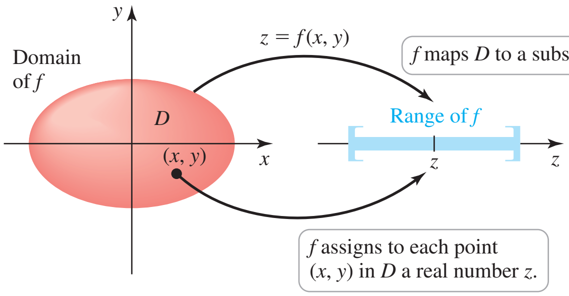


Figure 15.1

As with functions of one variable, a function of several variables may have a domain that is restricted by the context of the problem. For example, if the independent variables correspond to price or length or population, they take only nonnegative values, even though the associated function may be defined for negative values of the variables. If not stated otherwise, D is the set of all points for which the function is defined.

A polynomial in x and y consists of sums and products of polynomials in x and polynomials in y; for example, $f(x, y) = x^2y - 2xy - xy^2$. Such polynomials are defined for all values of x and y, so their domain is $\mathbb{R}^2$. A quotient of two polynomials in x and y, such as $h(x, y) = \dfrac{xy}{x - y}$, is a rational function in x and y. The domain of a rational function excludes points at which the denominator is zero, so the domain of h is $\{(x, y): x \neq y\}$.

EXAMPLE 1 Finding domains Find the domain of the function $g(x, y) = \sqrt{4 - x^2 - y^2}$.

SOLUTION Because g involves a square root, its domain consists of ordered pairs (x, y) for which $4 - x^2 - y^2 \geq 0$ or $x^2 + y^2 \leq 4$. Therefore, the domain of g is $\{(x, y): x^2 + y^2 \leq 4\}$, which is the set of points on or within the circle of radius 2 centered at the origin in the xy-plane (a *disk* of radius 2) (Figure 15.2).

Related Exercises 17–18 ◄

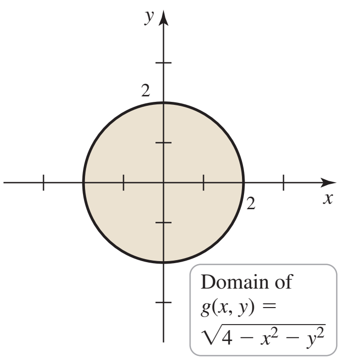


Figure 15.2

QUICK CHECK 1 Find the domains of $f(x, y) = \sin xy$ and $g(x, y) = \sqrt{(x^2 + 1)y}$. ◄

Graphs of Functions of Two Variables

The **graph** of a function f of two variables is the set of points (x, y, z) that satisfy the equation $z = f(x, y)$. More specifically, for each point (x, y) in the domain of f, the point $(x, y, f(x, y))$ lies on the graph of f (Figure 15.3). A similar definition applies to relations of the form $F(x, y, z) = 0$.

QUICK CHECK 2 Does the graph of a hyperboloid of one sheet represent a function? Does the graph of a cone with its axis parallel to the x-axis represent a function? ◄

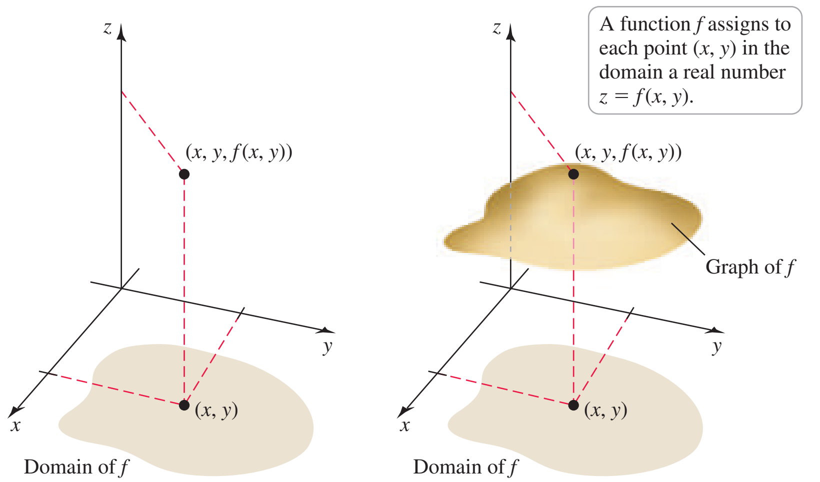


Figure 15.3

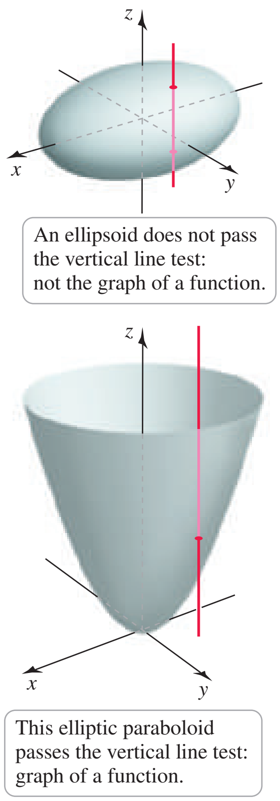


Figure 15.4

Like functions of one variable, functions of two variables must pass a **vertical line test**. A relation of the form $F(x, y, z) = 0$ is a function provided every line parallel to the z-axis intersects the graph of the relation at most once. For example, an ellipsoid (discussed in Section 13.6) is not the graph of a function because some vertical lines intersect the surface twice. On the other hand, an elliptic paraboloid of the form $z = ax^2 + by^2$ does represent a function (Figure 15.4).

EXAMPLE 2 **Graphing two-variable functions** Find the domain and range of the following functions. Then sketch a graph.

a. $f(x, y) = 2x + 3y - 12$ **b.** $g(x, y) = x^2 + y^2$ **c.** $h(x, y) = \sqrt{1 + x^2 + y^2}$

SOLUTION

a. Letting $z = f(x, y)$, we have the equation $z = 2x + 3y - 12$, or $2x + 3y - z = 12$, which describes a plane with a normal vector $\langle 2, 3, -1 \rangle$ (Section 13.5). The domain consists of all points in $\mathbb{R}^2$, and the range is $\mathbb{R}$. We sketch the surface by noting that the x-intercept is $(6, 0, 0)$ (setting $y = z = 0$); the y-intercept is $(0, 4, 0)$ and the z-intercept is $(0, 0, -12)$ (Figure 15.5).

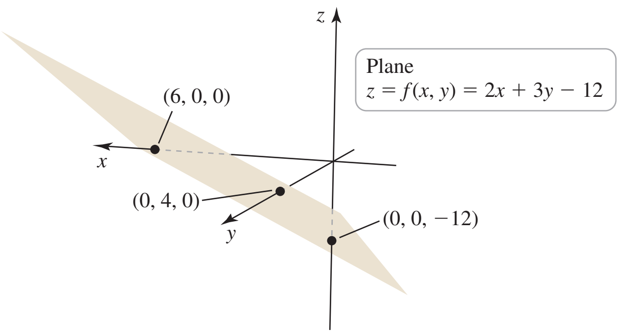


Figure 15.5

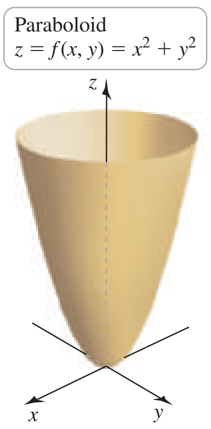


Figure 15.6

b. Letting $z = g(x, y)$, we have the equation $z = x^2 + y^2$, which describes an elliptic paraboloid that opens upward with vertex $(0, 0, 0)$. The domain is $\mathbb{R}^2$ and the range consists of all nonnegative real numbers (Figure 15.6).

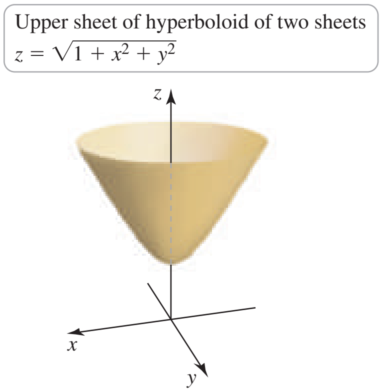


Figure 15.7

c. The domain of the function is $\mathbb{R}^2$ because the quantity under the square root is always positive. Note that $1 + x^2 + y^2 \geq 1$, so the range is $\{z: z \geq 1\}$. Squaring both sides of $z = \sqrt{1 + x^2 + y^2}$, we obtain $z^2 = 1 + x^2 + y^2$, or $-x^2 - y^2 + z^2 = 1$. This is the equation of a hyperboloid of two sheets that opens along the z-axis. Because the range is $\{z: z \geq 1\}$, the given function represents only the upper sheet of the hyperboloid (**Figure 15.7**; the lower sheet was introduced when we squared the original equation).

Related Exercises 25, 27, 29 ◄

QUICK CHECK 3 Find a function whose graph is the lower half of the hyperboloid $-x^2 - y^2 + z^2 = 1$. ◄

➤ To anticipate results that appear later in the chapter, notice how the streams in the topographic map—which flow downhill—cross the level curves roughly at right angles.

Level Curves

Functions of two variables are represented by surfaces in $\mathbb{R}^3$. However, such functions can be represented in another illuminating way, which is used to make topographic maps (**Figure 15.8**).

Figure 15.8

➤ A contour curve is a trace (Section 13.6) in the plane $z = z_0$.

➤ A level curve may not always be a single curve. It might consist of a point ($x^2 + y^2 = 0$) or it might consist of several lines or curves ($xy = 0$).

Consider a surface defined by the function $z = f(x, y)$ (**Figure 15.9**). Now imagine stepping onto the surface and walking along a path on which your elevation has the constant value $z = z_0$. The path you walk on the surface is part of a **contour curve**; the complete contour curve is the intersection of the surface and the horizontal plane $z = z_0$. When the contour curve is projected onto the xy-plane, the result is the curve $f(x, y) = z_0$. This curve in the xy-plane is called a **level curve**.

Imagine repeating this process with a different constant value of z, say, $z = z_1$. The path you walk this time, when projected onto the xy-plane, is part of another level curve $f(x, y) = z_1$. A collection of such level curves, corresponding to different values of z, provides a useful two-dimensional representation of the surface (**Figure 15.10**).

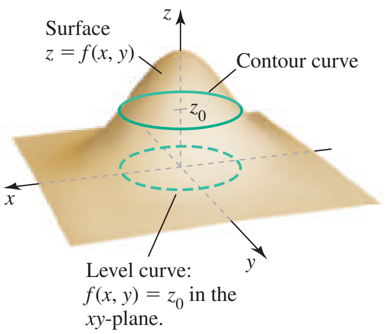


Figure 15.9

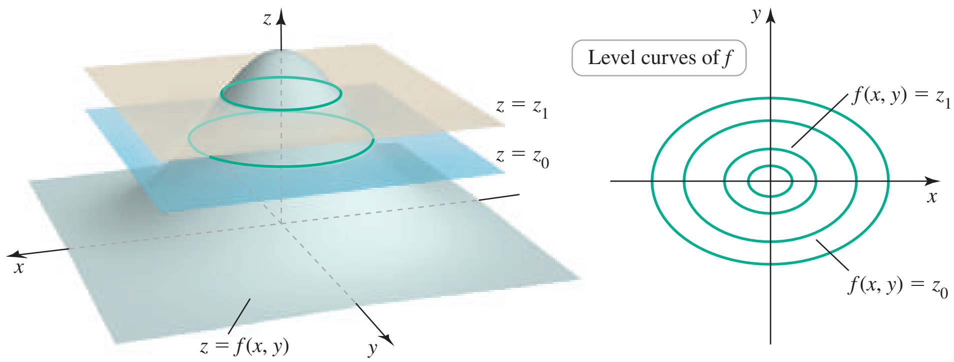


Figure 15.10

QUICK CHECK 4 Can two level curves of a function intersect? Explain. ◄

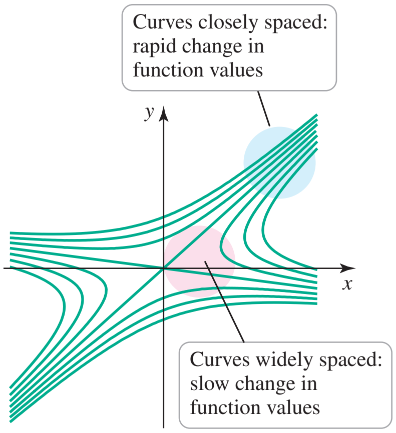


Figure 15.11

QUICK CHECK 5 Describe in words the level curves of the top half of the sphere $x^2 + y^2 + z^2 = 1$. ◄

Assuming two adjacent level curves always correspond to the same change in z, widely spaced level curves indicate gradual changes in z-values, while closely spaced level curves indicate rapid changes in some directions (Figure 15.11). Concentric closed level curves generally indicate either a peak or a depression on the surface.

EXAMPLE 3 **Level curves** Find and sketch the level curves of the following surfaces.

a. $f(x, y) = y - x^2 - 1$ **b.** $f(x, y) = e^{-x^2-y^2}$

SOLUTION

a. The level curves are described by the equation $y - x^2 - 1 = z_0$, where z_0 is a constant in the range of f. For all values of z_0, these curves are parabolas in the xy-plane, as seen by writing the equation in the form $y = x^2 + z_0 + 1$. For example:

- With $z_0 = 0$, the level curve is the parabola $y = x^2 + 1$; along this curve, the surface has an elevation (z-coordinate) of 0.
- With $z_0 = -1$, the level curve is $y = x^2$; along this curve, the surface has an elevation of -1.
- With $z_0 = 1$, the level curve is $y = x^2 + 2$, along which the surface has an elevation of 1.

As shown in Figure 15.12a, the level curves form a family of shifted parabolas. When these level curves are labeled with their z-coordinates, the graph of the surface $z = f(x, y)$ can be visualized (Figure 15.12b).

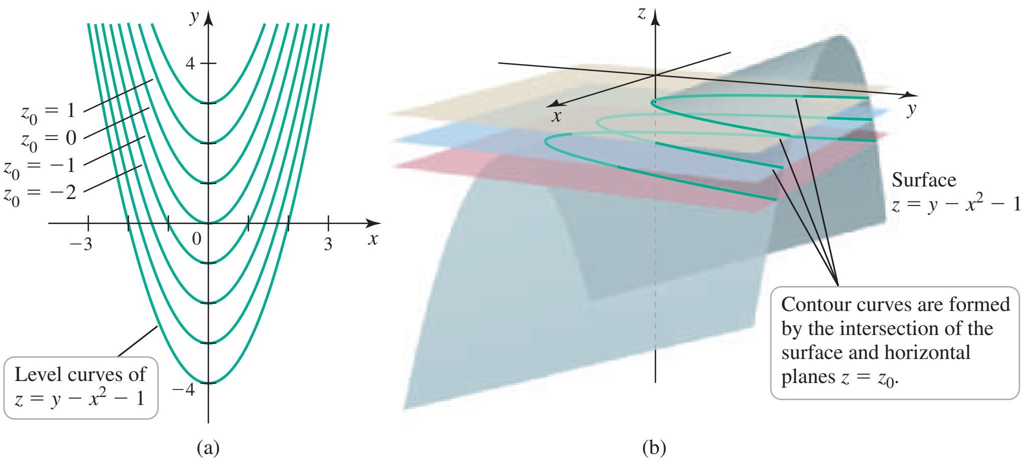


Figure 15.12

b. The level curves satisfy the equation $e^{-x^2-y^2} = z_0$, where z_0 is a positive constant. Taking the natural logarithm of both sides gives the equation $x^2 + y^2 = -\ln z_0$, which describes circular level curves. These curves can be sketched for all values of z_0 with $0 < z_0 \leq 1$ (because the right side of $x^2 + y^2 = -\ln z_0$ must be nonnegative). For example:

- With $z_0 = 1$, the level curve satisfies the equation $x^2 + y^2 = 0$, whose solution is the single point $(0, 0)$; at this point, the surface has an elevation of 1.
- With $z_0 = e^{-1}$, the level curve is $x^2 + y^2 = -\ln e^{-1} = 1$, which is a circle centered at $(0, 0)$ with a radius of 1; along this curve the surface has an elevation of $e^{-1} \approx 0.37$.

In general, the level curves are circles centered at $(0, 0)$; as the radii of the circles increase, the corresponding z-values decrease. Figure 15.13a shows the level curves, with larger z-values corresponding to darker shades. From these labeled level curves, we can reconstruct the graph of the surface (Figure 15.13b).

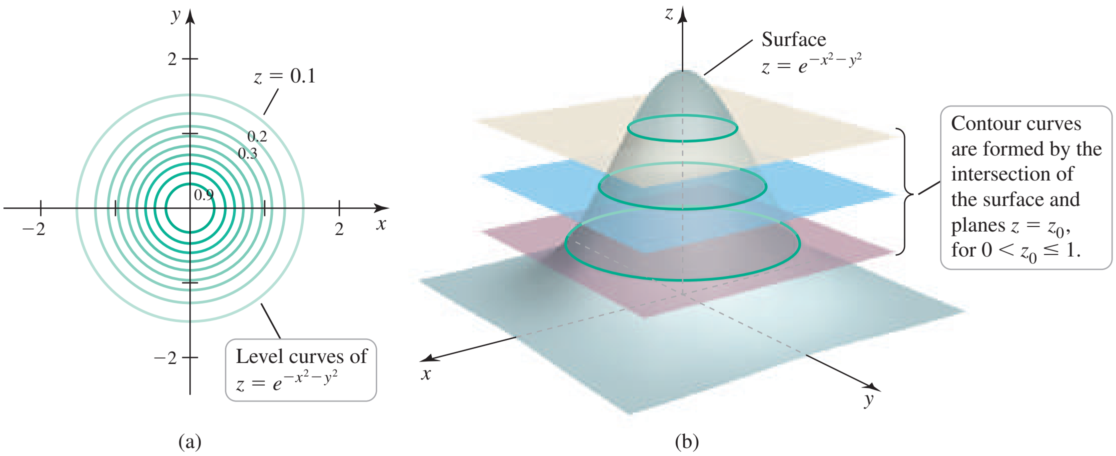


Figure 15.13

Related Exercises 37, 40 ◄

QUICK CHECK 6 Does the surface in Example 3b have a level curve for $z_0 = 0$? Explain. ◄

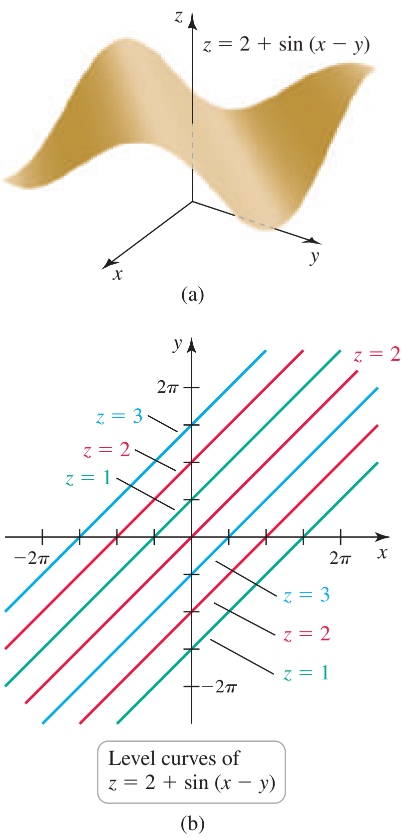


Figure 15.14

EXAMPLE 4 **Level curves** The graph of the function

$$f(x, y) = 2 + \sin(x - y)$$

is shown in **Figure 15.14a**. Sketch several level curves of the function.

SOLUTION The level curves are $f(x, y) = 2 + \sin(x - y) = z_0$, or $\sin(x - y) = z_0 - 2$. Because $-1 \le \sin(x - y) \le 1$, the admissible values of z_0 satisfy $-1 \le z_0 - 2 \le 1$, or, equivalently, $1 \le z_0 \le 3$. For example, when $z_0 = 2$, the level curves satisfy $\sin(x - y) = 0$. The solutions of this equation are $x - y = k\pi$, or $y = x - k\pi$, where k is an integer. Therefore, the surface has an elevation of 2 on this set of lines. With $z_0 = 1$ (the minimum value of z), the level curves satisfy $\sin(x - y) = -1$. The solutions are $x - y = -\pi/2 + 2k\pi$, where k is an integer; along these lines, the surface has an elevation of 1. Here we have an example in which each level curve is an infinite collection of lines of slope 1 (**Figure 15.14b**).

Related Exercise 43 ◄

Applications of Functions of Two Variables

The following examples offer two of many applications of functions of two variables.

EXAMPLE 5 **A probability function of two variables** Suppose on a particular day, the fraction of students on campus infected with flu is r, where $0 \le r \le 1$. If you have n random (possibly repeated) encounters with students during the day, the probability of meeting *at least* one infected person is $p(n, r) = 1 - (1 - r)^n$ (**Figure 15.15a**). Discuss this probability function.

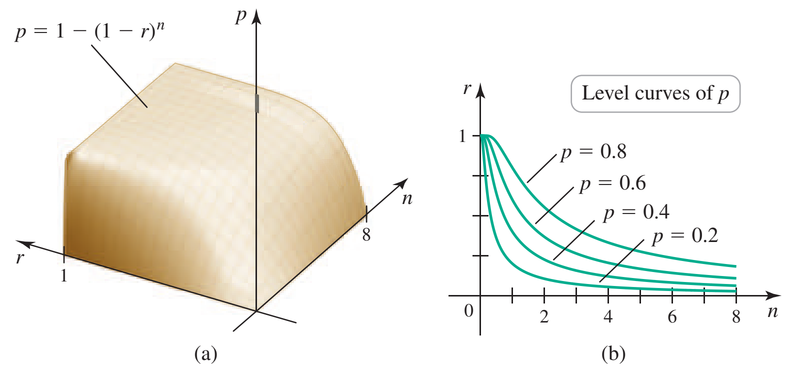


Figure 15.15

SOLUTION The independent variable r is restricted to the interval $[0, 1]$ because it is a fraction of the population. The other independent variable n is any nonnegative integer; for the purposes of graphing, we treat n as a real number in the interval $[0, 8]$. With $0 \le r \le 1$, note that $0 \le 1 - r \le 1$. If n is nonnegative, then $0 \le (1 - r)^n \le 1$, and

QUICK CHECK 7 In Example 5, if 50% of the population is infected, what is the probability of meeting at least one infected person in five encounters? ◂

Table 15.1

	n → 2	5	10	15	20
r = **0.05**	0.10	0.23	0.40	0.54	0.64
0.1	0.19	0.41	0.65	0.79	0.88
0.3	0.51	0.83	0.97	1	1
0.5	0.75	0.97	1	1	1
0.7	0.91	1	1	1	1

it follows that $0 \le p(n, r) \le 1$. Therefore, the range of the function is $[0, 1]$, which is consistent with the fact that p is a probability.

The level curves (**Figure 15.15b**) show that for a fixed value of n, the probability of at least one encounter increases with r; and for a fixed value of r, the probability increases with n. Therefore, as r increases or as n increases, the probability approaches 1 (at a surprising rate). If 10% of the population is infected $(r = 0.1)$ and you have $n = 10$ encounters, then the probability of at least one encounter with an infected person is $p(10, 0.1) \approx 0.651$, which is about 2 in 3.

A numerical view of this function is given in Table 15.1, where we see probabilities tabulated for various values of n and r (rounded to two digits). The numerical values confirm the preceding observations.

Related Exercise 44 ◂

EXAMPLE 6 Electric potential function in two variables The electric field at points in the xy-plane due to two point charges located at $(0, 0)$ and $(1, 0)$ is related to the electric potential function

$$\varphi(x, y) = \frac{2}{\sqrt{x^2 + y^2}} + \frac{2}{\sqrt{(x - 1)^2 + y^2}}.$$

Discuss the electric potential function.

➤ The electric potential function, often denoted φ (pronounced *fee* or *fie*), is a scalar-valued function from which the electric field can be computed. Potential functions are discussed in detail in Chapter 17.

SOLUTION The domain of the function contains all points of $\mathbb{R}^2$ except $(0, 0)$ and $(1, 0)$ where the charges are located. As these points are approached, the potential function becomes arbitrarily large (**Figure 15.16a**). The potential approaches zero as x or y increases in magnitude. These observations imply that the range of the potential function is all positive real numbers. The level curves of φ are closed curves, encircling either a single charge (at small distances) or both charges (at larger distances; **Figure 15.16b**).

➤ A function that grows without bound near a point, as in the case of the electric potential function, is said to have a *singularity* at that point. A singularity is analogous to a vertical asymptote in a function of one variable.

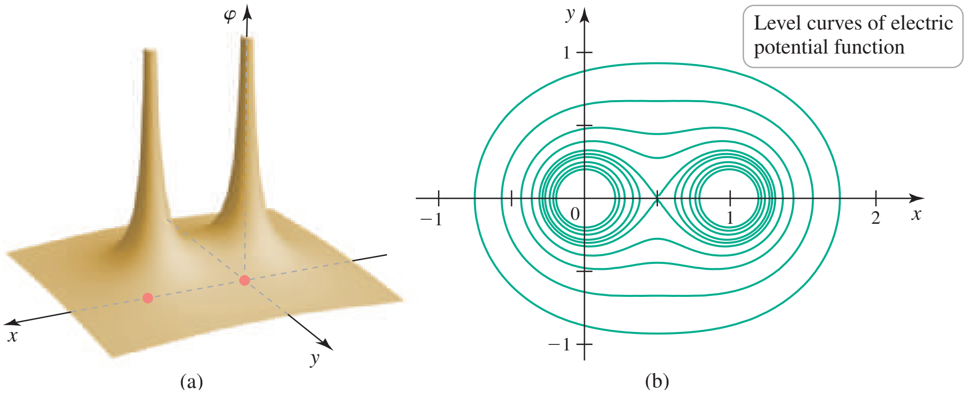


Figure 15.16

Related Exercise 45 ◂

QUICK CHECK 8 In Example 6, what is the electric potential at the point $\left(\frac{1}{2}, 0\right)$? ◂

Functions of More Than Two Variables

Many properties of functions of two independent variables extend naturally to functions of three or more variables. A function of three variables is defined explicitly in the form $w = f(x, y, z)$ and implicitly in the form $F(x, y, z, w) = 0$. With more than three independent variables, the variables are usually written $x_1, \ldots, x_n$. Table 15.2 shows the progression of functions of several variables.

Table 15.2

Number of Independent Variables	Explicit Form	Implicit Form	Graph Resides In . . .
1	$y = f(x)$	$F(x, y) = 0$	$\mathbb{R}^2$ (xy-plane)
2	$z = f(x, y)$	$F(x, y, z) = 0$	$\mathbb{R}^3$ (xyz-space)
3	$w = f(x, y, z)$	$F(x, y, z, w) = 0$	$\mathbb{R}^4$
n	$x_{n+1} = f(x_1, x_2, \ldots, x_n)$	$F(x_1, x_2, \ldots, x_n, x_{n+1}) = 0$	$\mathbb{R}^{n+1}$

The concepts of domain and range extend from the one- and two-variable cases in an obvious way.

DEFINITION Function, Domain, and Range with n Independent Variables

The **function** $x_{n+1} = f(x_1, x_2, \ldots, x_n)$ assigns a unique real number x_{n+1} to each point $(x_1, x_2, \ldots, x_n)$ in a set D in $\mathbb{R}^n$. The set D is the **domain** of f. The **range** is the set of real numbers x_{n+1} that are assumed as the points $(x_1, x_2, \ldots, x_n)$ vary over the domain.

EXAMPLE 7 **Finding domains** Find the domain of the following functions.

a. $g(x, y, z) = \sqrt{16 - x^2 - y^2 - z^2}$ **b.** $h(x, y, z) = \dfrac{12y^2}{z - y}$

SOLUTION

a. Values of the variables that make the argument of a square root negative must be excluded from the domain. In this case, the quantity under the square root is nonnegative provided

$$16 - x^2 - y^2 - z^2 \geq 0, \quad \text{or} \quad x^2 + y^2 + z^2 \leq 16.$$

Therefore, the domain of g is a closed ball in $\mathbb{R}^3$ of radius 4 centered at the origin.

➤ Recall that a closed ball of radius r is the set of all points on or within a sphere of radius r.

b. Values of the variables that make a denominator zero must be excluded from the domain. In this case, the denominator vanishes for all points in $\mathbb{R}^3$ that satisfy $z - y = 0$, or $y = z$. Therefore, the domain of h is the set $\{(x, y, z): y \neq z\}$. This set is $\mathbb{R}^3$ excluding the points on the plane $y = z$.

Related Exercises 51–52 ◄

QUICK CHECK 9 What is the domain of the function $w = f(x, y, z) = 1/xyz$? ◄

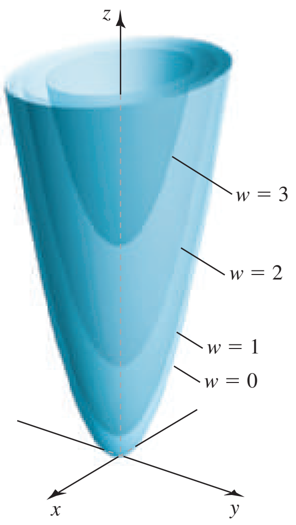


Figure 15.17

Graphs of Functions of More Than Two Variables

Graphing functions of *two* independent variables requires a three-dimensional coordinate system, which is the limit of ordinary graphing methods. Clearly, difficulties arise in graphing functions with three or more independent variables. For example, the graph of the function $w = f(x, y, z)$ resides in four dimensions. Here are two approaches to representing functions of three independent variables.

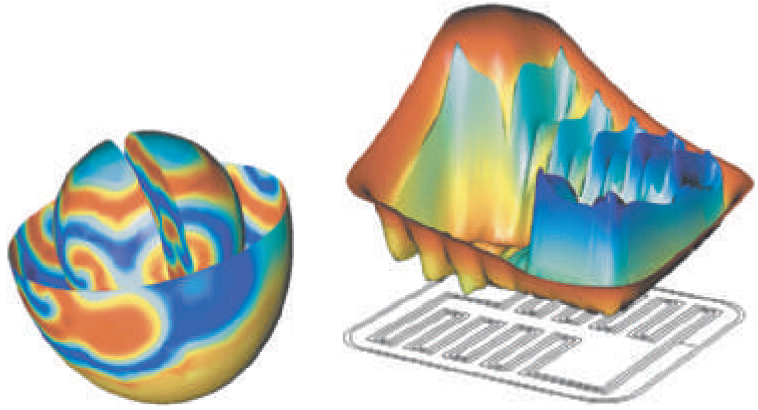
The idea of level curves can be extended. With the function $w = f(x, y, z)$, level curves become **level surfaces**, which are surfaces in $\mathbb{R}^3$ on which w is constant. For example, the level surfaces of the function

$$w = f(x, y, z) = \sqrt{z - x^2 - 2y^2}$$

satisfy $w = \sqrt{z - x^2 - 2y^2} = C$, where C is a nonnegative constant. This equation is satisfied when $z = x^2 + 2y^2 + C^2$. Therefore, the level surfaces are elliptic paraboloids, stacked one inside another (Figure 15.17).

Another approach to displaying functions of three variables is to use colors to represent the fourth dimension. Figure 15.18a shows the electrical activity of the heart at one snapshot in time. The three independent variables correspond to locations in the heart. At each point, the value of the electrical activity, which is the dependent variable, is coded by colors.

In Figure 15.18b, the dependent variable is the switching speed in an integrated circuit, again represented by colors, as it varies over points of the domain. Software to produce such images, once expensive and inefficient, has become much more accessible.



(a) (b)

Figure 15.18

SECTION 15.1 EXERCISES

Getting Started

1. A function is defined by $z = x^2y - xy^2$. Identify the independent and dependent variables.

2. What is the domain of $f(x, y) = x^2y - xy^2$?

3. What is the domain of $g(x, y) = \dfrac{1}{xy}$?

4. What is the domain of $h(x, y) = \sqrt{x - y}$?

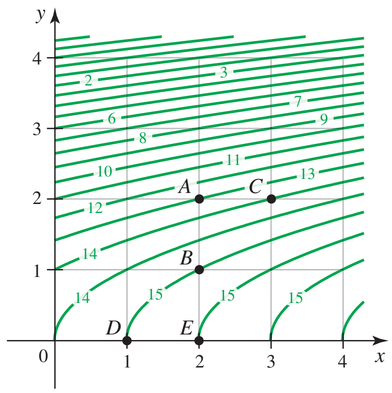
5. How many axes (or how many dimensions) are needed to graph the function $z = f(x, y)$? Explain.

6. Explain how to graph the level curves of a surface $z = f(x, y)$.

7. Given the function $f(x, y) = \sqrt{10 - x + y}$, evaluate $f(2, 1)$ and $f(-9, -3)$.

8. Given the function $g(x, y, z) = \dfrac{x + y}{z}$, evaluate $g(1, 5, 3)$ and $g(3, 7, 2)$.

9–10. *The function $z = f(x, y)$ gives the elevation z (in hundreds of feet) of a hillside above the point (x, y). Use the level curves of f to answer the following questions (see figure).*



9. Katie and Zeke are standing on the surface above the point $A(2, 2)$.
   a. At what elevation are Katie and Zeke standing?
   b. Katie hikes south to the point on the surface above $B(2, 1)$ and Zeke hikes east to the point on the surface above $C(3, 2)$. Who experienced the greater elevation change and what is the difference in their elevations?

10. Katie and Zeke are standing on the surface above $D(1, 0)$. Katie hikes on the surface above the level curve containing $D(1, 0)$ to $B(2, 1)$ and Zeke walks east along the surface to $E(2, 0)$. What can be said about the elevations of Katie and Zeke during their hikes?

11. Describe in words the level curves of the paraboloid $z = x^2 + y^2$.

12. How many axes (or how many dimensions) are needed to graph the level surfaces of $w = f(x, y, z)$? Explain.

13. The domain of $Q = f(u, v, w, x, y, z)$ lies in $\mathbb{R}^n$ for what value of n? Explain.

14. Give two methods for graphically representing a function with three independent variables.

Practice Exercises

15–24. Domains *Find the domain of the following functions.*

15. $f(x, y) = 2xy - 3x + 4y$

16. $f(x, y) = \cos(x^2 - y^2)$

17. $f(x, y) = \sqrt{25 - x^2 - y^2}$

18. $f(x, y) = \dfrac{1}{\sqrt{x^2 + y^2 - 25}}$

19. $f(x, y) = \sin \dfrac{x}{y}$

20. $f(x, y) = \dfrac{12}{y^2 - x^2}$

21. $g(x, y) = \ln(x^2 - y)$

22. $f(x, y) = \sin^{-1}(y - x^2)$

23. $g(x, y) = \sqrt{\dfrac{xy}{x^2 + y^2}}$

24. $h(x, y) = \sqrt{x - 2y + 4}$

25–33. Graphs of familiar functions *Use what you learned about surfaces in Sections 13.5 and 13.6 to sketch a graph of the following functions. In each case, identify the surface and state the domain and range of the function.*

25. $f(x, y) = 6 - x - 2y$

26. $g(x, y) = 4$

27. $p(x, y) = x^2 - y^2$

28. $h(x, y) = 2x^2 + 3y^2$

29. $G(x, y) = -\sqrt{1 + x^2 + y^2}$

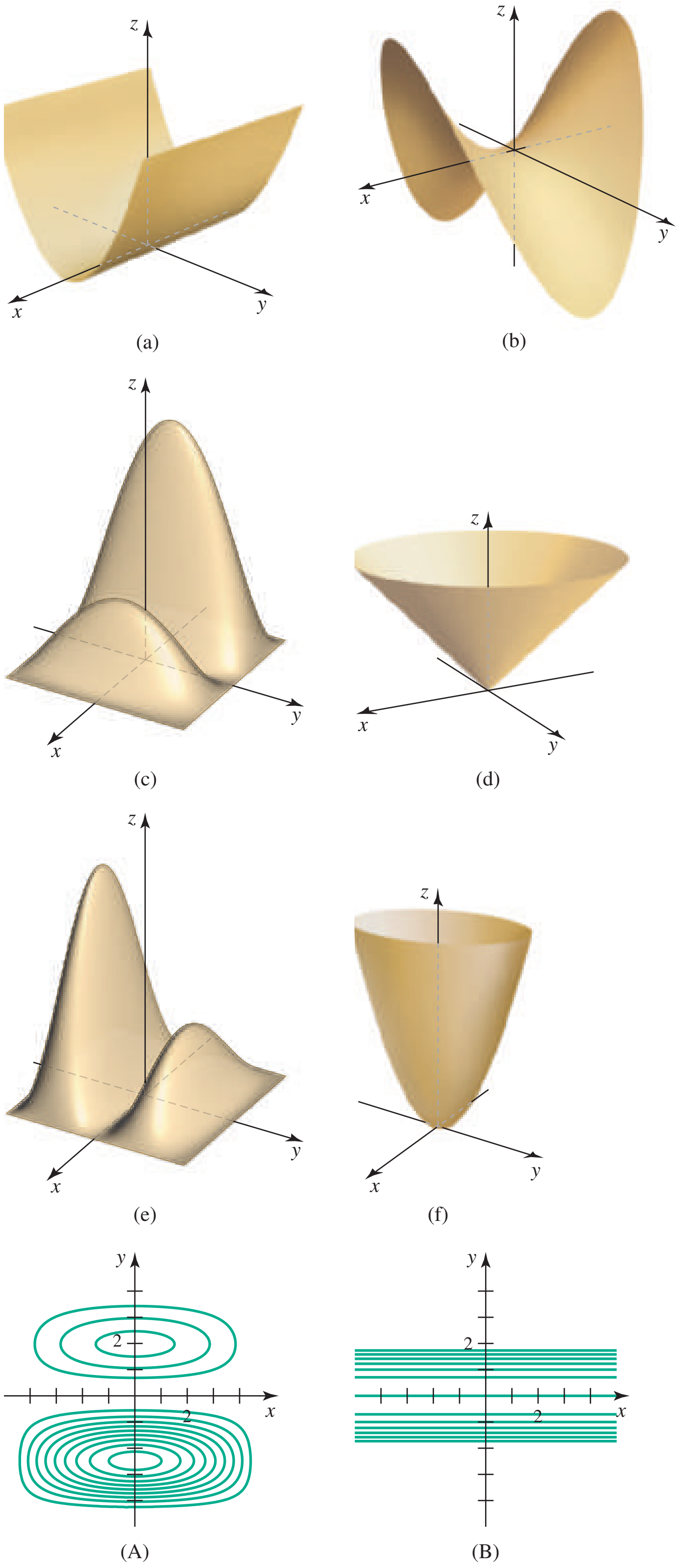
30. $F(x, y) = \sqrt{1 - x^2 - y^2}$

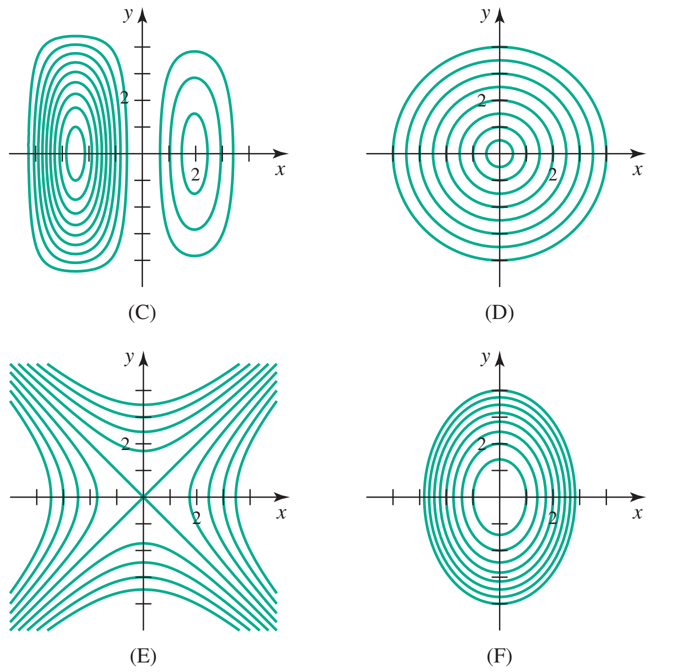
31. $P(x, y) = \sqrt{x^2 + y^2 - 1}$

32. $H(x, y) = \sqrt{x^2 + y^2}$

33. $g(x, y) = \sqrt{16 - 4x^2}$

34. Matching level curves with surfaces Match surfaces a–f in the figure with level curves A–F.

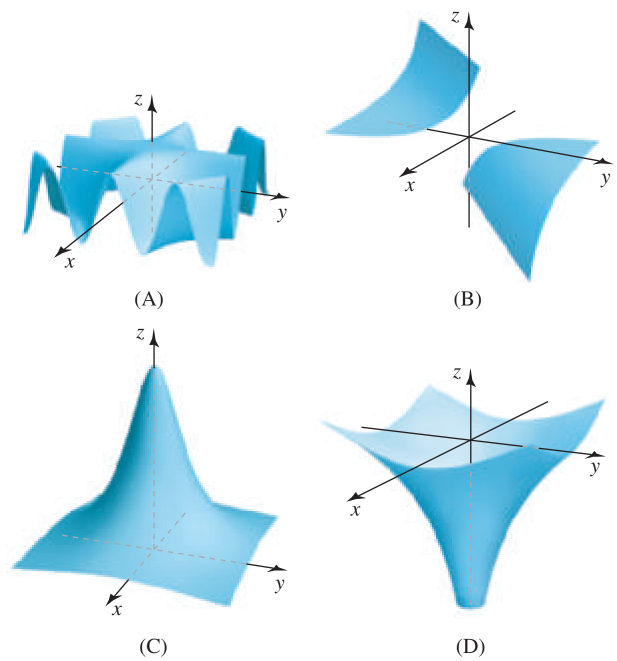




35. Matching surfaces Match functions a–d with surfaces A–D in the figure.

a. $f(x, y) = \cos xy$ **b.** $g(x, y) = \ln(x^2 + y^2)$

c. $h(x, y) = \dfrac{1}{x - y}$ **d.** $p(x, y) = \dfrac{1}{1 + x^2 + y^2}$



T 36–43. Level curves *Graph several level curves of the following functions using the given window. Label at least two level curves with their z-values.*

36. $z = x^2 + y^2$; $[-4, 4] \times [-4, 4]$

37. $z = x - y^2$; $[0, 4] \times [-2, 2]$

38. $z = 2x - y$; $[-2, 2] \times [-2, 2]$

39. $z = \sqrt{x^2 + 4y^2}$; $[-8, 8] \times [-8, 8]$

40. $z = e^{-x^2-2y^2}$; $[-2, 2] \times [-2, 2]$

41. $z = \sqrt{25 - x^2 - y^2}$; $[-6, 6] \times [-6, 6]$

42. $z = \sqrt{y - x^2 - 1}$; $[-5, 5] \times [-5, 5]$

43. $z = 3\cos(2x + y)$; $[-2, 2] \times [-2, 2]$

44. Earned run average A baseball pitcher's earned run average (ERA) is $A(e, i) = 9e/i$, where e is the number of earned runs given up by the pitcher and i is the number of innings pitched. Good pitchers have low ERAs. Assume $e \geq 0$ and $i > 0$ are real numbers.

a. The single-season major league record for the lowest ERA was set by Dutch Leonard of the Detroit Tigers in 1914. During that season, Dutch pitched a total of 224 innings and gave up just 24 earned runs. What was his ERA?

b. Determine the ERA of a relief pitcher who gives up 4 earned runs in one-third of an inning.

c. Graph the level curve $A(e, i) = 3$ and describe the relationship between e and i in this case.

T 45. Electric potential function The electric potential function for two positive charges, one at $(0, 1)$ with twice the strength of the charge at $(0, -1)$, is given by

$$\varphi(x, y) = \frac{2}{\sqrt{x^2 + (y - 1)^2}} + \frac{1}{\sqrt{x^2 + (y + 1)^2}}.$$

a. Graph the electric potential using the window $[-5, 5] \times [-5, 5] \times [0, 10]$.

b. For what values of x and y is the potential φ defined?

c. Is the electric potential greater at $(3, 2)$ or $(2, 3)$?

d. Describe how the electric potential varies along the line $y = x$.

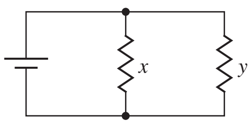
T 46. Cobb-Douglas production function The output Q of an economic system subject to two inputs, such as labor L and capital K, is often modeled by the Cobb-Douglas production function $Q(L, K) = cL^aK^b$, where a, b, and c are positive real numbers. When $a + b = 1$, the case is called *constant returns to scale*. Suppose $a = 1/3$, $b = 2/3$, and $c = 40$.

a. Graph the output function using the window $[0, 20] \times [0, 20] \times [0, 500]$.

b. If L is held constant at $L = 10$, write the function that gives the dependence of Q on K.

c. If K is held constant at $K = 15$, write the function that gives the dependence of Q on L.

T 47. Resistors in parallel Two resistors wired in parallel in an electrical circuit give an effective resistance of $R(x, y) = \dfrac{xy}{x + y}$, where x and y are the positive resistances of the individual resistors (typically measured in ohms).



a. Graph the resistance function using the window $[0, 10] \times [0, 10] \times [0, 5]$.

b. Estimate the maximum value of R, for $0 < x \leq 10$ and $0 < y \leq 10$.

c. Explain what it means to say that the resistance function is symmetric in x and y.

T 48. Level curves of a savings account Suppose you make a one-time deposit of P dollars into a savings account that earns interest at an annual rate of $p\%$ compounded continuously. The balance in the account after t years is $B(P, r, t) = Pe^{rt}$, where $r = p/100$ (for example, if the annual interest rate is 4%, then $r = 0.04$). Let the interest rate be fixed at $r = 0.04$.

a. With a target balance of \$2000, find the set of all points (P, t) that satisfy $B = 2000$. This curve gives all deposits P and times t that result in a balance of \$2000.

b. Repeat part (a) with $B = \$500$, \$1000, \$1500, and \$2500, and draw the resulting level curves of the balance function.

c. In general, on one level curve, if t increases, does P increase or decrease?

T 49. Level curves of a savings plan Suppose you make monthly deposits of P dollars into an account that earns interest at a *monthly* rate of $p\%$. The balance in the account after t years is $B(P, r, t) = P\left(\dfrac{(1 + r)^{12t} - 1}{r}\right)$, where $r = \dfrac{p}{100}$ (for example, if the annual interest rate is 9%, then $p = \dfrac{9}{12} = 0.75$ and $r = 0.0075$). Let the time of investment be fixed at $t = 20$ years.

a. With a target balance of \$20,000, find the set of all points (P, r) that satisfy $B = 20{,}000$. This curve gives all deposits P and monthly interest rates r that result in a balance of \$20,000 after 20 years.

b. Repeat part (a) with $B = \$5000$, \$10,000, \$15,000, and \$25,000, and draw the resulting level curves of the balance function.

50–56. Domains of functions of three or more variables *Find the domain of the following functions. If possible, give a description of the domains (for example, all points outside a sphere of radius* 1 *centered at the origin).*

50. $f(x, y, z) = 2xyz - 3xz + 4yz$

51. $g(x, y, z) = \dfrac{1}{x - z}$

52. $p(x, y, z) = \sqrt{x^2 + y^2 + z^2 - 9}$

53. $f(x, y, z) = \sqrt{y - z}$

54. $Q(x, y, z) = \dfrac{10}{1 + x^2 + y^2 + 4z^2}$

55. $F(x, y, z) = \sqrt{y - x^2}$

56. $f(w, x, y, z) = \sqrt{1 - w^2 - x^2 - y^2 - z^2}$

57. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. The domain of the function $f(x, y) = 1 - |x - y|$ is $\{(x, y): x \geq y\}$.

b. The domain of the function $Q = g(w, x, y, z)$ is a region in $\mathbb{R}^3$.

c. All level curves of the plane $z = 2x - 3y$ are lines.

58. Quarterback passer ratings One measurement of the quality of a quarterback in the National Football League is known as the *quarterback passer rating*. The rating formula is $R(c, t, i, y) = \dfrac{50 + 20c + 80t - 100i + 100y}{24}$, where $c\%$ of a quarterback's passes were completed, $t\%$ of his passes were thrown for touchdowns, $i\%$ of his passes were intercepted, and an average of y yards were gained per attempted pass.

a. In the 2016/17 NFL playoffs, Atlanta Falcons quarterback Matt Ryan completed 71.43% of his passes, 9.18% of his passes were thrown for touchdowns, none of his passes were intercepted, and he gained an average of 10.35 yards per passing attempt. What was his passer rating in the 2016 playoffs?

b. In the 2016 regular season, New England Patriots quarterback Tom Brady completed 67.36% of his passes, 6.48% of his passes were thrown for touchdowns, 0.46% of his passes were intercepted, and he gained an average of 8.23 yards per passing attempt. What was his passer rating in the 2016 regular season?

c. If c, t, and y remain fixed, what happens to the quarterback passer rating as i increases? Explain your answer with and without mathematics.

(*Source:* www.nfl.com)

T **59. Ideal Gas Law** Many gases can be modeled by the Ideal Gas Law, $PV = nRT$, which relates the temperature (T, measured in kelvins (K)), pressure (P, measured in pascals (Pa)), and volume (V, measured in m^3) of a gas. Assume the quantity of gas in question is $n = 1$ mole (mol). The gas constant has a value of $R = 8.3\ \text{m}^3\,\text{Pa/mol-K}$.

a. Consider T to be the dependent variable, and plot several level curves (called *isotherms*) of the temperature surface in the region $0 \le P \le 100{,}000$ and $0 \le V \le 0.5$.

b. Consider P to be the dependent variable, and plot several level curves (called *isobars*) of the pressure surface in the region $0 \le T \le 900$ and $0 < V \le 0.5$.

c. Consider V to be the dependent variable, and plot several level curves of the volume surface in the region $0 \le T \le 900$ and $0 < P \le 100{,}000$.

Explorations and Challenges

T **60. Water waves** A snapshot of a water wave moving toward shore is described by the function $z = 10 \sin(2x - 3y)$, where z is the height of the water surface above (or below) the xy-plane, which is the level of undisturbed water.

a. Graph the height function using the window $[-5, 5] \times [-5, 5] \times [-15, 15]$.

b. For what values of x and y is z defined?

c. What are the maximum and minimum values of the water height?

d. Give a vector in the xy-plane that is orthogonal to the level curves of the crests and troughs of the wave (which is parallel to the direction of wave propagation).

T **61. Approximate mountains** Suppose the elevation of Earth's surface over a 16-mi by 16-mi region is approximated by the function

$$z = 10e^{-(x^2+y^2)} + 5e^{-((x+5)^2+(y-3)^2)/10} + 4e^{-2((x-4)^2+(y+1)^2)}.$$

a. Graph the height function using the window $[-8, 8] \times [-8, 8] \times [0, 15]$.

b. Approximate the points (x, y) where the peaks in the landscape appear.

c. What are the approximate elevations of the peaks?

T **62–68. Graphing functions**

a. *Determine the domain and range of the following functions.*

b. *Graph each function using a graphing utility. Be sure to experiment with the window and orientation to give the best perspective on the surface.*

62. $g(x, y) = e^{-xy}$ **63.** $f(x, y) = |xy|$

64. $p(x, y) = 1 - |x - 1| + |y + 1|$

65. $h(x, y) = \dfrac{x + y}{x - y}$

66. $G(x, y) = \ln(2 + \sin(x + y))$

67. $F(x, y) = \tan^2(x - y)$ **68.** $P(x, y) = \cos x \sin 2y$

T **69–72. Peaks and valleys** *The following functions have exactly one isolated peak or one isolated depression (one local maximum or minimum). Use a graphing utility to approximate the coordinates of the peak or depression.*

69. $f(x, y) = x^2y^2 - 8x^2 - y^2 + 6$

70. $g(x, y) = (x^2 - x - 2)(y^2 + 2y)$

71. $h(x, y) = 1 - e^{-(x^2+y^2-2x)}$

72. $p(x, y) = 2 + |x - 1| + |y - 1|$

73. Level curves of planes Prove that the level curves of the plane $ax + by + cz = d$ are parallel lines in the xy-plane, provided $a^2 + b^2 \ne 0$ and $c \ne 0$.

74–77. Level surfaces *Find an equation for the family of level surfaces corresponding to f. Describe the level surfaces.*

74. $f(x, y, z) = \dfrac{1}{x^2 + y^2 + z^2}$ **75.** $f(x, y, z) = x^2 + y^2 - z$

76. $f(x, y, z) = x^2 - y^2 - z$ **77.** $f(x, y, z) = \sqrt{x^2 + 2z^2}$

78–81. Challenge domains *Find the domain of the following functions. Specify the domain mathematically, and then describe it in words or with a sketch.*

78. $g(x, y, z) = \dfrac{10}{x^2 - (y + z)x + yz}$

79. $f(x, y) = \sin^{-1}(x - y)^2$

80. $f(x, y, z) = \ln(z - x^2 - y^2 + 2x + 3)$

81. $h(x, y, z) = \sqrt[4]{z^2 - xz + yz - xy}$

82. Other balls The closed unit ball in $\mathbb{R}^3$ centered at the origin is the set $\{(x, y, z): x^2 + y^2 + z^2 \le 1\}$. Describe the following alternative unit balls.

a. $\{(x, y, z): |x| + |y| + |z| \le 1\}$

b. $\{(x, y, z): \max\{|x|, |y|, |z|\} \le 1\}$, where $\max\{a, b, c\}$ is the maximum value of a, b, and c

QUICK CHECK ANSWERS

1. $\mathbb{R}^2$; $\{(x, y): y \ge 0\}$ **2.** No; no **3.** $z = -\sqrt{1 + x^2 + y^2}$ **4.** No; otherwise the function would have two values at a single point. **5.** Concentric circles **6.** No; $z = 0$ is not in the range of the function. **7.** 0.97 **8.** 8 **9.** $\{(x, y, z): x \ne 0 \text{ and } y \ne 0 \text{ and } z \ne 0\}$ (which is $\mathbb{R}^3$, excluding the coordinate planes) ◄

15.2 Limits and Continuity

You have now seen examples of functions of several variables, but calculus has not yet entered the picture. In this section, we revisit topics encountered in single-variable calculus and see how they apply to functions of several variables. We begin with the fundamental concepts of limits and continuity.

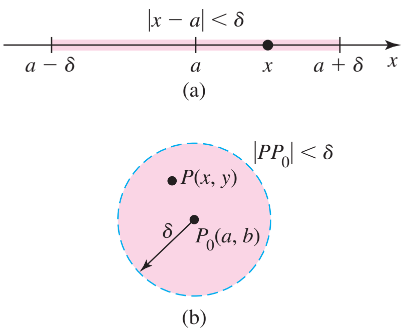


Figure 15.19

Limit of a Function of Two Variables

A function f of two variables has a limit L as $P(x, y)$ approaches a fixed point $P_0(a, b)$ if $|f(x, y) - L|$ can be made arbitrarily small for all P in the domain that are sufficiently close to P_0. If such a limit exists, we write

$$\lim_{(x, y) \to (a, b)} f(x, y) = \lim_{P \to P_0} f(x, y) = L.$$

To make this definition more precise, *close to* must be defined carefully.

A point x on the number line is close to another point a provided the distance $|x - a|$ is small (**Figure 15.19a**). In $\mathbb{R}^2$, a point $P(x, y)$ is close to another point $P_0(a, b)$ if the distance between them $|PP_0| = \sqrt{(x - a)^2 + (y - b)^2}$ is small (**Figure 15.19b**). When we say *for all P close to* P_0, it means that $|PP_0|$ is small for points P *on all sides of* P_0.

➤ The formal definition extends naturally to any number of variables. With n variables, the limit point is $P_0(a_1, \ldots, a_n)$, the variable point is $P(x_1, \ldots, x_n)$, and $|PP_0| = \sqrt{(x_1 - a_1)^2 + \cdots + (x_n - a_n)^2}$.

With this understanding of closeness, we can give a formal definition of a limit with two independent variables. This definition parallels the formal definition of a limit given in Section 2.7 (**Figure 15.20**).

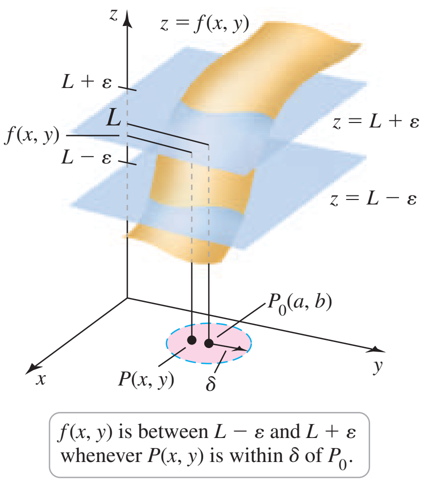


Figure 15.20

> **DEFINITION Limit of a Function of Two Variables**
>
> The function f has the **limit** L as $P(x, y)$ approaches $P_0(a, b)$, written
>
> $$\lim_{(x, y) \to (a, b)} f(x, y) = \lim_{P \to P_0} f(x, y) = L,$$
>
> if, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that
>
> $$|f(x, y) - L| < \varepsilon$$
>
> whenever (x, y) is in the domain of f and
>
> $$0 < |PP_0| = \sqrt{(x - a)^2 + (y - b)^2} < \delta.$$

The condition $|PP_0| < \delta$ means that the distance between $P(x, y)$ and $P_0(a, b)$ is less than δ as P approaches P_0 from all possible directions (**Figure 15.21**). Therefore, the limit exists only if $f(x, y)$ approaches L as P approaches P_0 *along all possible paths* in the domain of f. As shown in upcoming examples, this interpretation is critical in determining whether a limit exists.

As with functions of one variable, we first establish limits of the simplest functions.

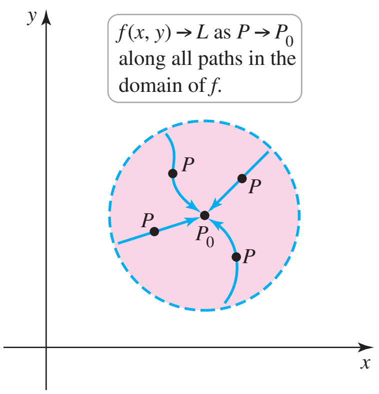


Figure 15.21

> **THEOREM 15.1 Limits of Constant and Linear Functions**
>
> Let a, b, and c be real numbers.
>
> **1.** Constant function $f(x, y) = c$: $\displaystyle\lim_{(x, y) \to (a, b)} c = c$
>
> **2.** Linear function $f(x, y) = x$: $\displaystyle\lim_{(x, y) \to (a, b)} x = a$
>
> **3.** Linear function $f(x, y) = y$: $\displaystyle\lim_{(x, y) \to (a, b)} y = b$

Proof:

1. Consider the constant function $f(x, y) = c$ and assume $\varepsilon > 0$ is given. To prove that the value of the limit is $L = c$, we must produce a $\delta > 0$ such that $|f(x, y) - L| < \varepsilon$

whenever $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$. For constant functions, we may use *any* $\delta > 0$. Then, for every (x, y) in the domain of f,

$$|f(x, y) - L| = |f(x, y) - c| = |c - c| = 0 < \varepsilon$$

whenever $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$.

2. Assume $\varepsilon > 0$ is given and take $\delta = \varepsilon$. The condition $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ implies that

$$\begin{aligned} 0 < \sqrt{(x-a)^2 + (y-b)^2} &< \varepsilon && \delta = \varepsilon \\ \sqrt{(x-a)^2} &< \varepsilon && (x-a)^2 \le (x-a)^2 + (y-b)^2 \\ |x - a| &< \varepsilon. && \sqrt{x^2} = |x| \text{ for real numbers } x \end{aligned}$$

Because $f(x, y) = x$ and $L = a$, we have shown that $|f(x, y) - L| < \varepsilon$ whenever $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$. Therefore, $\lim\limits_{(x, y)\to(a, b)} f(x, y) = L$, or $\lim\limits_{(x, y)\to(a, b)} x = a$. The proof that $\lim\limits_{(x, y)\to(a, b)} y = b$ is similar (Exercise 86).

Using the three basic limits in Theorem 15.1, we can compute limits of more complicated functions. The only tools needed are limit laws analogous to those given in Theorem 2.3. The proofs of these laws are examined in Exercises 88–89.

THEOREM 15.2 Limit Laws for Functions of Two Variables

Let L and M be real numbers and suppose $\lim\limits_{(x, y)\to(a, b)} f(x, y) = L$ and $\lim\limits_{(x, y)\to(a, b)} g(x, y) = M$. Assume c is a constant, and $n > 0$ is an integer.

1. **Sum** $\lim\limits_{(x, y)\to(a, b)} (f(x, y) + g(x, y)) = L + M$
2. **Difference** $\lim\limits_{(x, y)\to(a, b)} (f(x, y) - g(x, y)) = L - M$
3. **Constant multiple** $\lim\limits_{(x, y)\to(a, b)} cf(x, y) = cL$
4. **Product** $\lim\limits_{(x, y)\to(a, b)} f(x, y)g(x, y) = LM$
5. **Quotient** $\lim\limits_{(x, y)\to(a, b)} \dfrac{f(x, y)}{g(x, y)} = \dfrac{L}{M}$, provided $M \neq 0$
6. **Power** $\lim\limits_{(x, y)\to(a, b)} (f(x, y))^n = L^n$
7. **Root** $\lim\limits_{(x, y)\to(a, b)} (f(x, y))^{1/n} = L^{1/n}$, where we assume $L > 0$ if n is even.

➤ Recall that a polynomial in two variables consists of sums and products of polynomials in x and polynomials in y. A rational function is the quotient of two polynomials.

Combining Theorems 15.1 and 15.2 allows us to find limits of polynomial, rational, and algebraic functions in two variables.

EXAMPLE 1 Limits of two-variable functions Evaluate $\lim\limits_{(x, y)\to(2, 8)} (3x^2y + \sqrt{xy})$.

SOLUTION All the operations in this function appear in Theorem 15.2. Therefore, we can apply the limit laws directly.

$$\begin{aligned} \lim_{(x, y)\to(2, 8)} (3x^2y + \sqrt{xy}) &= \lim_{(x, y)\to(2, 8)} 3x^2y + \lim_{(x, y)\to(2, 8)} \sqrt{xy} && \text{Law 1} \\ &= 3 \lim_{(x, y)\to(2, 8)} x^2 \cdot \lim_{(x, y)\to(2, 8)} y \\ &\quad + \sqrt{\lim_{(x, y)\to(2, 8)} x \cdot \lim_{(x, y)\to(2, 8)} y} && \text{Laws 3, 4, 7} \\ &= 3 \cdot 2^2 \cdot 8 + \sqrt{2 \cdot 8} = 100 && \text{Law 6 and Theorem 15.1} \end{aligned}$$

Related Exercise 16 ◄

QUICK CHECK 1 Which of the following limits exist?

a. $\lim_{(x, y)\to(1, 1)} 3x^{12}y^2$

b. $\lim_{(x, y)\to(0, 0)} 3x^{-2}y^2$

c. $\lim_{(x, y)\to(1, 2)} \sqrt{x - y^2}$ ◄

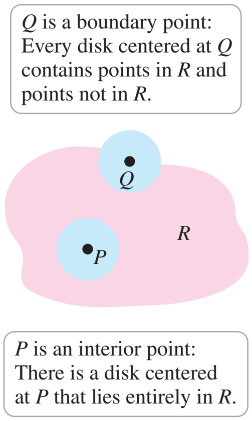


Figure 15.22

➤ The definitions of *interior point* and *boundary point* apply to regions in $\mathbb{R}^3$ if we replace *disk* by *ball*.

➤ Many sets, such as the annulus $\{(x, y): 2 \le x^2 + y^2 < 5\}$, are neither open nor closed.

QUICK CHECK 2 Give an example of a set that contains none of its boundary points. ◄

In Example 1, the value of the limit equals the value of the function at (a, b); in other words, $\lim_{(x, y)\to(a, b)} f(x, y) = f(a, b)$ and the limit can be evaluated by substitution. This is a property of *continuous* functions, discussed later in this section.

Limits at Boundary Points

This is an appropriate place to make some definitions that are used in the remainder of the text.

> **DEFINITION Interior and Boundary Points**
>
> Let R be a region in $\mathbb{R}^2$. An **interior point** P of R lies entirely within R, which means it is possible to find a disk centered at P that contains only points of R (Figure 15.22).
>
> A **boundary point** Q of R lies on the edge of R in the sense that *every* disk centered at Q contains at least one point in R and at least one point not in R.

For example, let R be the points in $\mathbb{R}^2$ satisfying $x^2 + y^2 < 9$. The boundary points of R lie on the circle $x^2 + y^2 = 9$. The interior points lie inside that circle and satisfy $x^2 + y^2 < 9$. Notice that the boundary points of a set need not lie in the set.

> **DEFINITION Open and Closed Sets**
>
> A region is **open** if it consists entirely of interior points. A region is **closed** if it contains all its boundary points.

An example of an open region in $\mathbb{R}^2$ is the open disk $\{(x, y): x^2 + y^2 < 9\}$. An example of a closed region in $\mathbb{R}^2$ is the square $\{(x, y): |x| \le 1, |y| \le 1\}$. Later in the text, we encounter interior and boundary points of three-dimensional sets such as balls, boxes, and pyramids.

Suppose $P_0(a, b)$ is a boundary point of the domain of f. The limit $\lim_{(x, y)\to(a, b)} f(x, y)$ exists, even if P_0 is not in the domain of f, provided $f(x, y)$ approaches the same value as (x, y) approaches (a, b) *along all paths that lie in the domain* (Figure 15.23).

Consider the function $f(x, y) = \dfrac{x^2 - y^2}{x - y}$ whose domain is $\{(x, y): x \ne y\}$. Provided $x \ne y$, we may cancel the factor $(x - y)$ from the numerator and denominator and write

$$f(x, y) = \frac{x^2 - y^2}{x - y} = \frac{(x - y)(x + y)}{x - y} = x + y.$$

The graph of f (Figure 15.24) is the plane $z = x + y$, with points corresponding to the line $x = y$ removed.

➤ Recall that this same method was used with functions of one variable. For example, after the common factor $x - 2$ is canceled, the function

$$g(x) = \frac{x^2 - 4}{x - 2}$$

becomes $g(x) = x + 2$, provided $x \ne 2$. In this case, 2 plays the role of a boundary point.

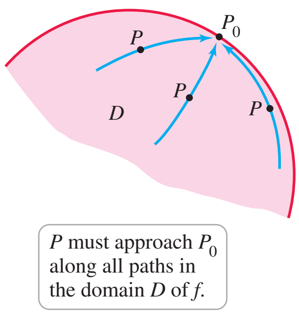


Figure 15.23

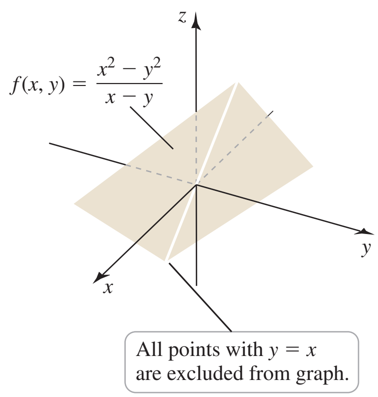


Figure 15.24

Now we examine $\displaystyle\lim_{(x,y)\to(4,4)} \frac{x^2 - y^2}{x - y}$, where $(4, 4)$ is a boundary point of the domain of f but does not lie in the domain. For this limit to exist, $f(x, y)$ must approach the same value along all paths to $(4, 4)$ that lie in the domain of f—that is, all paths approaching $(4, 4)$ that do not intersect $x = y$. To evaluate the limit, we proceed as follows:

$$\begin{aligned} \lim_{(x,y)\to(4,4)} \frac{x^2 - y^2}{x - y} &= \lim_{(x,y)\to(4,4)} (x + y) && \text{Assume } x \neq y \text{, cancel } x - y. \\ &= 4 + 4 = 8. && \text{Same limit along all paths in the domain} \end{aligned}$$

To emphasize, we let $(x, y) \to (4, 4)$ along all paths that do not intersect $x = y$, which lies outside the domain of f. Along all admissible paths, the function approaches 8.

QUICK CHECK 3 Can the limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{x}$ be evaluated by direct substitution? ◄

EXAMPLE 2 **Limits at boundary points** Evaluate $\displaystyle\lim_{(x,y)\to(4,1)} \frac{xy - 4y^2}{\sqrt{x} - 2\sqrt{y}}$.

SOLUTION Points in the domain of this function satisfy $x \geq 0$ and $y \geq 0$ (because of the square roots) and $x \neq 4y$ (to ensure the denominator is nonzero). We see that the point $(4, 1)$ lies on the boundary of the domain. Multiplying the numerator and denominator by the algebraic conjugate of the denominator, the limit is computed as follows:

$$\begin{aligned} \lim_{(x,y)\to(4,1)} \frac{xy - 4y^2}{\sqrt{x} - 2\sqrt{y}} &= \lim_{(x,y)\to(4,1)} \frac{(xy - 4y^2)(\sqrt{x} + 2\sqrt{y})}{(\sqrt{x} - 2\sqrt{y})(\sqrt{x} + 2\sqrt{y})} && \text{Multiply by conjugate.} \\ &= \lim_{(x,y)\to(4,1)} \frac{y(x - 4y)(\sqrt{x} + 2\sqrt{y})}{x - 4y} && \text{Simplify.} \\ &= \lim_{(x,y)\to(4,1)} y(\sqrt{x} + 2\sqrt{y}) && \text{Cancel } x - 4y\text{, assumed to be nonzero.} \\ &= 4. && \text{Evaluate limit.} \end{aligned}$$

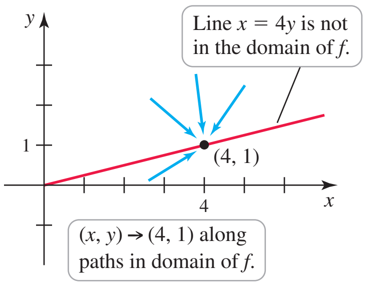


Figure 15.25

Because points on the line $x = 4y$ are outside the domain of the function, we assume $x - 4y \neq 0$. Along all other paths to $(4, 1)$, the function values approach 4 (**Figure 15.25**).

Related Exercises 26–27 ◄

➤ Notice that if we choose any path of the form $y = mx$, then $y \to 0$ as $x \to 0$. Therefore, $\displaystyle\lim_{(x,y)\to(0,0)}$ can be replaced by $\displaystyle\lim_{x\to 0}$ along this path. A similar argument applies to paths of the form $y = mx^p$, for $p > 0$.

EXAMPLE 3 **Nonexistence of a limit** Investigate the limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{(x + y)^2}{x^2 + y^2}$.

SOLUTION The domain of the function is $\{(x, y): (x, y) \neq (0, 0)\}$; therefore, the limit is at a boundary point outside the domain. Suppose we let (x, y) approach $(0, 0)$ along the line $y = mx$ for a fixed constant m. Substituting $y = mx$ and noting that $y \to 0$ as $x \to 0$, we have

$$\underset{(\text{along } y = mx)}{\lim_{(x,y)\to(0,0)}} \frac{(x + y)^2}{x^2 + y^2} = \lim_{x\to 0} \frac{(x + mx)^2}{x^2 + m^2x^2} = \lim_{x\to 0} \frac{x^2(1 + m)^2}{x^2(1 + m^2)} = \frac{(1 + m)^2}{1 + m^2}.$$

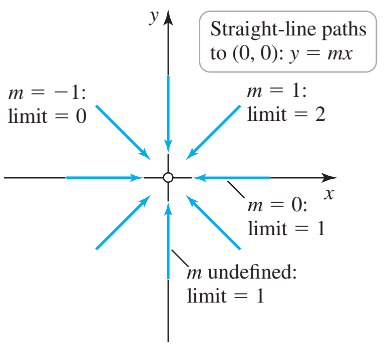


Figure 15.26

The constant m determines the direction of approach to $(0, 0)$. Therefore, depending on m, the function approaches different values as (x, y) approaches $(0, 0)$ (**Figure 15.26**). For example, if $m = 0$, the corresponding limit is 1, and if $m = -1$, the limit is 0. The reason for this behavior is revealed if we plot the surface and look at two level curves. The lines $y = x$ and $y = -x$ (excluding the origin) are level curves of the function for $z = 2$ and $z = 0$, respectively. (**Figure 15.27**). Therefore, as $(x, y) \to (0, 0)$ along $y = x$, $f(x, y) \to 2$, and as $(x, y) \to (0, 0)$ along $y = -x$, $f(x, y) \to 0$. Because the function approaches different values along different paths, we conclude that the *limit does not exist.*

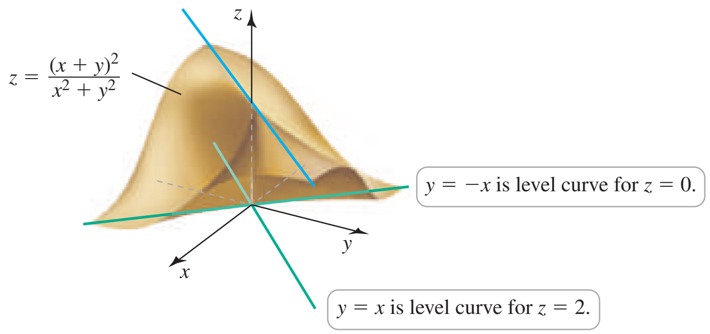


Figure 15.27

Related Exercise 30 ◀

The strategy used in Example 3 is an effective way to prove the nonexistence of a limit.

QUICK CHECK 4 What is the analog of the Two-Path Test for functions of a single variable? ◀

PROCEDURE Two-Path Test for Nonexistence of Limits

If $f(x, y)$ approaches two different values as (x, y) approaches (a, b) along two different paths in the domain of f, then $\lim_{(x, y)\to(a, b)} f(x, y)$ does not exist.

Continuity of Functions of Two Variables

The following definition of continuity for functions of two variables is analogous to the continuity definition for functions of one variable.

DEFINITION Continuity

The function f is continuous at the point (a, b) provided

1. f is defined at (a, b),
2. $\lim_{(x, y)\to(a, b)} f(x, y)$ exists, and
3. $\lim_{(x, y)\to(a, b)} f(x, y) = f(a, b)$.

A function of two (or more) variables is continuous at a point, provided its limit equals its value at that point (which implies the limit and the value both exist). The definition of continuity applies at boundary points of the domain of f, provided the limits in the definition are taken along all paths that lie in the domain. Because limits of polynomials and rational functions can be evaluated by substitution at points of their domains (that is, $\lim_{(x, y)\to(a, b)} f(x, y) = f(a, b)$), it follows that polynomials and rational functions are continuous at all points of their domains.

EXAMPLE 4 **Checking continuity** Determine the points at which the following function is continuous.

$$f(x, y) = \begin{cases} \dfrac{3xy^2}{x^2 + y^4} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

SOLUTION The function $\dfrac{3xy^2}{x^2 + y^4}$ is a rational function, so it is continuous at all points of its domain, which consists of all points of $\mathbb{R}^2$ except $(0, 0)$. To determine whether f is continuous at $(0, 0)$, we must show that

$$\lim_{(x, y)\to(0, 0)} \frac{3xy^2}{x^2 + y^4}$$

exists and equals $f(0, 0) = 0$ along all paths that approach $(0, 0)$.

➤ The choice of $x = my^2$ for paths to $(0, 0)$ is not obvious. Notice that if x is replaced with my^2 in f, the result involves the same power of y (in this case, y^4) in the numerator and denominator, which may be canceled.

You can verify that as (x, y) approaches $(0, 0)$ along paths of the form $y = mx$, where m is any constant, the function values approach $f(0, 0) = 0$. However, along parabolic paths of the form $x = my^2$ (where m is a nonzero constant), the limit behaves differently (**Figure 15.28**). This time we substitute $x = my^2$ and note that $x \to 0$ as $y \to 0$:

$$\lim_{\substack{(x,y)\to(0,0)\\ (\text{along } x = my^2)}} \frac{3xy^2}{x^2 + y^4} = \lim_{y\to 0} \frac{3(my^2)y^2}{(my^2)^2 + y^4} \quad \text{Substitute } x = my^2.$$

$$= \lim_{y\to 0} \frac{3my^4}{m^2y^4 + y^4} \quad \text{Simplify.}$$

$$= \lim_{y\to 0} \frac{3m}{m^2 + 1} \quad \text{Cancel } y^4.$$

$$= \frac{3m}{m^2 + 1}.$$

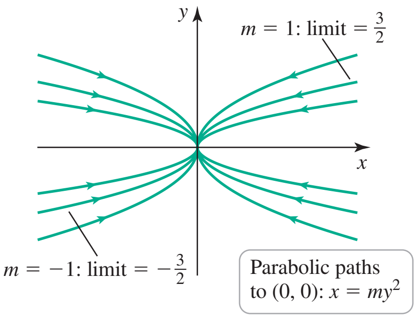


Figure 15.28

We see that along parabolic paths, the limit depends on the approach path. For example, with $m = 1$, along the path $x = y^2$ the function values approach $\frac{3}{2}$; with $m = -1$, along the path $x = -y^2$ the function values approach $-\frac{3}{2}$ (**Figure 15.29**). Because $f(x, y)$ approaches two different numbers along two different paths, the limit at $(0, 0)$ does not exist, and f is not continuous at $(0, 0)$.

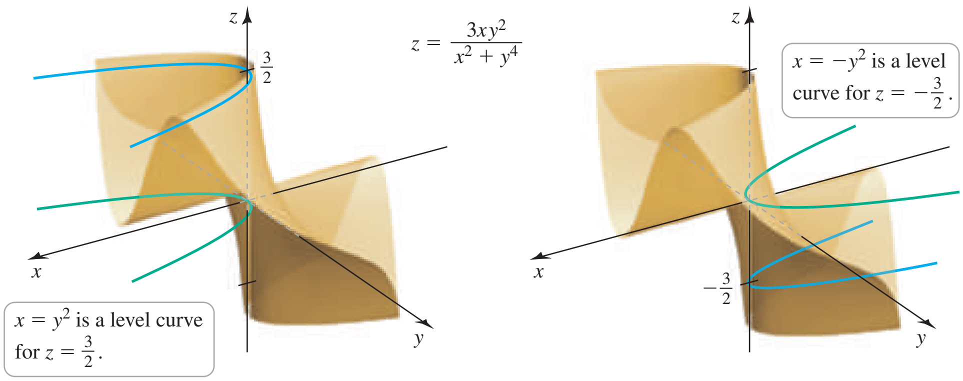


Figure 15.29

Related Exercises 41–42 ◄

QUICK CHECK 5 Which of the following functions are continuous at $(0, 0)$?

a. $f(x, y) = 2x^2y^5$

b. $f(x, y) = \dfrac{2x^2y^5}{x - 1}$

c. $f(x, y) = 2x^{-2}y^5$ ◄

Composite Functions Recall that for functions of a single variable, compositions of continuous functions are also continuous. The following theorem gives the analogous result for functions of two variables; it is proved in Appendix A.

> **THEOREM 15.3 Continuity of Composite Functions**
> If $u = g(x, y)$ is continuous at (a, b) and $z = f(u)$ is continuous at $g(a, b)$, then the composite function $z = f(g(x, y))$ is continuous at (a, b).

With Theorem 15.3, we can easily analyze the continuity of many functions. For example, $\sin x$, $\cos x$, and e^x are continuous functions of a single variable, for all real values of x. Therefore, compositions of these functions with polynomials in x and y (for example, $\sin(x^2y)$ and $e^{x^4 - y^2}$) are continuous for all real numbers x and y. Similarly, $\sqrt{x}$ is a continuous function of a single variable, for $x \geq 0$. Therefore, $\sqrt{u(x, y)}$ is continuous at (x, y) provided u is continuous at (x, y) and $u(x, y) \geq 0$. As long as we observe restrictions on domains, then compositions of continuous functions are also continuous.

EXAMPLE 5 **Continuity of composite functions.** Determine the points at which the following functions are continuous.

a. $h(x, y) = \ln(x^2 + y^2 + 4)$ **b.** $h(x, y) = e^{x/y}$

SOLUTION

a. This function is the composition $f(g(x, y))$, where

$$f(u) = \ln u \quad \text{and} \quad u = g(x, y) = x^2 + y^2 + 4.$$

As a polynomial, g is continuous for all (x, y) in $\mathbb{R}^2$. The function f is continuous for $u > 0$. Because $u = x^2 + y^2 + 4 > 0$ for all (x, y), it follows that h is continuous at all points of $\mathbb{R}^2$.

b. Letting $f(u) = e^u$ and $u = g(x, y) = x/y$, we have $h(x, y) = f(g(x, y))$. Note that f is continuous at all points of $\mathbb{R}$ and g is continuous at all points of $\mathbb{R}^2$ provided $y \neq 0$. Therefore, h is continuous on the set $\{(x, y): y \neq 0\}$.

Related Exercises 48–49 ◄

Functions of Three Variables

The work we have done with limits and continuity of functions of two variables extends to functions of three or more variables. Specifically, the limit laws of Theorem 15.2 apply to functions of the form $w = f(x, y, z)$. Polynomials and rational functions are continuous at all points of their domains, and limits of these functions may be evaluated by direct substitution at all points of their domains. Compositions of continuous functions of the form $f(g(x, y, z))$ are also continuous at points at which $g(x, y, z)$ is in the domain of f.

EXAMPLE 6 Functions of three variables

a. Evaluate $\displaystyle\lim_{(x, y, z)\to(2, \pi/2, 0)} \frac{x^2 \sin y}{z^2 + 4}$.

b. Find the points at which $h(x, y, z) = \sqrt{x^2 + y^2 + z^2 - 1}$ is continuous.

SOLUTION

a. This function consists of products and quotients of functions that are continuous at $(2, \pi/2, 0)$. Therefore, the limit is evaluated by direct substitution:

$$\lim_{(x, y, z)\to(2, \pi/2, 0)} \frac{x^2 \sin y}{z^2 + 4} = \frac{2^2 \sin(\pi/2)}{0^2 + 4} = 1.$$

b. This function is a composition in which the outer function $f(u) = \sqrt{u}$ is continuous for $u \geq 0$. The inner function

$$g(x, y, z) = x^2 + y^2 + z^2 - 1$$

is nonnegative provided $x^2 + y^2 + z^2 \geq 1$. Therefore, h is continuous at all points on or outside the unit sphere in $\mathbb{R}^3$.

Related Exercise 55 ◄

SECTION 15.2 EXERCISES

Getting Started

1. Explain what $\displaystyle\lim_{(x, y)\to(a, b)} f(x, y) = L$ means.

2. Explain why $f(x, y)$ must approach a unique number L as (x, y) approaches (a, b) along *all* paths in the domain in order for $\displaystyle\lim_{(x, y)\to(a, b)} f(x, y)$ to exist.

3. What does it means to say that limits of polynomials may be evaluated by direct substitution?

4. Suppose (a, b) is on the boundary of the domain of f. Explain how you would determine whether $\displaystyle\lim_{(x, y)\to(a, b)} f(x, y)$ exists.

5. Explain how examining limits along multiple paths may prove the nonexistence of a limit.

6. Explain why evaluating a limit along a finite number of paths does not prove the existence of a limit of a function of several variables.

7. What three conditions must be met for a function f to be continuous at the point (a, b)?

8. Let R be the unit disk $\{(x, y): x^2 + y^2 \leq 1\}$ with $(0, 0)$ removed. Is $(0, 0)$ a boundary point of R? Is R open or closed?

9. At what points of $\mathbb{R}^2$ is a rational function of two variables continuous?

10. Evaluate $\displaystyle\lim_{(x, y, z)\to(1, 1, -1)} xy^2z^3$.

11. Evaluate $\displaystyle\lim_{(x, y)\to(5, -5)} \frac{x^2 - y^2}{x + y}$.

12. Let $f(x) = \dfrac{x^2 - 2x - y^2 + 1}{x^2 - 2x + y^2 + 1}$. Use the Two-Path Test to show that $\displaystyle\lim_{(x,y)\to(1,0)} f(x)$ does not exist. (*Hint:* Examine $\displaystyle\lim_{\substack{(x,y)\to(1,0)\\(\text{along } y=0)}} f(x)$ and $\displaystyle\lim_{\substack{(x,y)\to(1,0)\\(\text{along } x=1)}} f(x)$ first.)

Practice Exercises

13–28. Limits of functions *Evaluate the following limits.*

13. $\displaystyle\lim_{(x,y)\to(2,9)} 101$

14. $\displaystyle\lim_{(x,y)\to(1,-3)} (3x + 4y - 2)$

15. $\displaystyle\lim_{(x,y)\to(-3,3)} (4x^2 - y^2)$

16. $\displaystyle\lim_{(x,y)\to(2,-1)} (xy^8 - 3x^2y^3)$

17. $\displaystyle\lim_{(x,y)\to(0,\pi)} \frac{\cos xy + \sin xy}{2y}$

18. $\displaystyle\lim_{(x,y)\to(e^2,4)} \ln \sqrt{xy}$

19. $\displaystyle\lim_{(x,y)\to(2,0)} \frac{x^2 - 3xy^2}{x + y}$

20. $\displaystyle\lim_{(u,v)\to(1,-1)} \frac{10uv - 2v^2}{u^2 + v^2}$

21. $\displaystyle\lim_{(x,y)\to(6,2)} \frac{x^2 - 3xy}{x - 3y}$

22. $\displaystyle\lim_{(x,y)\to(1,-2)} \frac{y^2 + 2xy}{y + 2x}$

23. $\displaystyle\lim_{(x,y)\to(3,1)} \frac{x^2 - 7xy + 12y^2}{x - 3y}$

24. $\displaystyle\lim_{(x,y)\to(-1,1)} \frac{2x^2 - xy - 3y^2}{x + y}$

25. $\displaystyle\lim_{(x,y)\to(2,2)} \frac{y^2 - 4}{xy - 2x}$

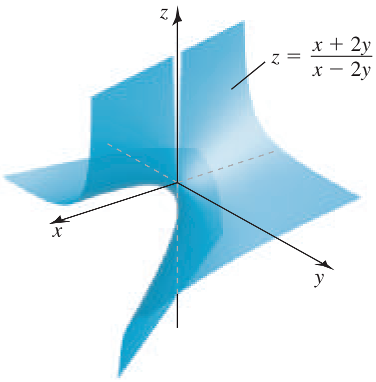
26. $\displaystyle\lim_{(x,y)\to(4,5)} \frac{\sqrt{x + y} - 3}{x + y - 9}$

27. $\displaystyle\lim_{(x,y)\to(1,2)} \frac{\sqrt{y} - \sqrt{x + 1}}{y - x - 1}$

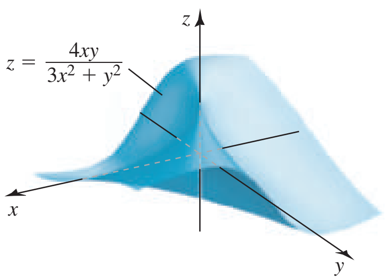
28. $\displaystyle\lim_{(u,v)\to(8,8)} \frac{u^{1/3} - v^{1/3}}{u^{2/3} - v^{2/3}}$

29–34. Nonexistence of limits *Use the Two-Path Test to prove that the following limits do not exist.*

29. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x + 2y}{x - 2y}$



30. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{4xy}{3x^2 + y^2}$



31. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^4 - 2x^2}{y^4 + x^2}$

32. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3 - y^2}{x^3 + y^2}$

33. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^3 + x^3}{xy^2}$

34. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y}{\sqrt{x^2 - y^2}}$

35–54. Continuity *At what points of $\mathbb{R}^2$ are the following functions continuous?*

35. $f(x, y) = x^2 + 2xy - y^3$

36. $f(x, y) = \dfrac{xy}{x^2y^2 + 1}$

37. $p(x, y) = \dfrac{4x^2y^2}{x^4 + y^2}$

38. $S(x, y) = \dfrac{2xy}{x^2 - y^2}$

39. $f(x, y) = \dfrac{2}{x(y^2 + 1)}$

40. $f(x, y) = \dfrac{x^2 + y^2}{x(y^2 - 1)}$

41. $f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

42. $f(x, y) = \begin{cases} \dfrac{y^4 - 2x^2}{y^4 + x^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

43. $f(x, y) = \sqrt{x^2 + y^2}$

44. $f(x, y) = e^{x^2 + y^2}$

45. $f(x, y) = \sin xy$

46. $g(x, y) = \ln (x - y)$

47. $h(x, y) = \cos (x + y)$

48. $p(x, y) = e^{x - y}$

49. $f(x, y) = \ln (x^2 + y^2)$

50. $f(x, y) = \sqrt{4 - x^2 - y^2}$

51. $g(x, y) = \sqrt[3]{x^2 + y^2 - 9}$

52. $h(x, y) = \dfrac{\sqrt{x - y}}{4}$

53. $f(x, y) = \begin{cases} \dfrac{\sin (x^2 + y^2)}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 1 & \text{if } (x, y) = (0, 0) \end{cases}$

54. $f(x, y) = \begin{cases} \dfrac{1 - \cos (x^2 + y^2)}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

55–60. Limits of functions of three variables *Evaluate the following limits.*

55. $\displaystyle\lim_{(x,y,z)\to(1,\ln 2,3)} ze^{xy}$

56. $\displaystyle\lim_{(x,y,z)\to(0,1,0)} \ln(1 + y)e^{xz}$

57. $\displaystyle\lim_{(x,y,z)\to(1,1,1)} \frac{yz - xy - xz - x^2}{yz + xy + xz - y^2}$

58. $\displaystyle\lim_{(x,y,z)\to(1,1,1)} \frac{x - \sqrt{xz} - \sqrt{xy} + \sqrt{yz}}{x - \sqrt{xz} + \sqrt{xy} - \sqrt{yz}}$

59. $\displaystyle\lim_{(x,y,z)\to(1,1,1)} \frac{x^2 + xy - xz - yz}{x - z}$

60. $\displaystyle\lim_{(x,y,z)\to(1,-1,1)} \frac{xz + 5x + yz + 5y}{x + y}$

61. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. If the limits $\lim_{(x,0)\to(0,0)} f(x,0)$ and $\lim_{(0,y)\to(0,0)} f(0,y)$ exist and equal L, then $\lim_{(x,y)\to(0,0)} f(x,y) = L$.

b. If $\lim_{(x,y)\to(a,b)} f(x,y)$ equals a finite number L, then f is continuous at (a,b).

c. If f is continuous at (a,b), then $\lim_{(x,y)\to(a,b)} f(x,y)$ exists.

d. If P is a boundary point of the domain of f, then P is in the domain of f.

62–76. Miscellaneous limits *Use the method of your choice to evaluate the following limits.*

62. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^2}{x^8 + y^2}$

63. $\displaystyle\lim_{(x,y)\to(0,1)} \frac{y \sin x}{x(y+1)}$

64. $\displaystyle\lim_{(x,y)\to(1,1)} \frac{x^2 + xy - 2y^2}{2x^2 - xy - y^2}$

65. $\displaystyle\lim_{(x,y)\to(1,0)} \frac{y \ln y}{x}$

66. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{|xy|}{xy}$

67. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{|x-y|}{|x+y|}$

68. $\displaystyle\lim_{(u,v)\to(-1,0)} \frac{uve^{-v}}{u^2 + v^2}$

69. $\displaystyle\lim_{(x,y)\to(2,0)} \frac{1 - \cos y}{xy^2}$

70. $\displaystyle\lim_{(x,y)\to(4,0)} x^2 y \ln xy$

71. $\displaystyle\lim_{(x,y)\to(1,0)} \frac{\sin xy}{xy}$

72. $\displaystyle\lim_{(x,y)\to(0,\pi/2)} \frac{1 - \cos xy}{4x^2y^3}$

73. $\displaystyle\lim_{(x,y)\to(0,2)} (2xy)^{xy}$

74. $\displaystyle\lim_{(x,y)\to(3,3)} \frac{x^2 + 2xy - 6x + y^2 - 6y}{x + y - 6}$

75. $\displaystyle\lim_{(x,y)\to(1,2)} \frac{x^2 + 2xy - x + y^2 - y - 6}{x + y - 3}$

76. $\displaystyle\lim_{(x,y)\to(0,0)} \tan^{-1} \frac{(2 + (x+y)^2 + (x-y)^2)}{2e^{x^2+y^2}}$

77. Piecewise function Let

$$f(x,y) = \begin{cases} \dfrac{\sin(x^2 + y^2 - 1)}{x^2 + y^2 - 1} & \text{if } x^2 + y^2 \neq 1 \\ b & \text{if } x^2 + y^2 = 1. \end{cases}$$

Find the value of b for which f is continuous at all points in $\mathbb{R}^2$.

78. Piecewise function Let

$$f(x,y) = \begin{cases} \dfrac{1 + 2xy - \cos xy}{xy} & \text{if } xy \neq 0 \\ a & \text{if } xy = 0. \end{cases}$$

Find the value of a for which f is continuous at all points in $\mathbb{R}^2$.

79–81. Limits using polar coordinates *Limits at* $(0,0)$ *may be easier to evaluate by converting to polar coordinates. Remember that the same limit must be obtained as* $r \to 0$ *along all paths in the domain to* $(0,0)$. *Evaluate the following limits or state that they do not exist.*

79. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2 + y^2}$

80. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x - y}{\sqrt{x^2 + y^2}}$

81. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 + y^2 + x^2y^2}{x^2 + y^2}$

Explorations and Challenges

82. Sine limits Verify that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin x + \sin y}{x + y} = 1$.

83. Nonexistence of limits Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{ax^m y^n}{bx^{m+n} + cy^{m+n}}$ does not exist when a, b, and c are nonzero real numbers and m and n are positive integers.

84. Nonexistence of limits Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{ax^{2(p-n)}y^n}{bx^{2p} + cy^p}$ does not exist when a, b, and c are nonzero real numbers and n and p are positive integers with $p \geq n$.

85. Filling in a function value The domain of $f(x,y) = e^{-1/(x^2+y^2)}$ excludes $(0,0)$. How should f be defined at $(0,0)$ to make it continuous there?

86. Limit proof Use the formal definition of a limit to prove that $\lim_{(x,y)\to(a,b)} y = b$. (*Hint:* Take $\delta = \varepsilon$.)

87. Limit proof Use the formal definition of a limit to prove that

$$\lim_{(x,y)\to(a,b)} (x + y) = a + b. \text{ (\textit{Hint:} Take } \delta = \varepsilon/2.)$$

88. Proof of Limit Law 1 Use the formal definition of a limit to prove that

$$\lim_{(x,y)\to(a,b)} (f(x,y) + g(x,y)) = \lim_{(x,y)\to(a,b)} f(x,y) + \lim_{(x,y)\to(a,b)} g(x,y).$$

89. Proof of Limit Law 3 Use the formal definition of a limit to prove that $\lim_{(x,y)\to(a,b)} cf(x,y) = c \lim_{(x,y)\to(a,b)} f(x,y)$.

QUICK CHECK ANSWERS

1. The limit exists only for (a). **2.** $\{(x,y): x^2 + y^2 < 2\}$ **3.** If a factor of x is first canceled, then the limit may be evaluated by substitution. **4.** If the left and right limits at a point are not equal, then the two-sided limit does not exist. **5.** (a) and (b) are continuous at $(0,0)$. ◄

15.3 Partial Derivatives

The derivative of a function of one variable, $y = f(x)$, measures the rate of change of y with respect to x, and it gives slopes of tangent lines. The analogous idea for functions of several variables presents a new twist: Derivatives may be defined with respect to any of the independent variables. For example, we can compute the derivative of $f(x, y)$ with respect to x or y. The resulting derivatives are called *partial derivatives*; they still represent rates of change and they are associated with slopes of tangents. Therefore, much of what you have learned about derivatives applies to functions of several variables. However, much is also different.

Derivatives with Two Variables

Consider a function f defined on a domain D in the xy-plane. Suppose f represents the elevation of the land (above sea level) over D. Imagine that you are on the surface $z = f(x, y)$ at the point $(a, b, f(a, b))$ and you are asked to determine the slope of the surface where you are standing. Your answer should be, *it depends*!

Figure 15.30a shows a function that resembles the landscape in Figure 15.30b. Suppose you are standing at the point $P(0, 0, f(0, 0))$, which lies on the pass or the saddle. The surface behaves differently depending on the direction in which you walk. If you walk east (positive x-direction), the elevation increases and your path takes you upward on the surface. If you walk north (positive y-direction), the elevation decreases and your path takes you downward on the surface. In fact, in every direction you walk from the point P, the function values change at different rates. So how should the slope or the rate of change at a given point be defined?

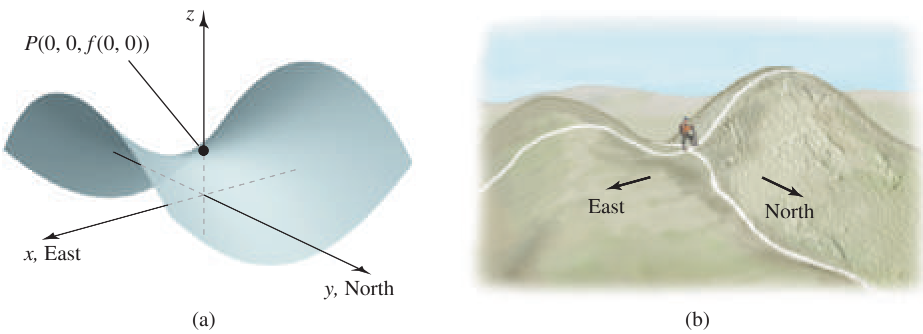


Figure 15.30

The answer to this question involves *partial derivatives*, which arise when we hold all but one independent variable fixed and then compute an ordinary derivative with respect to the remaining variable. Suppose we move along the surface $z = f(x, y)$, starting at the point $(a, b, f(a, b))$ in such a way that $y = b$ is fixed and only x varies. The resulting path is a curve (a trace) on the surface that varies in the x-direction (Figure 15.31). This curve is the intersection of the surface with the vertical plane $y = b$; it is described by $z = f(x, b)$, which is a function of the single variable x. We know how to compute the slope of this curve: It is the ordinary derivative of $f(x, b)$ with respect to x. This derivative is called the *partial derivative of f with respect to x*, denoted $\partial f/\partial x$ or f_x. When evaluated at (a, b), its value is defined by the limit

$$f_x(a, b) = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h},$$

provided this limit exists. Notice that the y-coordinate is fixed at $y = b$ in this limit. If we replace (a, b) with the variable point (x, y), then f_x becomes a function of x and y.

In a similar way, we can move along the surface $z = f(x, y)$ from the point $(a, b, f(a, b))$ in such a way that $x = a$ is fixed and only y varies. Now the result is

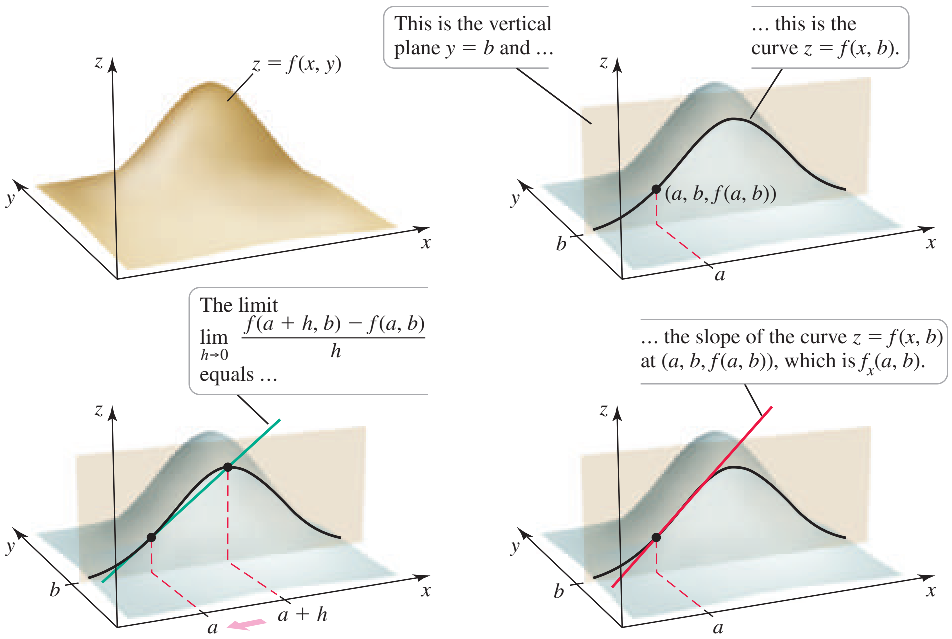


Figure 15.31

a trace described by $z = f(a, y)$, which is the intersection of the surface and the plane $x = a$ (Figure 15.32). The slope of this curve at (a, b) is given by the ordinary derivative of $f(a, y)$ with respect to y. This derivative is called the *partial derivative of f with respect to y*, denoted $\partial f/\partial y$ or f_y. When evaluated at (a, b), it is defined by the limit

$$f_y(a, b) = \lim_{h\to 0} \frac{f(a, b + h) - f(a, b)}{h},$$

provided this limit exists. If we replace (a, b) with the variable point (x, y), then f_y becomes a function of x and y.

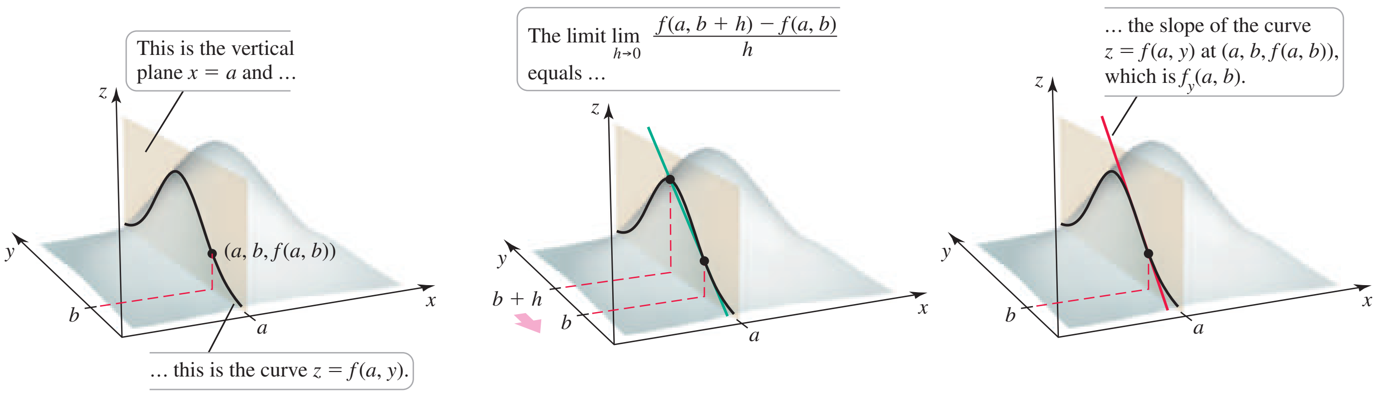


Figure 15.32

DEFINITION **Partial Derivatives**

The **partial derivative of f with respect to x at the point (a, b)** is

$$f_x(a, b) = \lim_{h\to 0} \frac{f(a + h, b) - f(a, b)}{h}.$$

The **partial derivative of f with respect to y at the point (a, b)** is

$$f_y(a, b) = \lim_{h\to 0} \frac{f(a, b + h) - f(a, b)}{h},$$

provided these limits exist.

➤ Recall that f' is a function, while $f'(a)$ is the value of the derivative at $x = a$. In the same way, f_x and f_y are functions of x and y, while $f_x(a, b)$ and $f_y(a, b)$ are their values at (a, b).

Notation The partial derivatives evaluated at a point (a, b) are denoted in any of the following ways:

$$\frac{\partial f}{\partial x}(a, b) = \left.\frac{\partial f}{\partial x}\right|_{(a,b)} = f_x(a, b) \quad \text{and} \quad \frac{\partial f}{\partial y}(a, b) = \left.\frac{\partial f}{\partial y}\right|_{(a,b)} = f_y(a, b).$$

Notice that the d in the ordinary derivative df/dx has been replaced with ∂ in the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$. The notation $\partial/\partial x$ is an instruction or operator: It says, "Take the partial derivative with respect to x of the function that follows."

Calculating Partial Derivatives We begin by calculating partial derivatives using the limit definition. The procedure in Example 1 should look familiar. It echoes the method used in Chapter 3 when we first introduced ordinary derivatives.

EXAMPLE 1 **Partial derivatives from the definition** Suppose $f(x, y) = x^2y$. Use the limit definition of partial derivatives to compute $f_x(x, y)$ and $f_y(x, y)$.

SOLUTION We compute the partial derivatives at an arbitrary point (x, y) in the domain. The partial derivative with respect to x is

$$\begin{aligned}
f_x(x, y) &= \lim_{h\to 0}\frac{f(x+h, y) - f(x, y)}{h} && \text{Definition of } f_x \text{ at } (x, y)\\
&= \lim_{h\to 0}\frac{(x+h)^2y - x^2y}{h} && \text{Substitute for } f(x+h, y) \text{ and } f(x, y).\\
&= \lim_{h\to 0}\frac{(x^2 + 2xh + h^2 - x^2)y}{h} && \text{Factor and expand.}\\
&= \lim_{h\to 0}(2x + h)y && \text{Simplify and cancel } h.\\
&= 2xy. && \text{Evaluate limit.}
\end{aligned}$$

In a similar way, the partial derivative with respect to y is

$$\begin{aligned}
f_y(x, y) &= \lim_{h\to 0}\frac{f(x, y+h) - f(x, y)}{h} && \text{Definition of } f_y \text{ at } (x, y)\\
&= \lim_{h\to 0}\frac{x^2(y+h) - x^2y}{h} && \text{Substitute for } f(x, y+h) \text{ and } f(x, y).\\
&= \lim_{h\to 0}\frac{x^2(y+h-y)}{h} && \text{Factor.}\\
&= x^2. && \text{Simplify and evaluate limit.}
\end{aligned}$$

Related Exercise 11 ◄

A careful examination of Example 1 reveals a shortcut for evaluating partial derivatives. To compute the partial derivative of f with respect to x, we treat y as a constant and take an ordinary derivative with respect to x:

$$\frac{\partial}{\partial x}(x^2y) = y\underbrace{\frac{\partial}{\partial x}(x^2)}_{2x} = 2xy. \quad \text{Treat } y \text{ as a constant.}$$

Similarly, we treat x (and therefore x^2) as a constant to evaluate the partial derivative of f with respect to y:

$$\frac{\partial}{\partial y}(x^2y) = x^2\underbrace{\frac{\partial}{\partial y}(y)}_{1} = x^2. \quad \text{Treat } x \text{ as a constant.}$$

The next two examples illustrate the process.

EXAMPLE 2 **Partial derivatives** Let $f(x, y) = x^3 - y^2 + 4$.

a. Compute $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.

b. Evaluate each derivative at $(2, -4)$.

SOLUTION

a. We compute the partial derivative with respect to x assuming y is a constant; the Power Rule gives

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(\underbrace{x^3}_{\text{variable}} - \underbrace{y^2 + 4}_{\substack{\text{constant with}\\ \text{respect to } x}}) = 3x^2 + 0 = 3x^2.$$

The partial derivative with respect to y is computed by treating x as a constant; using the Power Rule gives

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(\underbrace{x^3}_{\substack{\text{constant}\\ \text{with respect}\\ \text{to } y}} - \underbrace{y^2}_{\text{variable}} + \underbrace{4}_{\text{constant}}) = -2y.$$

QUICK CHECK 1 Compute f_x and f_y for $f(x, y) = 2xy$. ◄

b. It follows that $f_x(2, -4) = (3x^2)\big|_{(2,-4)} = 12$ and $f_y(2, -4) = (-2y)\big|_{(2,-4)} = 8$.

Related Exercise 16 ◄

EXAMPLE 3 **Partial derivatives** Compute the partial derivatives of the following functions.

a. $f(x, y) = \sin xy$ **b.** $g(x, y) = x^2e^{xy}$

SOLUTION

➤ Recall that

$$\frac{d}{dx}(\sin 2x) = 2\cos 2x.$$

Replacing 2 with the constant y, we have

$$\frac{\partial}{\partial x}(\sin xy) = y\cos xy.$$

a. Treating y as a constant and differentiating with respect to x, we have

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(\sin xy) = y\cos xy.$$

Holding x fixed and differentiating with respect to y, we have

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(\sin xy) = x\cos xy.$$

b. To compute the partial derivative with respect to x, we call on the Product Rule. Holding y fixed, we have

$$\begin{aligned}\frac{\partial g}{\partial x} &= \frac{\partial}{\partial x}(x^2e^{xy}) \\ &= \frac{\partial}{\partial x}(x^2)e^{xy} + x^2\frac{\partial}{\partial x}(e^{xy}) && \text{Product Rule} \\ &= 2xe^{xy} + x^2ye^{xy} && \text{Evaluate partial derivatives.} \\ &= xe^{xy}(2 + xy). && \text{Simplify.}\end{aligned}$$

➤ Because x and y are *independent* variables,

$$\frac{\partial}{\partial x}(y) = 0 \quad \text{and} \quad \frac{\partial}{\partial y}(x) = 0.$$

Treating x as a constant, the partial derivative with respect to y is

$$\frac{\partial g}{\partial y} = \frac{\partial}{\partial y}(x^2e^{xy}) = x^2\underbrace{\frac{\partial}{\partial y}(e^{xy})}_{xe^{xy}} = x^3e^{xy}.$$

Related Exercises 17, 21 ◄

Higher-Order Partial Derivatives

Just as we have higher-order derivatives of functions of one variable, we also have higher-order partial derivatives. For example, given a function f and its partial derivative f_x, we can take the derivative of f_x with respect to x or with respect to y, which accounts for two of the four possible *second-order partial derivatives*. Table 15.3 summarizes the notation for second partial derivatives.

Table 15.3

Notation 1	Notation 2	What we say . . .
$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$	$(f_x)_x = f_{xx}$	*d squared f dx squared or f-x-x*
$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$	$(f_y)_y = f_{yy}$	*d squared f dy squared or f-y-y*
$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$	$(f_y)_x = f_{yx}$	*f-y-x*
$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$	$(f_x)_y = f_{xy}$	*f-x-y*

QUICK CHECK 2 Which of the following expressions are equivalent to each other: (a) f_{xy}, (b) f_{yx}, or (c) $\frac{\partial^2 f}{\partial y \partial x}$? Write $\frac{\partial^2 f}{\partial p \partial q}$ in subscript notation. ◄

The order of differentiation can make a difference in the **mixed partial derivatives** f_{xy} and f_{yx}. So it is important to use the correct notation to reflect the order in which derivatives are taken. For example, the notations $\frac{\partial^2 f}{\partial x \partial y}$ and f_{yx} both mean $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$; that is, differentiate first with respect to y, then with respect to x.

EXAMPLE 4 **Second partial derivatives** Find the four second partial derivatives of $f(x, y) = 3x^4y - 2xy + 5xy^3$.

SOLUTION First, we compute

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(3x^4y - 2xy + 5xy^3) = 12x^3y - 2y + 5y^3$$

and

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(3x^4y - 2xy + 5xy^3) = 3x^4 - 2x + 15xy^2.$$

For the second partial derivatives, we have

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(12x^3y - 2y + 5y^3) = 36x^2y,$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(3x^4 - 2x + 15xy^2) = 30xy,$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(3x^4 - 2x + 15xy^2) = 12x^3 - 2 + 15y^2, \text{ and}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(12x^3y - 2y + 5y^3) = 12x^3 - 2 + 15y^2.$$

QUICK CHECK 3 Compute f_{xxx} and f_{xxy} for $f(x, y) = x^3y$. ◄

Related Exercises 39–40 ◄

Equality of Mixed Partial Derivatives Notice that the two mixed partial derivatives in Example 4 are equal; that is, $f_{xy} = f_{yx}$. It turns out that most of the functions we encounter in this text have this property. Sufficient conditions for equality of mixed partial derivatives are given in a theorem attributed to the French mathematician Alexis Clairaut (1713–1765). The proof is found in advanced texts.

THEOREM 15.4 (Clairaut) Equality of Mixed Partial Derivatives
Assume f is defined on an open set D of $\mathbb{R}^2$, and that f_{xy} and f_{yx} are continuous throughout D. Then $f_{xy} = f_{yx}$ at all points of D.

Assuming sufficient continuity, Theorem 15.4 can be extended to higher derivatives of f. For example, $f_{xyx} = f_{xxy} = f_{yxx}$.

Functions of Three Variables

Everything we learned about partial derivatives of functions with two variables carries over to functions of three or more variables, as illustrated in Example 5.

EXAMPLE 5 Partial derivatives with more than two variables Find f_x, f_y, and f_z when $f(x, y, z) = e^{-xy} \cos z$.

SOLUTION To find f_x, we treat y and z as constants and differentiate with respect to x:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(\underbrace{e^{-xy}}_{y\text{ is constant}} \cdot \underbrace{\cos z}_{\text{constant}}) = -ye^{-xy} \cos z.$$

Holding x and z constant and differentiating with respect to y, we have

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(\underbrace{e^{-xy}}_{x\text{ is constant}} \cdot \underbrace{\cos z}_{\text{constant}}) = -xe^{-xy} \cos z.$$

To find f_z, we hold x and y constant and differentiate with respect to z:

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}(\underbrace{e^{-xy}}_{\text{constant}} \cos z) = -e^{-xy} \sin z.$$

QUICK CHECK 4 Compute f_{xz} and f_{zz} for $f(x, y, z) = xyz - x^2 z + yz^2$. ◄

Related Exercises 55–56 ◄

Applications of Partial Derivatives When functions are used in realistic applications (for example, to describe velocity, pressure, investment fund balance, or population), they often involve more than one independent variable. For this reason, partial derivatives appear frequently in mathematical modeling.

EXAMPLE 6 Ideal Gas Law The pressure P, volume V, and temperature T of an ideal gas are related by the equation $PV = kT$, where $k > 0$ is a constant depending on the amount of gas.

a. Determine the rate of change of the pressure with respect to the volume at constant temperature. Interpret the result.

b. Determine the rate of change of the pressure with respect to the temperature at constant volume. Interpret the result.

c. Explain these results using level curves.

➤ Implicit differentiation can also be used with partial derivatives. Instead of solving for P, we could differentiate both sides of $PV = kT$ with respect to V holding T fixed. Using the Product Rule, $P_V V + P = 0$, which implies that $P_V = -P/V$. Substituting $P = kT/V$, we have $P_V = -kT/V^2$.

➤ In the Ideal Gas Law, temperature is a positive variable because it is measured in kelvins.

SOLUTION Expressing the pressure as a function of volume and temperature, we have $P = k\dfrac{T}{V}$.

a. We find the partial derivative $\partial P/\partial V$ by holding T constant and differentiating P with respect to V:

$$\frac{\partial P}{\partial V} = \frac{\partial}{\partial V}\left(k\frac{T}{V}\right) = kT\frac{\partial}{\partial V}(V^{-1}) = -\frac{kT}{V^2}.$$

Recognizing that P, V, and T are always positive, we see that $\dfrac{\partial P}{\partial V} < 0$, which means that the pressure is a decreasing function of volume at a constant temperature.

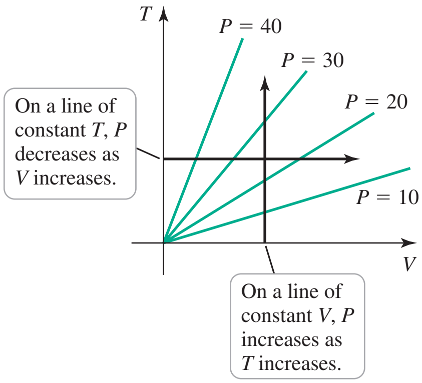


Figure 15.33

QUICK CHECK 5 Explain why, in Figure 15.33, the slopes of the level curves increase as the pressure increases. ◄

b. The partial derivative $\partial P/\partial T$ is found by holding V constant and differentiating P with respect to T:

$$\frac{\partial P}{\partial T} = \frac{\partial}{\partial T}\left(k\frac{T}{V}\right) = \frac{k}{V}.$$

In this case, $\partial P/\partial T > 0$, which says that the pressure is an increasing function of temperature at constant volume.

c. The level curves (Section 15.1) of the pressure function are curves in the VT-plane that satisfy $k\frac{T}{V} = P_0$, where P_0 is a constant. Solving for T, the level curves are given by $T = \frac{1}{k}P_0V$. Because $\frac{P_0}{k}$ is a positive constant, the level curves are lines in the first quadrant (**Figure 15.33**) with slope P_0/k. The fact that $\frac{\partial P}{\partial V} < 0$ (from part (a)) means that if we hold $T > 0$ fixed and move in the direction of increasing V on a *horizontal* line, we cross level curves corresponding to decreasing pressures. Similarly, $\frac{\partial P}{\partial T} > 0$ (from part (b)) means that if we hold $V > 0$ fixed and move in the direction of increasing T on a *vertical* line, we cross level curves corresponding to increasing pressures.

Related Exercise 69 ◄

Differentiability

We close this section with a technical matter that bears on the remainder of the chapter. Although we know how to compute partial derivatives of a function of several variables, we have not said what it means for such a function to be *differentiable* at a point. It is tempting to conclude that if the partial derivatives f_x and f_y exist at a point, then f is differentiable there. However, it is not so simple.

Recall that a function f of one variable is differentiable at $x = a$ provided the limit

$$f'(a) = \lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x}$$

exists. If f is differentiable at a, it means that the curve is smooth at the point $(a, f(a))$ (no jumps, corners, or cusps); furthermore, the curve has a unique tangent line at that point with slope $f'(a)$. Differentiability for a function of several variables should carry the same properties: The surface should be smooth at the point in question, and something analogous to a unique tangent line should exist at the point.

Staying with the one-variable case, we define the quantity

$$\varepsilon = \underbrace{\frac{f(a + \Delta x) - f(a)}{\Delta x}}_{\text{slope of secant line}} - \underbrace{f'(a)}_{\text{slope of tangent line}},$$

where ε is viewed as a function of Δx. Notice that ε is the difference between the slopes of secant lines and the slope of the tangent line at the point $(a, f(a))$. If f is differentiable at a, then this difference approaches zero as $\Delta x \to 0$; therefore, $\lim_{\Delta x \to 0} \varepsilon = 0$. Multiplying both sides of the definition of ε by Δx gives

$$\varepsilon\Delta x = f(a + \Delta x) - f(a) - f'(a)\,\Delta x.$$

Rearranging, we have the change in the function $y = f(x)$:

$$\Delta y = f(a + \Delta x) - f(a) = f'(a)\,\Delta x + \underbrace{\varepsilon}_{\varepsilon \to 0 \text{ as } \Delta x \to 0}\Delta x.$$

➤ Notice that $f'(a)\,\Delta x$ is the approximate change in the function given by a linear approximation.

This expression says that in the one-variable case, if f is differentiable at a, then the change in f between a and a nearby point $a + \Delta x$ is represented by $f'(a)\,\Delta x$ plus a quantity $\varepsilon\,\Delta x$, where $\lim_{\Delta x \to 0} \varepsilon = 0$.

The analogous requirement with several variables is the definition of differentiability for functions of two (or more) variables.

DEFINITION Differentiability

The function $z = f(x, y)$ is **differentiable at (a, b)** provided $f_x(a, b)$ and $f_y(a, b)$ exist and the change $\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b)$ equals

$$\Delta z = f_x(a, b)\,\Delta x + f_y(a, b)\,\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y,$$

where for fixed a and b, ε_1 and ε_2 are functions that depend only on Δx and Δy, with $(\varepsilon_1, \varepsilon_2) \to (0, 0)$ as $(\Delta x, \Delta y) \to (0, 0)$. A function is **differentiable** on an open set R if it is differentiable at every point of R.

Several observations are needed here. First, the definition extends to functions of more than two variables. Second, we show how differentiability is related to linear approximation and the existence of a *tangent plane* in Section 15.6. Finally, the conditions of the definition are generally difficult to verify. The following theorem may be useful in checking differentiability.

THEOREM 15.5 Conditions for Differentiability

Suppose the function f has partial derivatives f_x and f_y defined on an open set containing (a, b), with f_x and f_y continuous at (a, b). Then f is differentiable at (a, b).

As shown in Example 7, the existence of f_x and f_y at (a, b) is not enough to ensure differentiability of f at (a, b). However, by Theorem 15.5, if f_x and f_y are continuous at (a, b) (and defined in an open set containing (a, b)), then we can conclude f is differentiable there. Polynomials and rational functions are differentiable at all points of their domains, as are compositions of exponential, logarithmic, and trigonometric functions with other differentiable functions. The proof of this theorem is given in Appendix A.

We close with the analog of Theorem 3.1, which states that differentiability implies continuity.

THEOREM 15.6 Differentiable Implies Continuous

If a function f is differentiable at (a, b), then it is continuous at (a, b).

Proof: By the definition of differentiability,

$$\Delta z = f_x(a, b)\,\Delta x + f_y(a, b)\,\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y,$$

where $(\varepsilon_1, \varepsilon_2) \to (0, 0)$ as $(\Delta x, \Delta y) \to (0, 0)$. Because f is assumed to be differentiable, we see that as Δx and Δy approach 0,

$$\lim_{(\Delta x, \Delta y) \to (0, 0)} \Delta z = 0.$$

Also, because $\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b)$, it follows that

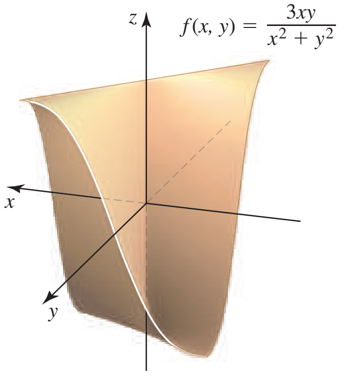
$$\lim_{(\Delta x, \Delta y) \to (0, 0)} f(a + \Delta x, b + \Delta y) = f(a, b),$$

which implies continuity of f at (a, b). ◄

➤ Recall that continuity requires that $\lim_{(x, y) \to (a, b)} f(x, y) = f(a, b)$, which is equivalent to $\lim_{(\Delta x, \Delta y) \to (0, 0)} f(a + \Delta x, b + \Delta y) = f(a, b)$.

EXAMPLE 7 A nondifferentiable function Discuss the differentiability and continuity of the function

$$f(x, y) = \begin{cases} \dfrac{3xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$



f is not continuous at $(0, 0)$, even though $f_x(0, 0) = f_y(0, 0) = 0$.

Figure 15.34

➤ The relationships between the existence and continuity of partial derivatives and whether a function is differentiable are further explored in Exercises 96–97.

SOLUTION As a rational function, f is continuous and differentiable at all points $(x, y) \neq (0, 0)$. The interesting behavior occurs at the origin. Using calculations similar to those in Example 4 in Section 15.2, it can be shown that if the origin is approached along the line $y = mx$, then

$$\lim_{\substack{(x, y) \to (0, 0) \\ (\text{along } y = mx)}} \frac{3xy}{x^2 + y^2} = \frac{3m}{m^2 + 1}.$$

Therefore, the value of the limit depends on the direction of approach, which implies that the limit does not exist, and f is not continuous at $(0, 0)$. By Theorem 15.6, it follows that f is not differentiable at $(0, 0)$. **Figure 15.34** shows the discontinuity of f at the origin.

Let's look at the first partial derivatives of f at $(0, 0)$. A short calculation shows that

$$f_x(0, 0) = \lim_{h \to 0} \frac{f(0 + h, 0) - f(0, 0)}{h} = \lim_{h \to 0} \frac{0 - 0}{h} = 0,$$
$$f_y(0, 0) = \lim_{h \to 0} \frac{f(0, 0 + h) - f(0, 0)}{h} = \lim_{h \to 0} \frac{0 - 0}{h} = 0.$$

Despite the fact that its first partial derivatives exist at $(0, 0)$, f is not differentiable at $(0, 0)$. As noted earlier, the existence of first partial derivatives at a point is not enough to ensure differentiability at that point.

Related Exercises 77–78 ◄

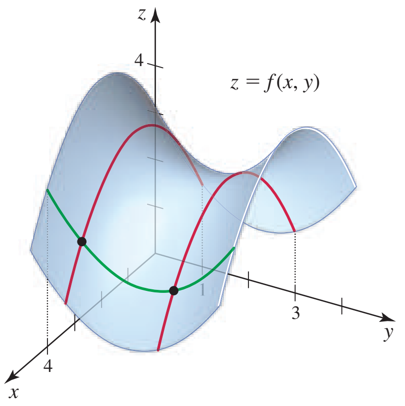
SECTION 15.3 EXERCISES

Getting Started

1. Suppose you are standing on the surface $z = f(x, y)$ at the point $(a, b, f(a, b))$. Interpret the meaning of $f_x(a, b)$ and $f_y(a, b)$ in terms of slopes or rates of change.

2. Let $f(x, y) = 3x^2 + y^3$.
   a. Compute f_x and f_y.
   b. Evaluate each derivative at $(1, 3)$.
   c. Find the four second partial derivatives of f.

3. Given the graph of a function $z = f(x, y)$ and its traces in the planes $x = 4$, $y = 1$, and $y = 3$ (see figure), determine whether the following partial derivatives are positive or negative.



   a. $f_x(4, 1)$ **b.** $f_y(4, 1)$ **c.** $f_x(4, 3)$ **d.** $f_y(4, 3)$

4. Find f_x and f_y when $f(x, y) = y^8 + 2x^6 + 2xy$.

5. Find f_x and f_y when $f(x, y) = 3x^2y + 2$.

6. Find the four second partial derivatives of $f(x, y) = x^2y^3$.

7. Verify that $f_{xy} = f_{yx}$, for $f(x, y) = 2x^3 + 3y^2 + 1$.

8. Verify that $f_{xy} = f_{yx}$, for $f(x, y) = xe^y$.

9. Find f_x, f_y, and f_z, for $f(x, y, z) = xy + xz + yz$.

10. The volume of a right circular cylinder with radius r and height h is $V = \pi r^2 h$. Is the volume an increasing or decreasing function of the radius at a fixed height (assume $r > 0$ and $h > 0$)?

Practice Exercises

11–14. Evaluating partial derivatives using limits *Use the limit definition of partial derivatives to evaluate $f_x(x, y)$ and $f_y(x, y)$ for the following functions.*

11. $f(x, y) = 5xy$
12. $f(x, y) = x + y^2 + 4$
13. $f(x, y) = \dfrac{x}{y}$
14. $f(x, y) = \sqrt{xy}$

15–37. Partial derivatives *Find the first partial derivatives of the following functions.*

15. $f(x, y) = xe^y$
16. $f(x, y) = 4x^3y^2 + 3x^2y^3 + 10$
17. $f(x, y) = e^{x^2y}$
18. $f(x, y) = (3xy + 4y^2 + 1)^5$
19. $f(w, z) = \dfrac{w}{w^2 + z^2}$
20. $f(s, t) = \dfrac{s - t}{s + t}$
21. $f(x, y) = x \cos xy$
22. $f(x, y) = \tan^{-1} \dfrac{x^2}{y^2}$
23. $s(y, z) = z^2 \tan yz$
24. $g(x, z) = x \ln (z^2 + x^2)$
25. $G(s, t) = \dfrac{\sqrt{st}}{s + t}$
26. $F(p, q) = \sqrt{p^2 + pq + q^2}$

27. $f(x, y) = x^{2y}$

28. $g(x, y) = \cos^5(x^2y^3)$

29. $h(x, y) = x - \sqrt{x^2 - 4y}$

30. $h(u, v) = \sqrt{\dfrac{uv}{u - v}}$

31. $f(x, y) = \int_x^{y^3} e^{t^2}\,dt$

32. $g(x, y) = y \sin^{-1}\sqrt{xy}$

33. $f(x, y) = 1 - \tan^{-1}(x^2 + y^2)$

34. $f(x, y) = \ln(1 + e^{-xy})$

35. $h(x, y) = (1 + 2y)^x$

36. $f(x, y) = 1 - \cos(2(x + y)) + \cos^2(x + y)$

37. $f(x, y) = \int_x^y h(s)\,ds$, where h is continuous for all real numbers

38–47. Second partial derivatives *Find the four second partial derivatives of the following functions.*

38. $f(x, y) = x^2 \sin y$

39. $h(x, y) = x^3 + xy^2 + 1$

40. $f(x, y) = 2x^5y^2 + x^2y$

41. $f(x, y) = y^3 \sin 4x$

42. $f(x, y) = \sin^2(x^3y)$

43. $p(u, v) = \ln(u^2 + v^2 + 4)$

44. $Q(r, s) = \dfrac{e^{r^3s}}{s}$

45. $F(r, s) = re^s$

46. $H(x, y) = \sqrt{4 + x^2 + y^2}$

47. $f(x, y) = \tan^{-1}(x^3y^2)$

48–53. Equality of mixed partial derivatives *Verify that $f_{xy} = f_{yx}$ for the following functions.*

48. $f(x, y) = 3x^2y^{-1} - 2x^{-1}y^2$

49. $f(x, y) = e^{x+y}$

50. $f(x, y) = \sqrt{xy}$

51. $f(x, y) = \cos xy$

52. $f(x, y) = e^{\sin xy}$

53. $f(x, y) = (2x - y^3)^4$

54–62. Partial derivatives with more than two variables *Find the first partial derivatives of the following functions.*

54. $G(r, s, t) = \sqrt{rs + rt + st}$

55. $h(x, y, z) = \cos(x + y + z)$

56. $g(x, y, z) = 2x^2y - 3xz^4 + 10y^2z^2$

57. $F(u, v, w) = \dfrac{u}{v + w}$

58. $Q(x, y, z) = \tan xyz$

59. $G(r, s, t) = \sqrt{rs^3t^5}$

60. $g(w, x, y, z) = \cos(w + x)\sin(y - z)$

61. $h(w, x, y, z) = \dfrac{wz}{xy}$

62. $F(w, x, y, z) = w\sqrt{x + 2y + 3z}$

63. **Exploiting patterns** Let $R(t) = \dfrac{at + b}{ct + d}$ and

$$g(x, y, z) = \frac{4x - 2y - 2z}{-6x + 3y - 3z}.$$

a. Verify that $R'(t) = \dfrac{ad - bc}{(ct + d)^2}$.

b. Use the derivative $R'(t)$ to find the first partial derivatives of g.

64–67. Estimating partial derivatives from a table *The following table shows values of a function $f(x, y)$ for values of x from 2 to 2.5 and values of y from 3 to 3.5. Use this table to estimate the values of the following partial derivatives.*

y \ x	**2**	**2.1**	**2.2**	**2.3**	**2.4**	**2.5**
3	4.243	4.347	4.450	4.550	4.648	4.743
3.1	4.384	4.492	4.598	4.701	4.802	4.902
3.2	4.525	4.637	4.746	4.853	4.957	5.060
3.3	4.667	4.782	4.895	5.005	5.112	5.218
3.4	4.808	4.930	5.043	5.156	5.267	5.376
3.5	4.950	5.072	5.191	5.308	5.422	5.534

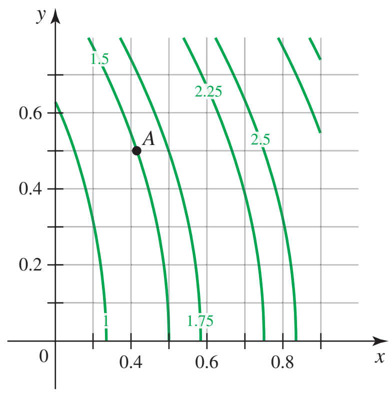
64. $f_x(2, 3)$

65. $f_y(2, 3)$

66. $f_x(2.2, 3.4)$

67. $f_y(2.4, 3.3)$

68. **Estimating partial derivatives from a graph** Use the level curves of f (see figure) to estimate the values of f_x and f_y at $A(0.42, 0.5)$.



69. **Gas law calculations** Consider the Ideal Gas Law $PV = kT$, where $k > 0$ is a constant. Solve this equation for V in terms of P and T.

a. Determine the rate of change of the volume with respect to the pressure at constant temperature. Interpret the result.

b. Determine the rate of change of the volume with respect to the temperature at constant pressure. Interpret the result.

c. Assuming $k = 1$, draw several level curves of the volume function, and interpret the results as in Example 6.

70. **Body mass index** The body mass index (BMI) for an adult human is given by the function $B = \dfrac{w}{h^2}$, where w is the weight measured in kilograms and h is the height measured in meters.

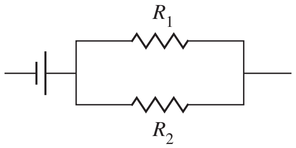
a. Find the rate of change of the BMI with respect to weight at a constant height.

b. For fixed h, is the BMI an increasing or decreasing function of w? Explain.

c. Find the rate of change of the BMI with respect to height at a constant weight.

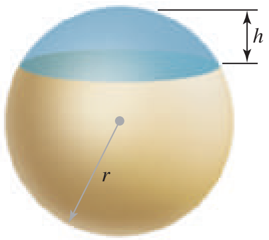
d. For fixed w, is the BMI an increasing or decreasing function of h? Explain.

71. Resistors in parallel Two resistors in an electrical circuit with resistance R_1 and R_2 wired in parallel with a constant voltage give an effective resistance of R, where $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$.



a. Find $\frac{\partial R}{\partial R_1}$ and $\frac{\partial R}{\partial R_2}$ by solving for R and differentiating.
b. Find $\frac{\partial R}{\partial R_1}$ and $\frac{\partial R}{\partial R_2}$ by differentiating implicitly.
c. Describe how an increase in R_1 with R_2 constant affects R.
d. Describe how a decrease in R_2 with R_1 constant affects R.

72. Spherical caps The volume of the cap of a sphere of radius r and thickness h is $V = \frac{\pi}{3}h^2(3r - h)$, for $0 \le h \le 2r$.



$V = \frac{\pi}{3}h^2(3r - h)$

a. Compute the partial derivatives V_h and V_r.
b. For a sphere of any radius, is the rate of change of volume with respect to r greater when $h = 0.2r$ or when $h = 0.8r$?
c. For a sphere of any radius, for what value of h is the rate of change of volume with respect to r equal to 1?
d. For a fixed radius r, for what value of h $(0 \le h \le 2r)$ is the rate of change of volume with respect to h the greatest?

73–76. Heat equation *The flow of heat along a thin conducting bar is governed by the one-dimensional heat equation (with analogs for thin plates in two dimensions and for solids in three dimensions):*

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},$$

where u is a measure of the temperature at a location x on the bar at time t and the positive constant k is related to the conductivity of the material. Show that the following functions satisfy the heat equation with $k = 1$.

73. $u(x, t) = 4e^{-4t}\cos 2x$

74. $u(x, t) = 10e^{-t}\sin x$

75. $u(x, t) = Ae^{-a^2t}\cos ax$, for any real numbers a and A

76. $u(x, t) = e^{-t}(2\sin x + 3\cos x)$

77–78. Nondifferentiability? *Consider the following functions f.*

a. *Is f continuous at* $(0, 0)$?
b. *Is f differentiable at* $(0, 0)$?
c. *If possible, evaluate* $f_x(0, 0)$ *and* $f_y(0, 0)$.
d. *Determine whether* f_x *and* f_y *are continuous at* $(0, 0)$.
e. *Explain why Theorems 15.5 and 15.6 are consistent with the results in parts (a)–(d).*

77. $f(x, y) = \begin{cases} -\dfrac{xy}{x^2 + y^2} & \text{if } (x, y) \ne (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

78. $f(x, y) = \begin{cases} \dfrac{2xy^2}{x^2 + y^4} & \text{if } (x, y) \ne (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

79. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. $\frac{\partial}{\partial x}(y^{10}) = 10y^9$.
b. $\frac{\partial^2}{\partial x \partial y}(\sqrt{xy}) = \frac{1}{\sqrt{xy}}$.
c. If f has continuous partial derivatives of all orders, then $f_{xxy} = f_{yxx}$.

80. Mixed partial derivatives

a. Consider the function $w = f(x, y, z)$. List all possible second partial derivatives that could be computed.
b. Let $f(x, y, z) = x^2y + 2xz^2 - 3y^2z$ and determine which second partial derivatives are equal.
c. How many second partial derivatives does $p = g(w, x, y, z)$ have?

Explorations and Challenges

81. Partial derivatives and level curves Consider the function $z = x/y^2$.

a. Compute z_x and z_y.
b. Sketch the level curves for $z = 1, 2, 3$, and 4.
c. Move along the horizontal line $y = 1$ in the xy-plane and describe how the corresponding z-values change. Explain how this observation is consistent with z_x as computed in part (a).
d. Move along the vertical line $x = 1$ in the xy-plane and describe how the corresponding z-values change. Explain how this observation is consistent with z_y as computed in part (a).

82. Volume of a box A box with a square base of length x and height h has a volume $V = x^2h$.

a. Compute the partial derivatives V_x and V_h.
b. For a box with $h = 1.5$ m, use linear approximation to estimate the change in volume if x increases from $x = 0.5$ m to $x = 0.51$ m.
c. For a box with $x = 0.5$ m, use linear approximation to estimate the change in volume if h decreases from $h = 1.5$ m to $h = 1.49$ m.
d. For a fixed height, does a 10% change in x always produce (approximately) a 10% change in V? Explain.
e. For a fixed base length, does a 10% change in h always produce (approximately) a 10% change in V? Explain.

83. Electric potential function The electric potential in the xy-plane associated with two positive charges, one at $(0, 1)$ with twice the magnitude of the charge at $(0, -1)$, is

$$\varphi(x, y) = \frac{2}{\sqrt{x^2 + (y - 1)^2}} + \frac{1}{\sqrt{x^2 + (y + 1)^2}}.$$

a. Compute φ_x and φ_y.
b. Describe how φ_x and φ_y behave as $x, y \to \pm\infty$.
c. Evaluate $\varphi_x(0, y)$, for all $y \ne \pm 1$. Interpret this result.
d. Evaluate $\varphi_y(x, 0)$, for all x. Interpret this result.

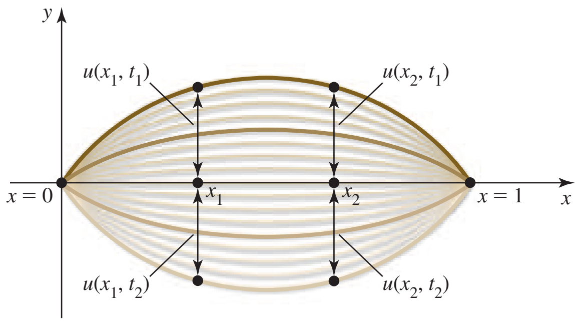
T **84. Cobb-Douglas production function** The output Q of an economic system subject to two inputs, such as labor L and capital K, is often modeled by the Cobb-Douglas production function $Q(L, K) = cL^a K^b$. Suppose $a = \frac{1}{3}$, $b = \frac{2}{3}$, and $c = 1$.

a. Evaluate the partial derivatives Q_L and Q_K.
b. Suppose $L = 10$ is fixed and K increases from $K = 20$ to $K = 20.5$. Use linear approximation to estimate the change in Q.
c. Suppose $K = 20$ is fixed and L decreases from $L = 10$ to $L = 9.5$. Use linear approximation to estimate the change in Q.
d. Graph the level curves of the production function in the first quadrant of the LK-plane for $Q = 1, 2$, and 3.
e. Use the graph of part (d). If you move along the vertical line $L = 2$ in the positive K-direction, how does Q change? Is this consistent with Q_K computed in part (a)?
f. Use the graph of part (d). If you move along the horizontal line $K = 2$ in the positive L-direction, how does Q change? Is this consistent with Q_L computed in part (a)?

85. An identity Show that if $f(x, y) = \dfrac{ax + by}{cx + dy}$, where a, b, c, and d are real numbers with $ad - bc = 0$, then $f_x = f_y = 0$, for all x and y in the domain of f. Give an explanation.

86. Wave on a string Imagine a string that is fixed at both ends (for example, a guitar string). When plucked, the string forms a standing wave. The displacement u of the string varies with position x and with time t. Suppose it is given by $u = f(x, t) = 2 \sin(\pi x) \sin(\pi t/2)$, for $0 \le x \le 1$ and $t \ge 0$ (see figure). At a fixed point in time, the string forms a wave on $[0, 1]$. Alternatively, if you focus on a point on the string (fix a value of x), that point oscillates up and down in time.

a. What is the period of the motion in time?
b. Find the rate of change of the displacement with respect to time at a constant position (which is the vertical velocity of a point on the string).
c. At a fixed time, what point on the string is moving fastest?
d. At a fixed position on the string, when is the string moving fastest?
e. Find the rate of change of the displacement with respect to position at a constant time (which is the slope of the string).
f. At a fixed time, where is the slope of the string greatest?



87–89. Wave equation *Traveling waves (for example, water waves or electromagnetic waves) exhibit periodic motion in both time and position. In one dimension, some types of wave motion are governed by the one-dimensional wave equation*

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},$$

where $u(x, t)$ is the height or displacement of the wave surface at position x and time t, and c is the constant speed of the wave. Show that the following functions are solutions of the wave equation.

87. $u(x, t) = \cos(2(x + ct))$

88. $u(x, t) = 5 \cos(2(x + ct)) + 3 \sin(x - ct)$

89. $u(x, t) = Af(x + ct) + Bg(x - ct)$, where A and B are constants, and f and g are twice differentiable functions of one variable.

90–93. Laplace's equation *A classical equation of mathematics is Laplace's equation, which arises in both theory and applications. It governs ideal fluid flow, electrostatic potentials, and the steady-state distribution of heat in a conducting medium. In two dimensions, Laplace's equation is*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Show that the following functions are ***harmonic****; that is, they satisfy Laplace's equation.*

90. $u(x, y) = e^{-x} \sin y$

91. $u(x, y) = x(x^2 - 3y^2)$

92. $u(x, y) = e^{ax} \cos ay$, for any real number a

93. $u(x, y) = \tan^{-1}\left(\dfrac{y}{x - 1}\right) - \tan^{-1}\left(\dfrac{y}{x + 1}\right)$

94–95. Differentiability *Use the definition of differentiability to prove that the following functions are differentiable at $(0, 0)$. You must produce functions ε_1 and ε_2 with the required properties.*

94. $f(x, y) = x + y$

95. $f(x, y) = xy$

96–97. Nondifferentiability? *Consider the following functions f.*

a. *Is f continuous at $(0, 0)$?*
b. *Is f differentiable at $(0, 0)$?*
c. *If possible, evaluate $f_x(0, 0)$ and $f_y(0, 0)$.*
d. *Determine whether f_x and f_y are continuous at $(0, 0)$.*
e. *Explain why Theorems 15.5 and 15.6 are consistent with the results in parts (a)–(d).*

96. $f(x, y) = 1 - |xy|$

97. $f(x, y) = \sqrt{|xy|}$

98. Cauchy-Riemann equations In the advanced subject of complex variables, a function typically has the form $f(x, y) = u(x, y) + iv(x, y)$, where u and v are real-valued functions and $i = \sqrt{-1}$ is the imaginary unit. A function $f = u + iv$ is said to be *analytic* (analogous to differentiable) if it satisfies the Cauchy-Riemann equations: $u_x = v_y$ and $u_y = -v_x$.

a. Show that $f(x, y) = (x^2 - y^2) + i(2xy)$ is analytic.
b. Show that $f(x, y) = x(x^2 - 3y^2) + iy(3x^2 - y^2)$ is analytic.
c. Show that if $f = u + iv$ is analytic, then $u_{xx} + u_{yy} = 0$ and $v_{xx} + v_{yy} = 0$. Assume u and v satisfy the conditions in Theorem 15.4.

99. Derivatives of an integral Let h be continuous for all real numbers. Find f_x and f_y when $f(x, y) = \int_1^{xy} h(s)\, ds$.

QUICK CHECK ANSWERS

1. $f_x = 2y$; $f_y = 2x$ **2.** (a) and (c) are the same; f_{qp} **3.** $f_{xxx} = 6y$; $f_{xxy} = 6x$ **4.** $f_{xz} = y - 2x$; $f_{zz} = 2y$ **5.** The equations of the level curves are $T = \frac{1}{k} P_0 V$. As the pressure P_0 increases, the slopes of these lines increase. ◄

15.4 The Chain Rule

In this section, we combine ideas based on the Chain Rule (Section 3.7) with what we know about partial derivatives (Section 15.3) to develop new methods for finding derivatives of functions of several variables. To illustrate the importance of these methods, consider the following situation.

Economists modeling manufacturing systems often work with *production functions* that relate the productivity (output) of the system to all the variables on which it depends (input). A simplified production function might take the form $P = F(L, K, R)$, where L, K, and R represent the availability of labor, capital, and natural resources, respectively. However, the variables L, K, and R may be intermediate variables that depend on other variables. For example, it might be that L is a function of the unemployment rate u, K is a function of the prime interest rate i, and R is a function of time t (seasonal availability of resources). Even in this simplified model, we see that productivity, which is the dependent variable, is ultimately related to many other variables (**Figure 15.35**). Of critical interest to an economist is how changes in one variable determine changes in other variables. For instance, if the unemployment rate increases by 0.1% and the interest rate decreases by 0.2%, what is the effect on productivity? In this section, we develop the tools needed to answer such questions.

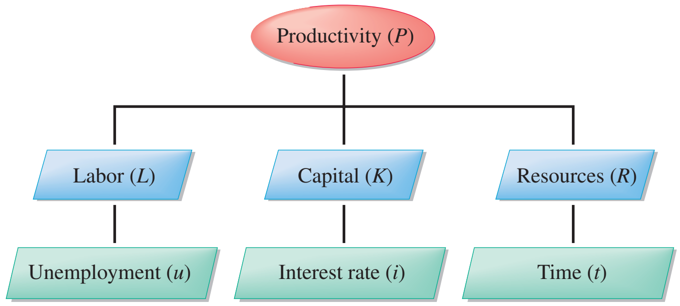


Figure 15.35

The Chain Rule with One Independent Variable

Recall the basic Chain Rule: If y is a function of u and u is a function of t, then $\dfrac{dy}{dt} = \dfrac{dy}{du}\dfrac{du}{dt}$. We first extend the Chain Rule to composite functions of the form $z = f(x, y)$, where x and y are functions of t. What is $\dfrac{dz}{dt}$?

We illustrate the relationships among the variables t, x, y, and z using a *tree diagram* (**Figure 15.36**). To find dz/dt, first notice that z depends on x, which in turn depends on t. The change in z with respect to x is the partial derivative $\partial z/\partial x$, and the change in x with respect to t is the ordinary derivative dx/dt. These derivatives appear on the corresponding branches of the tree diagram. Using the Chain Rule idea, the product of these derivatives gives the change in z with respect to t through x.

Similarly, z also depends on y. The change in z with respect to y is $\partial z/\partial y$, and the change in y with respect to t is dy/dt. The product of these derivatives, which appear on the corresponding branches of the tree, gives the change in z with respect to t through y. Summing the contributions to dz/dt along each branch of the tree leads to the following theorem, the proof of which is found in Appendix A.

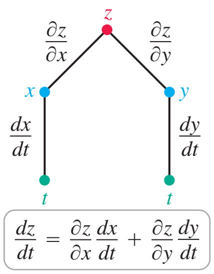


Figure 15.36

➤ A subtle observation about notation should be made. If $z = f(x, y)$, where x and y are functions of another variable t, it is common to write $z = f(t)$ to show that z ultimately depends on t. However, these two functions denoted f are actually different. We *should* write (or at least remember) that in fact $z = F(t)$, where F is a function other than f. This distinction is often overlooked for the sake of convenience.

QUICK CHECK 1 Explain why Theorem 15.7 reduces to the Chain Rule for a function of one variable in the case that $z = f(x)$ and $x = g(t)$. ◄

THEOREM 15.7 Chain Rule (One Independent Variable)

Let z be a differentiable function of x and y on its domain, where x and y are differentiable functions of t on an interval I. Then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

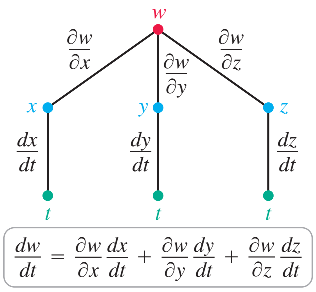


Figure 15.37

Before presenting examples, several comments are in order.

- With $z = f(x(t), y(t))$, the dependent variable is z and the sole independent variable is t. The variables x and y are **intermediate variables**.
- The choice of notation for partial and ordinary derivatives in the Chain Rule is important. We write the ordinary derivatives dx/dt and dy/dt because x and y depend only on t. We write the partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$ because z is a function of both x and y. Finally, we write dz/dt as an ordinary derivative because z ultimately depends only on t.
- Theorem 15.7 generalizes directly to functions of more than two intermediate variables (**Figure 15.37**). For example, if $w = f(x, y, z)$, where x, y, and z are functions of the single independent variable t, then

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}.$$

➤ If f, x, and y are simple, as in Example 1, it is possible to substitute $x(t)$ and $y(t)$ into f, producing a function of t only, and then differentiate with respect to t. But this approach quickly becomes impractical with more complicated functions, and the Chain Rule offers a great advantage.

EXAMPLE 1 **Chain Rule with one independent variable** Let $z = x^2 - 3y^2 + 20$, where $x = 2\cos t$ and $y = 2\sin t$.

a. Find $\dfrac{dz}{dt}$ and evaluate it at $t = \pi/4$.

b. Interpret the result geometrically.

SOLUTION

a. Computing the intermediate derivatives and applying the Chain Rule (Theorem 15.7), we find that

$$\begin{aligned}
\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\
&= \underbrace{(2x)}_{\frac{\partial z}{\partial x}}\,\underbrace{(-2\sin t)}_{\frac{dx}{dt}} + \underbrace{(-6y)}_{\frac{\partial z}{\partial y}}\,\underbrace{(2\cos t)}_{\frac{dy}{dt}} && \text{Evaluate derivatives.} \\
&= -4x\sin t - 12y\cos t && \text{Simplify.} \\
&= -8\cos t\sin t - 24\sin t\cos t && \text{Substitute } x = 2\cos t,\ y = 2\sin t. \\
&= -16\sin 2t. && \text{Simplify; } \sin 2t = 2\sin t\cos t.
\end{aligned}$$

Substituting $t = \pi/4$ gives $\left.\dfrac{dz}{dt}\right|_{t=\pi/4} = -16.$

b. The parametric equations $x = 2\cos t$, $y = 2\sin t$, for $0 \le t \le 2\pi$, describe a circle C of radius 2 in the xy-plane. Imagine walking on the surface $z = x^2 - 3y^2 + 20$ directly above the circle C consistent with positive (counterclockwise) orientation of C. Your path rises and falls as you walk (**Figure 15.38**); the rate of change of your elevation z with respect to t is given by dz/dt. For example, when $t = \pi/4$, the corresponding point on the surface is $(\sqrt{2}, \sqrt{2}, 16)$. At that point, z decreases at a rate of -16 (by part (a)) as you walk on the surface above C.

Related Exercises 10, 12 ◄

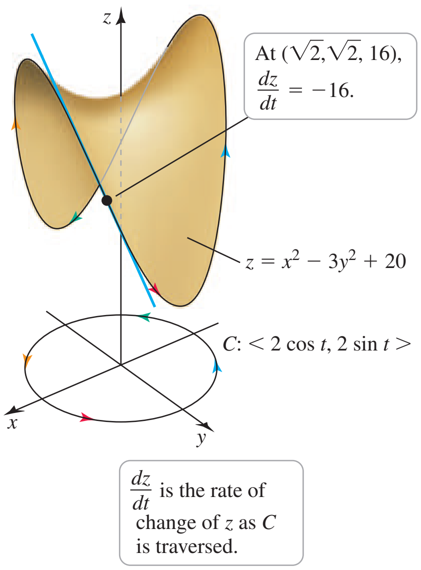


Figure 15.38

The Chain Rule with Several Independent Variables

The ideas behind the Chain Rule of Theorem 15.7 can be modified to cover a variety of situations in which functions of several variables are composed with one another. For example, suppose z depends on two intermediate variables x and y, each of which depends on the independent variables s and t. Once again, a tree diagram (**Figure 15.39**) helps organize the relationships among variables. The dependent variable z now ultimately depends on the two independent variables s and t, so it makes sense to ask about the rates of change of z with respect to either s or t, which are $\partial z/\partial s$ and $\partial z/\partial t$, respectively.

To compute $\partial z/\partial s$, we note that there are two paths in the tree (in red in Figure 15.39) that connect z to s and contribute to $\partial z/\partial s$. Along one path, z changes with respect to x

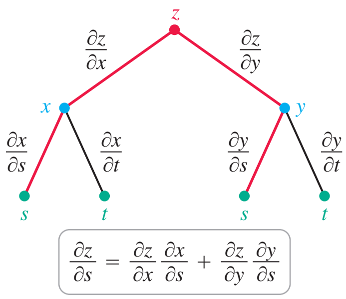


Figure 15.39

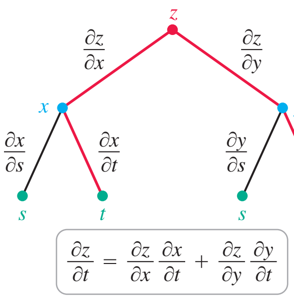


Figure 15.40

QUICK CHECK 2 Suppose $w = f(x, y, z)$, where $x = g(s, t)$, $y = h(s, t)$, and $z = p(s, t)$. Extend Theorem 15.8 to write a formula for $\partial w/\partial t$. ◄

(with rate of change $\partial z/\partial x$) and x changes with respect to s (with rate of change $\partial x/\partial s$). Along the other path, z changes with respect to y (with rate of change $\partial z/\partial y$) and y changes with respect to s (with rate of change $\partial y/\partial s$). We use a Chain Rule calculation along each path and combine the results. A similar argument leads to $\partial z/\partial t$ (**Figure 15.40**).

> **THEOREM 15.8 Chain Rule (Two Independent Variables)**
> Let z be a differentiable function of x and y, where x and y are differentiable functions of s and t. Then
>
> $$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}.$$

EXAMPLE 2 Chain Rule with two independent variables Let $z = \sin 2x \cos 3y$, where $x = s + t$ and $y = s - t$. Evaluate $\partial z/\partial s$ and $\partial z/\partial t$.

SOLUTION The tree diagram in Figure 15.39 gives the Chain Rule formula for $\partial z/\partial s$: We form products of the derivatives along the red branches connecting z to s and add the results. The partial derivative is

$$\begin{aligned}\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}\\ &= \underbrace{2\cos 2x \cos 3y}_{\frac{\partial z}{\partial x}} \cdot \underbrace{1}_{\frac{\partial x}{\partial s}} + \underbrace{(-3\sin 2x \sin 3y)}_{\frac{\partial z}{\partial y}} \cdot \underbrace{1}_{\frac{\partial y}{\partial s}}\\ &= 2\cos(2\underbrace{(s+t)}_{x})\cos(3\underbrace{(s-t)}_{y}) - 3\sin(2\underbrace{(s+t)}_{x})\sin(3\underbrace{(s-t)}_{y}).\end{aligned}$$

Following the branches of Figure 15.40 connecting z to t, we have

$$\begin{aligned}\frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}\\ &= \underbrace{2\cos 2x \cos 3y}_{\frac{\partial z}{\partial x}} \cdot \underbrace{1}_{\frac{\partial x}{\partial t}} + \underbrace{(-3\sin 2x \sin 3y)}_{\frac{\partial z}{\partial y}} \cdot \underbrace{-1}_{\frac{\partial y}{\partial t}}\\ &= 2\cos(2\underbrace{(s+t)}_{x})\cos(3\underbrace{(s-t)}_{y}) + 3\sin(2\underbrace{(s+t)}_{x})\sin(3\underbrace{(s-t)}_{y}).\end{aligned}$$

Related Exercise 22 ◄

EXAMPLE 3 More variables Let w be a function of x, y, and z, each of which is a function of s and t.

a. Draw a labeled tree diagram showing the relationships among the variables.

b. Write the Chain Rule formula for $\dfrac{\partial w}{\partial s}$.

SOLUTION

a. Because w is a function of x, y, and z, the upper branches of the tree (**Figure 15.41**) are labeled with the partial derivatives w_x, w_y, and w_z. Each of x, y, and z is a function of two variables, so the lower branches of the tree also require partial derivative labels.

b. Extending Theorem 15.8, we take the three paths through the tree that connect w to s (red branches in Figure 15.41). Multiplying the derivatives that appear on each path and adding gives the result

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}.$$

Related Exercises 25–26 ◄

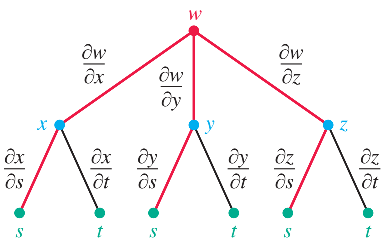


Figure 15.41

QUICK CHECK 3 If Q is a function of w, x, y, and z, each of which is a function of r, s, and t, how many dependent variables, intermediate variables, and independent variables are there? ◄

It is probably clear by now that we can create a Chain Rule for any set of relationships among variables. The key is to draw an accurate tree diagram and label the branches of the tree with the appropriate derivatives.

EXAMPLE 4 **A different kind of tree** Let w be a function of z, where z is a function of x and y, and each of x and y is a function of t. Draw a labeled tree diagram and write the Chain Rule formula for dw/dt.

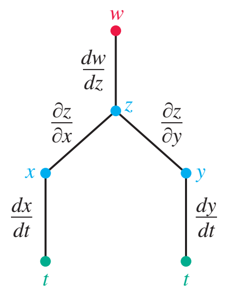


Figure 15.42

SOLUTION The dependent variable w is related to the independent variable t through two paths in the tree: $w \to z \to x \to t$ and $w \to z \to y \to t$ (Figure 15.42). At the top of the tree, w is a function of the single variable z, so the rate of change is the ordinary derivative dw/dz. The tree below z looks like Figure 15.36. Multiplying the derivatives on each of the two branches connecting w to t and adding the results, we have

$$\frac{dw}{dt} = \frac{dw}{dz}\frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{dw}{dz}\frac{\partial z}{\partial y}\frac{dy}{dt} = \frac{dw}{dz}\left(\frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}\right).$$

Related Exercise 31 ◀

Implicit Differentiation

Using the Chain Rule for partial derivatives, the technique of implicit differentiation can be put in a larger perspective. Recall that if x and y are related through an implicit relationship, such as $\sin xy + \pi y^2 = x$, then dy/dx is computed using implicit differentiation (Section 3.8). Another way to compute dy/dx is to define the function $F(x, y) = \sin xy + \pi y^2 - x$. Notice that the original relationship $\sin xy + \pi y^2 = x$ is $F(x, y) = 0$.

To find dy/dx, we treat x as the independent variable and differentiate both sides of $F(x, y(x)) = 0$ with respect to x. The derivative of the right side is 0. On the left side, we use the Chain Rule of Theorem 15.7:

$$\frac{\partial F}{\partial x}\underbrace{\frac{dx}{dx}}_{1} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0.$$

Noting that $dx/dx = 1$ and solving for dy/dx, we obtain the following theorem.

➤ The question of whether a relationship of the form $F(x, y) = 0$ or $F(x, y, z) = 0$ determines one or more functions is addressed by a theorem of advanced calculus called the Implicit Function Theorem.

THEOREM 15.9 **Implicit Differentiation**
Let F be differentiable on its domain and suppose $F(x, y) = 0$ defines y as a differentiable function of x. Provided $F_y \neq 0$,

$$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$

EXAMPLE 5 **Implicit differentiation** Find dy/dx when $F(x, y) = \sin xy + \pi y^2 - x = 0$.

➤ The method of Theorem 15.9 generalizes to computing $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ with functions of the form $F(x, y, z) = 0$ (Exercise 56).

SOLUTION Computing the partial derivatives of F with respect to x and y, we find that

$$F_x = y\cos xy - 1 \quad \text{and} \quad F_y = x\cos xy + 2\pi y.$$

Therefore,

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{y\cos xy - 1}{x\cos xy + 2\pi y}.$$

As with many implicit differentiation calculations, the result is left in terms of both x and y. The same result is obtained using the methods of Section 3.8.

Related Exercises 37 ◀

QUICK CHECK 4 Use the method of Example 5 to find dy/dx when $F(x, y) = x^2 + xy - y^3 - 7 = 0$. Compare your solution to Example 3 in Section 3.8. Which method is easier? ◀

EXAMPLE 6 **Fluid flow** A basin of circulating water is represented by the square region $\{(x, y): 0 \le x \le 1, 0 \le y \le 1\}$, where x is positive in the eastward direction and y is positive in the northward direction. The velocity components of the water are

$$\text{the east-west velocity } u(x, y) = 2 \sin \pi x \cos \pi y \text{ and}$$
$$\text{the north-south velocity } v(x, y) = -2 \cos \pi x \sin \pi y;$$

these velocity components produce the flow pattern shown in Figure 15.43. The *streamlines* shown in the figure are the paths followed by small parcels of water. The speed of the water at a point (x, y) is given by the function $s(x, y) = \sqrt{u(x, y)^2 + v(x, y)^2}$. Find $\partial s/\partial x$ and $\partial s/\partial y$, the rates of change of the water speed in the x- and y-directions, respectively.

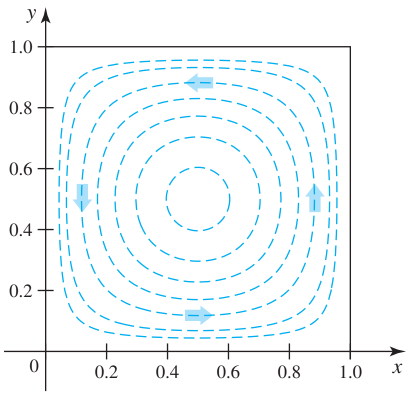


Figure 15.43

SOLUTION The dependent variable s depends on the independent variables x and y through the intermediate variables u and v (Figure 15.44). Theorem 15.8 applies here in the form

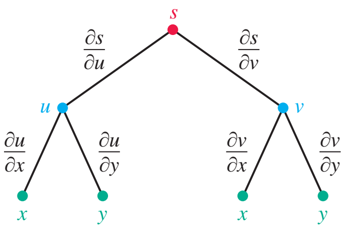


Figure 15.44

$$\frac{\partial s}{\partial x} = \frac{\partial s}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial s}{\partial v}\frac{\partial v}{\partial x} \quad \text{and} \quad \frac{\partial s}{\partial y} = \frac{\partial s}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial s}{\partial v}\frac{\partial v}{\partial y}.$$

The derivatives $\partial s/\partial u$ and $\partial s/\partial v$ are easier to find if we square the speed function to obtain $s^2 = u^2 + v^2$ and then use implicit differentiation. To compute $\partial s/\partial u$, we differentiate both sides of $s^2 = u^2 + v^2$ with respect to u:

$$2s\frac{\partial s}{\partial u} = 2u, \quad \text{which implies that} \quad \frac{\partial s}{\partial u} = \frac{u}{s}.$$

Similarly, differentiating $s^2 = u^2 + v^2$ with respect to v gives

$$2s\frac{\partial s}{\partial v} = 2v, \quad \text{which implies that} \quad \frac{\partial s}{\partial v} = \frac{v}{s}.$$

Now the Chain Rule leads to $\dfrac{\partial s}{\partial x}$:

$$\begin{aligned}
\frac{\partial s}{\partial x} &= \frac{\partial s}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial s}{\partial v}\frac{\partial v}{\partial x} \\
&= \underbrace{\frac{u}{s}}_{\frac{\partial s}{\partial u}}\underbrace{(2\pi \cos \pi x \cos \pi y)}_{\frac{\partial u}{\partial x}} + \underbrace{\frac{v}{s}}_{\frac{\partial s}{\partial v}}\underbrace{(2\pi \sin \pi x \sin \pi y)}_{\frac{\partial v}{\partial x}} \\
&= \frac{2\pi}{s}(u \cos \pi x \cos \pi y + v \sin \pi x \sin \pi y).
\end{aligned}$$

A similar calculation shows that

$$\frac{\partial s}{\partial y} = -\frac{2\pi}{s}(u \sin \pi x \sin \pi y + v \cos \pi x \cos \pi y).$$

As a final step, you could replace s, u, and v with their definitions in terms of x and y.

Related Exercises 41–42 ◄

EXAMPLE 7 **Second derivatives** Let $z = f(x, y) = \dfrac{x}{y}$, where $x = s + t^2$ and $y = s^2 - t$. Compute $\dfrac{\partial^2 z}{\partial s^2} = z_{ss}$, $\dfrac{\partial^2 z}{\partial t \partial s} = z_{st}$, and $\dfrac{\partial^2 z}{\partial t^2} = z_{tt}$, and express the results in terms of s and t. We use subscripts for partial derivatives in this example to simplify the notation.

SOLUTION First, we need some ground rules. In this example, it is possible to express f in terms of s and t by substituting, after which the result could be differentiated directly to find the required partial derivatives. Unfortunately, this maneuver is not always possible in practice (see Exercises 72 and 73). Therefore, to make this example as useful as possible, we develop general formulas for the second partial derivatives and make substitutions only in the last step.

Figures 15.39 and 15.40 show the relationships among the variables, and Example 2 demonstrates the calculation of the first partial derivatives. Throughout these calculations, it is important to remember the meaning of differentiation with respect to s and t:

$$(\ )_s = (\ )_x\, x_s + (\ )_y\, y_s \text{ and } (\ )_t = (\ )_x\, x_t + (\ )_y\, y_t.$$

Let's compute the first partial derivatives:

$$z_s = z_x x_s + z_y y_s \text{ and } z_t = z_x x_t + z_y y_t.$$

Differentiating z_s with respect to s, we have

$$\begin{aligned} z_{ss} &= (z_x x_s + z_y y_s)_s \\ &= (z_x)_s\, x_s + z_x x_{ss} + (z_y)_s\, y_s + z_y y_{ss} && \text{Product Rule (twice)} \\ &= \underbrace{(z_{xx} x_s + z_{xy} y_s)}_{(z_x)_s} x_s + z_x x_{ss} && \text{Differentiate } z_x \text{ and } z_y \text{ with respect to } s. \\ &\quad + \underbrace{(z_{yx} x_s + z_{yy} y_s)}_{(z_y)_s} y_s + z_y y_{ss} \\ &= z_{xx} x_s^2 + 2 z_{xy} x_s y_s + z_{yy} y_s^2 + z_x x_{ss} + z_y y_{ss}. && \text{Simplify with } z_{xy} = z_{yx}. \end{aligned}$$

At this point, we substitute

$$z_x = \frac{1}{y},\ z_y = -\frac{x}{y^2},\ z_{xx} = 0,\ z_{xy} = -\frac{1}{y^2},\ z_{yy} = \frac{2x}{y^3},\ x_s = 1,\ x_{ss} = 0,\ y_s = 2s, \text{ and } y_{ss} = 2$$

and simplify to find that

$$z_{ss} = \frac{2(s^3 + 3st + 3s^2t^2 + t^3)}{(s^2 - t)^3}.$$

Differentiating z_s with respect to t, a similar procedure produces z_{st}:

$$\begin{aligned} z_{st} &= (z_x x_s + z_y y_s)_t \\ &= (z_x)_t\, x_s + z_x x_{st} + (z_y)_t\, y_s + z_y y_{st} && \text{Product Rule (twice)} \\ &= \underbrace{(z_{xx} x_t + z_{xy} y_t)}_{(z_x)_t} x_s + z_x x_{st} && \text{Differentiate } z_x \text{ and } z_y \text{ with respect to } t. \\ &\quad + \underbrace{(z_{yx} x_t + z_{yy} y_t)}_{(z_y)_t} y_s + z_y y_{st} \\ &= z_{xx} x_s x_t + z_{xy} x_s y_t + z_{xy} x_t y_s + z_{yy} y_s y_t + z_x x_{st} + z_y y_{st}. && \text{Simplify with } z_{xy} = z_{yx}. \end{aligned}$$

Substituting in terms of s and t with $x_{st} = 0$ and $y_{st} = 0$, we have

$$z_{st} = -\frac{3s^2 + t + 4s^3 t}{(s^2 - t)^3}.$$

An analogous calculation gives

$$z_{tt} = \frac{2s(1 + s^3)}{(s^2 - t)^3}.$$

Related Exercise 45 ◄

SECTION 15.4 EXERCISES

Getting Started

1. Suppose $z = f(x, y)$, where x and y are functions of t. How many dependent, intermediate, and independent variables are there?

2. Let z be a function of x and y, while x and y are functions of t. Explain how to find dz/dt.

3. Suppose w is a function of x, y, and z, which are each functions of t. Explain how to find dw/dt.

4. Let $z = f(x, y)$, $x = g(s, t)$, and $y = h(s, t)$. Explain how to find $\partial z/\partial t$.

5. Given that $w = F(x, y, z)$, and x, y, and z are functions of r and s, sketch a Chain Rule tree diagram with branches labeled with the appropriate derivatives.

6. Suppose $F(x, y) = 0$ and y is a differentiable function of x. Explain how to find dy/dx.

7. Evaluate dz/dt, where $z = x^2 + y^3$, $x = t^2$, and $y = t$, using Theorem 15.7. Check your work by writing z as a function of t and evaluating dz/dt.

8. Evaluate dz/dt, where $z = xy^2$, $x = t^2$, and $y = t$, using Theorem 15.7. Check your work by writing z as a function of t and evaluating dz/dt.

Practice Exercises

9–18. Chain Rule with one independent variable *Use Theorem 15.7 to find the following derivatives.*

9. dz/dt, where $z = x \sin y$, $x = t^2$, and $y = 4t^3$

10. dz/dt, where $z = x^2y - xy^3$, $x = t^2$, and $y = t^{-2}$

11. dw/dt, where $w = \cos 2x \sin 3y$, $x = t/2$, and $y = t^4$

12. dz/dt, where $z = \sqrt{r^2 + s^2}$, $r = \cos 2t$, and $s = \sin 2t$

13. dz/dt, where $z = (x + 2y)^{10}$, $x = \sin^2 t$, $y = (3t + 4)^5$

14. $\dfrac{dz}{dt}$, where $z = \dfrac{x^{20}}{y^{10}}$, $x = \tan^{-1} t$, $y = \ln(t^2 + 1)$

15. dw/dt, where $w = xy \sin z$, $x = t^2$, $y = 4t^3$, and $z = t + 1$

16. dQ/dt, where $Q = \sqrt{x^2 + y^2 + z^2}$, $x = \sin t$, $y = \cos t$, and $z = \cos t$

17. dV/dt, where $V = xyz$, $x = e^t$, $y = 2t + 3$, and $z = \sin t$

18. $\dfrac{dU}{dt}$, where $U = \dfrac{xy^2}{z^8}$, $x = e^t$, $y = \sin 3t$, and $z = 4t + 1$

19–26. Chain Rule with several independent variables *Find the following derivatives.*

19. z_s and z_t, where $z = x^2 \sin y$, $x = s - t$, and $y = t^2$

20. z_s and z_t, where $z = \sin(2x + y)$, $x = s^2 - t^2$, and $y = s^2 + t^2$

21. z_s and z_t, where $z = xy - x^2y$, $x = s + t$, and $y = s - t$

22. z_s and z_t, where $z = \sin x \cos 2y$, $x = s + t$, and $y = s - t$

23. z_s and z_t, where $z = e^{x+y}$, $x = st$, and $y = s + t$

24. z_s and z_t, where $z = \sin xy$, $x = s^2t$, and $y = (s + t)^{10}$

25. w_s and w_t, where $w = \dfrac{x - z}{y + z}$, $x = s + t$, $y = st$, and $z = s - t$

26. w_r, w_s, and w_t, where $w = \sqrt{x^2 + y^2 + z^2}$, $x = st$, $y = rs$, and $z = rt$

27. Changing cylinder The volume of a right circular cylinder with radius r and height h is $V = \pi r^2 h$.

a. Assume r and h are functions of t. Find $V'(t)$.
b. Suppose $r = e^t$ and $h = e^{-2t}$, for $t \geq 0$. Use part (a) to find $V'(t)$.
c. Does the volume of the cylinder in part (b) increase or decrease as t increases?

28. Changing pyramid The volume of a pyramid with a square base x units on a side and a height of h is $V = \frac{1}{3}x^2 h$.

a. Assume x and h are functions of t. Find $V'(t)$.
b. Suppose $x = \dfrac{t}{t + 1}$ and $h = \dfrac{1}{t + 1}$, for $t \geq 0$. Use part (a) to find $V'(t)$.
c. Does the volume of the pyramid in part (b) increase or decrease as t increases?

29–30. Derivative practice two ways *Find the indicated derivative in two ways:*

a. *Replace x and y to write z as a function of t, and differentiate.*
b. *Use the Chain Rule.*

29. $z'(t)$, where $z = \dfrac{1}{x} + \dfrac{1}{y}$, $x = t^2 + 2t$, and $y = t^3 - 2$

30. $z'(t)$, where $z = \ln(x + y)$, $x = te^t$, and $y = e^t$

31–34. Making trees *Use a tree diagram to write the required Chain Rule formula.*

31. w is a function of z, where z is a function of p, q, and r, each of which is a function of t. Find dw/dt.

32. $w = f(x, y, z)$, where $x = g(t)$, $y = h(s, t)$, and $z = p(r, s, t)$. Find $\partial w/\partial t$.

33. $u = f(v)$, where $v = g(w, x, y)$, $w = h(z)$, $x = p(t, z)$, and $y = q(t, z)$. Find $\partial u/\partial z$.

34. $u = f(v, w, x)$, where $v = g(r, s, t)$, $w = h(r, s, t)$, $x = p(r, s, t)$, and $r = F(z)$. Find $\partial u/\partial z$.

35–40. Implicit differentiation *Use Theorem 15.9 to evaluate dy/dx. Assume each equation implicitly defines y as a differentiable function of x.*

35. $x^2 - 2y^2 - 1 = 0$

36. $x^3 + 3xy^2 - y^5 = 0$

37. $2 \sin xy = 1$

38. $ye^{xy} - 2 = 0$

39. $\sqrt{x^2 + 2xy + y^4} = 3$

40. $y \ln(x^2 + y^2 + 4) = 3$

41–42. Fluid flow *The x- and y-components of a fluid moving in two dimensions are given by the following functions u and v. The speed of the fluid at (x, y) is $s(x, y) = \sqrt{u(x, y)^2 + v(x, y)^2}$. Use the Chain Rule to find $\partial s/\partial x$ and $\partial s/\partial y$.*

41. $u(x, y) = 2y$ and $v(x, y) = -2x$; $x \geq 0$ and $y \geq 0$

42. $u(x, y) = x(1 - x)(1 - 2y)$ and $v(x, y) = y(y - 1)(1 - 2x)$; $0 \leq x \leq 1, 0 \leq y \leq 1$

43–48. Second derivatives *For the following sets of variables, find all the relevant second derivatives. In all cases, first find general expressions for the second derivatives and then substitute variables at the last step.*

43. $f(x, y) = x^2y$, where $x = s + t$ and $y = s - t$

44. $f(x, y) = x^2y - xy^2$, where $x = st$ and $y = s/t$

45. $f(x, y) = y/x$, where $x = s^2 + t^2$ and $y = s^2 - t^2$

46. $f(x, y) = e^{x-y}$, where $x = s^2$ and $y = 3t^2$

47. $f(x, y, z) = xy + xz - yz$, where $x = s^2 - 2s$, $y = 2/s^2$, and $z = 3s^2 - 2$

48. $f(x, y) = xy$, where $x = s + 2t - u$ and $y = s + 2t + u$

49. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample. Assume all partial derivatives exist.

a. If $z = (x + y)\sin xy$, where x and y are functions of s, then $\dfrac{\partial z}{\partial s} = \dfrac{dz}{dx}\dfrac{dx}{ds}$.

b. Given that $w = f(x(s, t), y(s, t), z(s, t))$, the rate of change of w with respect to t is dw/dt.

50–54. Derivative practice *Find the indicated derivative for the following functions.*

50. $\partial z/\partial p$, where $z = x/y$, $x = p + q$, and $y = p - q$

51. dw/dt, where $w = xyz$, $x = 2t^4$, $y = 3t^{-1}$, and $z = 4t^{-3}$

52. $\partial w/\partial x$, where $w = \cos z - \cos x \cos y + \sin x \sin y$, and $z = x + y$

53. $\dfrac{\partial z}{\partial x}$, where $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = 1$

54. $\partial z/\partial x$, where $xy - z = 1$

55. Change on a line Suppose $w = f(x, y, z)$ and ℓ is the line $\mathbf{r}(t) = \langle at, bt, ct\rangle$, for $-\infty < t < \infty$.

a. Find $w'(t)$ on ℓ (in terms of a, b, c, w_x, w_y, and w_z).
b. Apply part (a) to find $w'(t)$ when $f(x, y, z) = xyz$.
c. Apply part (a) to find $w'(t)$ when $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$.
d. For a general twice differentiable function $w = f(x, y, z)$, find $w''(t)$.

56. Implicit differentiation rule with three variables Assume $F(x, y, z(x, y)) = 0$ implicitly defines z as a differentiable function of x and y. Extend Theorem 15.9 to show that

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \quad\text{and}\quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}.$$

57–59. Implicit differentiation with three variables *Use the result of Exercise 56 to evaluate $\partial z/\partial x$ and $\partial z/\partial y$ for the following relations.*

57. $xy + xz + yz = 3$

58. $x^2 + 2y^2 - 3z^2 = 1$

59. $xyz + x + y - z = 0$

60. More than one way Let $e^{xyz} = 2$. Find z_x and z_y in three ways (and check for agreement).

a. Use the result of Exercise 56.
b. Take logarithms of both sides and differentiate $xyz = \ln 2$.
c. Solve for z and differentiate $z = \dfrac{\ln 2}{xy}$.

61–64. Walking on a surface *Consider the following surfaces specified in the form $z = f(x, y)$ and the oriented curve C in the xy-plane.*

a. In each case, find $z'(t)$.
b. Imagine that you are walking on the surface directly above the curve C in the direction of positive orientation. Find the values of t for which you are walking uphill (that is, z is increasing).

61. $z = x^2 + 4y^2 + 1$, C: $x = \cos t$, $y = \sin t$; $0 \le t \le 2\pi$

62. $z = 4x^2 - y^2 + 1$, C: $x = \cos t$, $y = \sin t$; $0 \le t \le 2\pi$

63. $z = \sqrt{1 - x^2 - y^2}$, C: $x = e^{-t}$, $y = e^{-t}$; $t \ge \dfrac{1}{2}\ln 2$

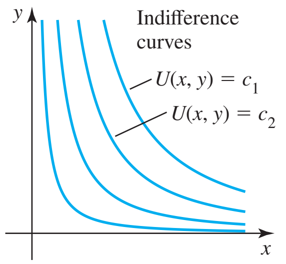
64. $z = 2x^2 + y^2 + 1$, C: $x = 1 + \cos t$, $y = \sin t$; $0 \le t \le 2\pi$

65. Conservation of energy A projectile with mass m is launched into the air on a parabolic trajectory. For $t \ge 0$, its horizontal and vertical coordinates are $x(t) = u_0 t$ and $y(t) = -\dfrac{1}{2}gt^2 + v_0 t$, respectively, where u_0 is the initial horizontal velocity, v_0 is the initial vertical velocity, and g is the acceleration due to gravity. Recalling that $u(t) = x'(t)$ and $v(t) = y'(t)$ are the components of the velocity, the energy of the projectile (kinetic plus potential) is

$$E(t) = \frac{1}{2}m(u^2 + v^2) + mgy.$$

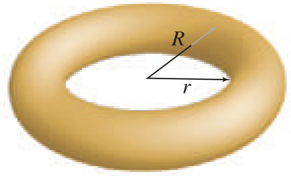
Use the Chain Rule to compute $E'(t)$ and show that $E'(t) = 0$, for all $t \ge 0$. Interpret the result.

66. Utility functions in economics Economists use *utility functions* to describe consumers' relative preference for two or more commodities (for example, vanilla vs. chocolate ice cream or leisure time vs. material goods). The Cobb-Douglas family of utility functions has the form $U(x, y) = x^a y^{1-a}$, where x and y are the amounts of two commodities and $0 < a < 1$ is a parameter. Level curves on which the utility function is constant are called *indifference curves*; the preference is the same for all combinations of x and y along an indifference curve (see figure).



a. The marginal utilities of the commodities x and y are defined to be $\partial U/\partial x$ and $\partial U/\partial y$, respectively. Compute the marginal utilities for the utility function $U(x, y) = x^a y^{1-a}$.
b. The marginal rate of substitution (MRS) is the slope of the indifference curve at the point (x, y). Use the Chain Rule to show that for $U(x, y) = x^a y^{1-a}$, the MRS is $-\dfrac{a}{1-a}\dfrac{y}{x}$.
c. Find the MRS for the utility function $U(x, y) = x^{0.4}y^{0.6}$ at $(x, y) = (8, 12)$.

67. Constant volume tori The volume of a solid torus is given by $V = \dfrac{\pi^2}{4}(R + r)(R - r)^2$, where r and R are the inner and outer radii and $R > r$ (see figure).



a. If R and r increase at the same rate, does the volume of the torus increase, decrease, or remain constant?
b. If R and r decrease at the same rate, does the volume of the torus increase, decrease, or remain constant?

68. Body surface area One of several empirical formulas that relates the surface area S of a human body to the height h and weight w of the body is the Mosteller formula $S(h, w) = \frac{1}{60}\sqrt{hw}$, where h is measured in cm, w is measured in kg, and S is measured in square meters. Suppose h and w are functions of t.

a. Find $S'(t)$.
b. Show that the condition under which the surface area remains constant as h and w change is $wh'(t) + hw'(t) = 0$.
c. Show that part (b) implies that for constant surface area, h and w must be inversely related; that is, $h = C/w$, where C is a constant.

69. The Ideal Gas Law The pressure, temperature, and volume of an ideal gas are related by $PV = kT$, where $k > 0$ is a constant. Any two of the variables may be considered independent, which determines the dependent variable.

a. Use implicit differentiation to compute the partial derivatives $\frac{\partial P}{\partial V}$, $\frac{\partial T}{\partial P}$, and $\frac{\partial V}{\partial T}$.
b. Show that $\frac{\partial P}{\partial V}\frac{\partial T}{\partial P}\frac{\partial V}{\partial T} = -1$. (See Exercise 75 for a generalization.)

70. Variable density The density of a thin circular plate of radius 2 is given by $\rho(x, y) = 4 + xy$. The edge of the plate is described by the parametric equations $x = 2\cos t$, $y = 2\sin t$, for $0 \le t \le 2\pi$.

a. Find the rate of change of the density with respect to t on the edge of the plate.
b. At what point(s) on the edge of the plate is the density a maximum?

T **71. Spiral through a domain** Suppose you follow the helical path $C: x = \cos t, y = \sin t, z = t$, for $t \ge 0$, through the domain of the function $w = f(x, y, z) = \frac{xyz}{z^2 + 1}$.

a. Find $w'(t)$ along C.
b. Estimate the point (x, y, z) on C at which w has its maximum value.

Explorations and Challenges

72. Change of coordinates Recall that Cartesian and polar coordinates are related through the transformation equations

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \quad \text{or} \quad \begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = y/x. \end{cases}$$

a. Evaluate the partial derivatives x_r, y_r, x_θ, and y_θ.
b. Evaluate the partial derivatives r_x, r_y, θ_x, and θ_y.
c. For a function $z = f(x, y)$, find z_r and z_θ, where x and y are expressed in terms of r and θ.
d. For a function $z = g(r, \theta)$, find z_x and z_y, where r and θ are expressed in terms of x and y.
e. Show that $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$.

73. Change of coordinates continued An important derivative operation in many applications is called the Laplacian; in Cartesian coordinates, for $z = f(x, y)$, the Laplacian is $z_{xx} + z_{yy}$. Determine the Laplacian in polar coordinates using the following steps.

a. Begin with $z = g(r, \theta)$ and write z_x and z_y in terms of polar coordinates (see Exercise 72).
b. Use the Chain Rule to find $z_{xx} = \frac{\partial}{\partial x}(z_x)$. There should be two major terms, which, when expanded and simplified, result in five terms.
c. Use the Chain Rule to find $z_{yy} = \frac{\partial}{\partial y}(z_y)$. There should be two major terms, which, when expanded and simplified, result in five terms.
d. Combine parts (b) and (c) to show that

$$z_{xx} + z_{yy} = z_{rr} + \frac{1}{r}z_r + \frac{1}{r^2}z_{\theta\theta}.$$

74. Geometry of implicit differentiation Suppose x and y are related by the equation $F(x, y) = 0$. Interpret the solution of this equation as the set of points (x, y) that lie on the intersection of the surface $z = F(x, y)$ with the xy-plane $(z = 0)$.

a. Make a sketch of a surface and its intersection with the xy-plane. Give a geometric interpretation of the result that $\frac{dy}{dx} = -\frac{F_x}{F_y}$.
b. Explain geometrically what happens at points where $F_y = 0$.

75. General three-variable relationship In the implicit relationship $F(x, y, z) = 0$, any two of the variables may be considered independent, which then determines the dependent variable. To avoid confusion, we may use a subscript to indicate which variable is held fixed in a derivative calculation; for example, $\left(\frac{\partial z}{\partial x}\right)_y$ means that y is held fixed in taking the partial derivative of z with respect to x. (In this context, the subscript does *not* mean a derivative.)

a. Differentiate $F(x, y, z) = 0$ with respect to x, holding y fixed, to show that $\left(\frac{\partial z}{\partial x}\right)_y = -\frac{F_x}{F_z}$.
b. As in part (a), find $\left(\frac{\partial y}{\partial z}\right)_x$ and $\left(\frac{\partial x}{\partial y}\right)_z$.
c. Show that $\left(\frac{\partial z}{\partial x}\right)_y\left(\frac{\partial y}{\partial z}\right)_x\left(\frac{\partial x}{\partial y}\right)_z = -1$.
d. Find the relationship analogous to part (c) for the case $F(w, x, y, z) = 0$.

76. Second derivative Let $f(x, y) = 0$ define y as a twice differentiable function of x.

a. Show that $y''(x) = -\frac{f_{xx}f_y^2 - 2f_xf_yf_{xy} + f_{yy}f_x^2}{f_y^3}$.
b. Verify part (a) using the function $f(x, y) = xy - 1$.

77. Subtleties of the Chain Rule Let $w = f(x, y, z) = 2x + 3y + 4z$, which is defined for all (x, y, z) in $\mathbb{R}^3$. Suppose we are interested in the partial derivative w_x on a subset of $\mathbb{R}^3$, such as the plane P given by $z = 4x - 2y$. The point to be made is that the result is not unique unless we specify which variables are considered independent.

a. We could proceed as follows. On the plane P, consider x and y as the independent variables, which means z depends on x and y, so we write $w = f(x, y, z(x, y))$. Differentiate with respect to x, holding y fixed, to show that $\left(\frac{\partial w}{\partial x}\right)_y = 18$, where the subscript y indicates that y is held fixed.

b. Alternatively, on the plane P, we could consider x and z as the independent variables, which means y depends on x and z, so we write $w = f(x, y(x, z), z)$ and differentiate with respect to x, holding z fixed. Show that $\left(\frac{\partial w}{\partial x}\right)_z = 8$, where the subscript z indicates that z is held fixed.

c. Make a sketch of the plane $z = 4x - 2y$ and interpret the results of parts (a) and (b) geometrically.

d. Repeat the arguments of parts (a) and (b) to find $\left(\frac{\partial w}{\partial y}\right)_x$, $\left(\frac{\partial w}{\partial y}\right)_z$, $\left(\frac{\partial w}{\partial z}\right)_x$, and $\left(\frac{\partial w}{\partial z}\right)_y$.

QUICK CHECK ANSWERS

1. If $z = f(x(t))$, then $\frac{\partial z}{\partial y} = 0$, and the original Chain Rule results. **2.** $\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$ **3.** One dependent variable, four intermediate variables, and three independent variables **4.** $\frac{dy}{dx} = \frac{2x + y}{3y^2 - x}$; in this case, using $\frac{dy}{dx} = -\frac{F_x}{F_y}$ is more efficient. ◀

15.5 Directional Derivatives and the Gradient

Partial derivatives tell us a lot about the rate of change of a function on its domain. However, they do not *directly* answer some important questions. For example, suppose you are standing at a point $(a, b, f(a, b))$ on the surface $z = f(x, y)$. The partial derivatives f_x and f_y tell you the rate of change (or slope) of the surface at that point in the directions parallel to the x-axis and y-axis, respectively. But you could walk in an infinite number of directions from that point and find a different rate of change in every direction. With this observation in mind, we pose several questions.

- Suppose you are standing on a surface and you walk in a direction *other* than a coordinate direction—say, northwest or south-southeast. What is the rate of change of the function in such a direction?
- Suppose you are standing on a surface and you release a ball at your feet and let it roll. In which direction will it roll?
- If you are hiking up a mountain, in what direction should you walk after each step if you want to follow the steepest path?

These questions are answered in this section by introducing the *directional derivative*, followed by one of the central concepts of calculus—the *gradient*.

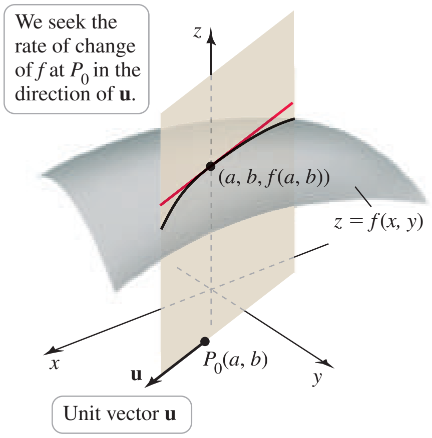


Figure 15.45

Directional Derivatives

Let $(a, b, f(a, b))$ be a point on the surface $z = f(x, y)$ and let **u** be a unit vector in the xy-plane (Figure 15.45). Our aim is to find the rate of change of f in the direction **u** at $P_0(a, b)$. In general, this rate of change is neither $f_x(a, b)$ nor $f_y(a, b)$ (unless $\mathbf{u} = \langle 1, 0 \rangle$ or $\mathbf{u} = \langle 0, 1 \rangle$), but it turns out to be a combination of $f_x(a, b)$ and $f_y(a, b)$.

Figure 15.46a shows the unit vector $\mathbf{u} = \langle u_1, u_2 \rangle$; its x- and y-components are u_1 and u_2, respectively. The derivative we seek must be computed along the line ℓ in the xy-plane through P_0 in the direction of **u**. A neighboring point P, which is h units from P_0 along ℓ, has coordinates $P(a + hu_1, b + hu_2)$ (Figure 15.46b).

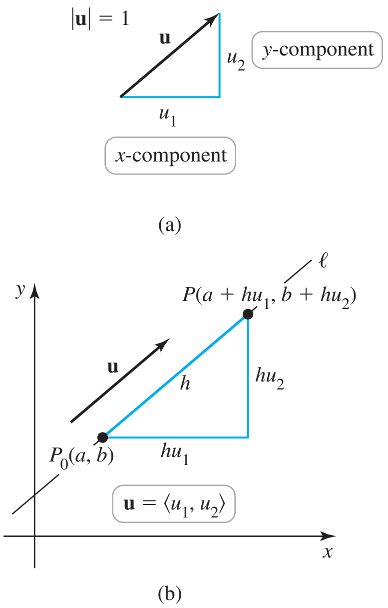


Figure 15.46

Now imagine the plane Q perpendicular to the xy-plane, containing ℓ. This plane cuts the surface $z = f(x, y)$ in a curve C. Consider two points on C corresponding to P_0 and P; they have z-coordinates $f(a, b)$ and $f(a + hu_1, b + hu_2)$ (Figure 15.47). The slope of the secant line between these points is

$$\frac{f(a + hu_1, b + hu_2) - f(a, b)}{h}.$$

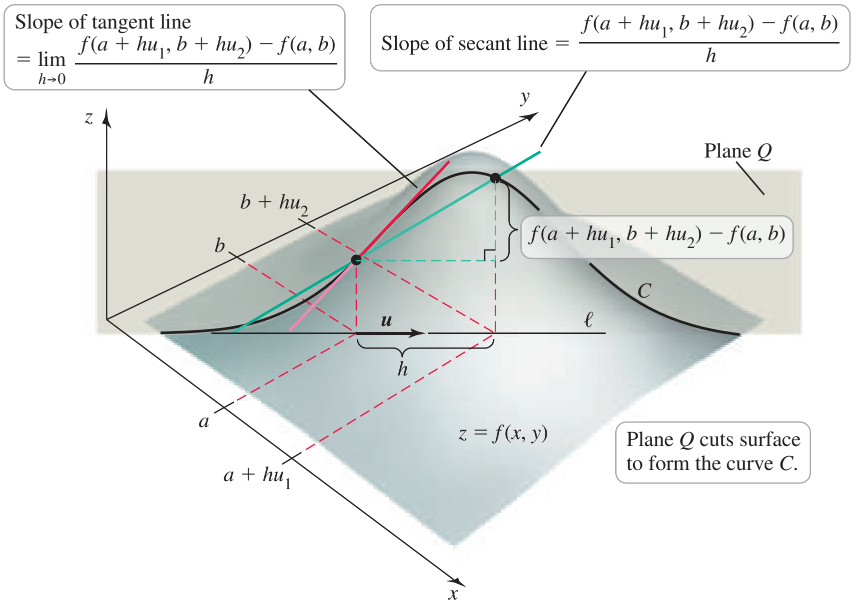


Figure 15.47

The derivative of f in the direction of $\mathbf{u}$ is obtained by letting $h \to 0$; when the limit exists, it is called the *directional derivative of f at* (a, b) *in the direction of* $\mathbf{u}$. It gives the slope of the line tangent to the curve C in the plane Q.

➤ The definition of the directional derivative looks like the definition of the ordinary derivative if we write it as

$$\lim_{P \to P_0} \frac{f(P) - f(P_0)}{|P - P_0|},$$

where P approaches P_0 along the line ℓ.

DEFINITION Directional Derivative

Let f be differentiable at (a, b) and let $\mathbf{u} = \langle u_1, u_2 \rangle$ be a unit vector in the xy-plane. The **directional derivative of f at (a, b) in the direction of u** is

$$D_{\mathbf{u}} f(a, b) = \lim_{h \to 0} \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h},$$

provided the limit exists.

As motivation, it is instructive to see how the directional derivative includes the ordinary derivative in one variable. Setting $u_2 = 0$ in the definition of the directional derivative and ignoring the second variable gives the rate of change of f in the x-direction. The directional derivative then becomes

$$\lim_{h \to 0} \frac{f(a + hu_1) - f(a)}{h}.$$

Multiplying the numerator and denominator of this quotient by u_1, we have

$$u_1 \underbrace{\lim_{h \to 0} \frac{f(a + hu_1) - f(a)}{hu_1}}_{f'(a)} = u_1 f'(a).$$

Only because $\mathbf{u}$ is a unit vector and $u_1 = 1$ does the directional derivative reduce to the ordinary derivative $f'(a)$ in the x-direction. A similar argument may be used in the y-direction. Choosing $\mathbf{u}$ to be a unit vector gives the simplest formulas for the directional derivative.

As with ordinary derivatives, we would prefer to evaluate directional derivatives without taking limits. Fortunately, there is an easy way to express a directional derivative in terms of partial derivatives.

QUICK CHECK 1 Explain why, when $\mathbf{u} = \langle 1, 0 \rangle$ in the definition of the directional derivative, the result is $f_x(a, b)$, and when $\mathbf{u} = \langle 0, 1 \rangle$, the result is $f_y(a, b)$. ◄

The key is to define a function that is equal to f along the line ℓ through (a, b) in the direction of the unit vector $\mathbf{u} = \langle u_1, u_2 \rangle$. The points on ℓ satisfy the parametric equations

$$x = a + su_1 \quad \text{and} \quad y = b + su_2,$$

➤ To see that s is an arc length parameter, note that the line ℓ may be written in the form

$$\mathbf{r}(s) = \langle a + su_1, b + su_2 \rangle.$$

Therefore, $\mathbf{r}'(s) = \langle u_1, u_2 \rangle$ and $|\mathbf{r}'(s)| = 1$. It follows by the discussion in Section 14.4 that s is an arc length parameter.

➤ Note that $g'(s)$ does not correctly measure the slope of f along ℓ unless $\mathbf{u}$ is a unit vector.

where $-\infty < s < \infty$. Because $\mathbf{u}$ is a unit vector, the parameter s corresponds to arc length. As s increases, the points (x, y) move along ℓ in the direction of $\mathbf{u}$ with $s = 0$ corresponding to (a, b). Now we define the function

$$g(s) = f(\underbrace{a + su_1}_{x}, \underbrace{b + su_2}_{y}),$$

which gives the values of f along ℓ. The derivative of f along ℓ is $g'(s)$ (see margin note), and when evaluated at $s = 0$, it is the directional derivative of f at (a, b); that is, $g'(0) = D_{\mathbf{u}} f(a, b)$.

Noting that $\dfrac{dx}{ds} = u_1$ and $\dfrac{dy}{ds} = u_2$, we apply the Chain Rule to find that

$$\begin{aligned} D_{\mathbf{u}} f(a, b) = g'(0) &= \left(\frac{\partial f}{\partial x}\underbrace{\frac{dx}{ds}}_{u_1} + \frac{\partial f}{\partial y}\underbrace{\frac{dy}{ds}}_{u_2}\right)\Bigg|_{s=0} && \text{Chain Rule} \\ &= f_x(a, b)u_1 + f_y(a, b)u_2 && s = 0 \text{ corresponds to } (a, b). \\ &= \langle f_x(a, b), f_y(a, b) \rangle \cdot \langle u_1, u_2 \rangle. && \text{Identify dot product.} \end{aligned}$$

We see that the directional derivative is a weighted average of the partial derivatives $f_x(a, b)$ and $f_y(a, b)$, with the components of $\mathbf{u}$ serving as the weights. In other words, knowing the slope of the surface in the x- and y-directions allows us to find the slope in any direction. Notice that the directional derivative can be written as a dot product, which provides a practical formula for computing directional derivatives.

QUICK CHECK 2 In the parametric description $x = a + su_1$ and $y = b + su_2$, where $\mathbf{u} = \langle u_1, u_2 \rangle$ is a unit vector, show that any positive change Δs in s produces a line segment of length Δs. ◀

THEOREM 15.10 Directional Derivative
Let f be differentiable at (a, b) and let $\mathbf{u} = \langle u_1, u_2 \rangle$ be a unit vector in the xy-plane. The **directional derivative of f at (a, b) in the direction of $\mathbf{u}$** is

$$D_{\mathbf{u}} f(a, b) = \langle f_x(a, b), f_y(a, b) \rangle \cdot \langle u_1, u_2 \rangle.$$

EXAMPLE 1 Computing directional derivatives Consider the paraboloid $z = f(x, y) = \frac{1}{4}(x^2 + 2y^2) + 2$. Let P_0 be the point $(3, 2)$ and consider the unit vectors

$$\mathbf{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle \quad \text{and} \quad \mathbf{v} = \left\langle \frac{1}{2}, -\frac{\sqrt{3}}{2} \right\rangle.$$

a. Find the directional derivative of f at P_0 in the directions of $\mathbf{u}$ and $\mathbf{v}$.

b. Graph the surface and interpret the directional derivatives.

SOLUTION

a. We see that $f_x = x/2$ and $f_y = y$; evaluated at $(3, 2)$, we have $f_x(3, 2) = 3/2$ and $f_y(3, 2) = 2$. The directional derivatives in the directions $\mathbf{u}$ and $\mathbf{v}$ are

$$\begin{aligned} D_{\mathbf{u}} f(3, 2) &= \langle f_x(3, 2), f_y(3, 2) \rangle \cdot \langle u_1, u_2 \rangle \\ &= \frac{3}{2}\cdot\frac{1}{\sqrt{2}} + 2\cdot\frac{1}{\sqrt{2}} = \frac{7}{2\sqrt{2}} \approx 2.47 \text{ and} \\ D_{\mathbf{v}} f(3, 2) &= \langle f_x(3, 2), f_y(3, 2) \rangle \cdot \langle v_1, v_2 \rangle \\ &= \frac{3}{2}\cdot\frac{1}{2} + 2\left(-\frac{\sqrt{3}}{2}\right) = \frac{3}{4} - \sqrt{3} \approx -0.98. \end{aligned}$$

➤ It is understood that the line tangent to C in the direction of $\mathbf{u}$ lies in the vertical plane Q containing $\mathbf{u}$.

b. In the direction of $\mathbf{u}$, the directional derivative is approximately 2.47. Because it is positive, the function is increasing at $(3, 2)$ in this direction. Equivalently, if Q is the vertical plane containing $\mathbf{u}$, and C is the curve along which the surface intersects Q, then the slope of the line tangent to C is approximately 2.47 (**Figure 15.48a**). In the direction of $\mathbf{v}$, the directional derivative is approximately -0.98. Because it is negative, the function is decreasing in this direction. In this case, the vertical plane Q contains $\mathbf{v}$ and again C is the curve along which the surface intersects Q; the slope of the line tangent to C is approximately -0.98 (**Figure 15.48b**).

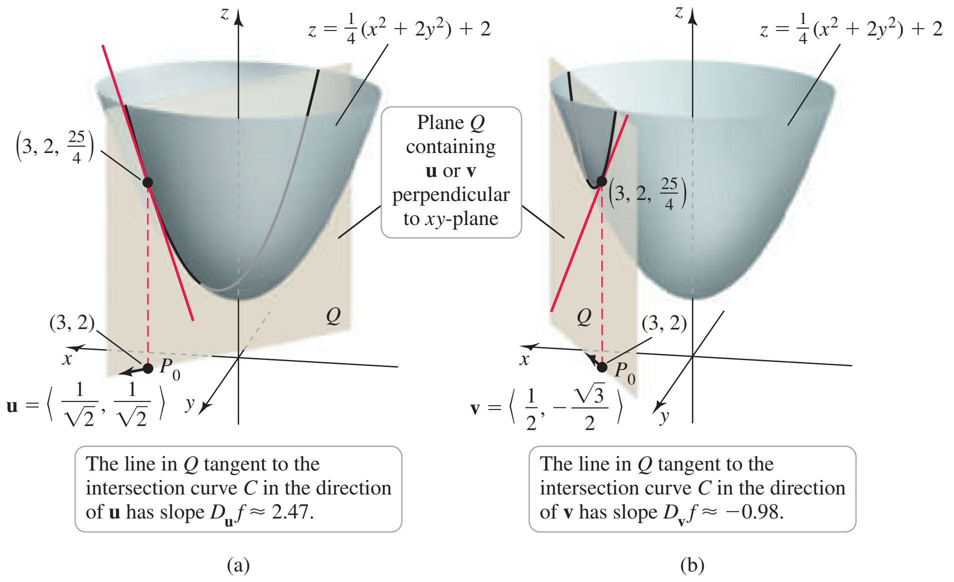


Figure 15.48

Related Exercises 11–12 ◄

QUICK CHECK 3 In Example 1, evaluate $D_{-\mathbf{u}}\, f(3, 2)$ and $D_{-\mathbf{v}}\, f(3, 2)$. ◄

The Gradient Vector

We have seen that the directional derivative can be written as a dot product: $D_{\mathbf{u}}\, f(a, b) = \langle f_x(a, b), f_y(a, b) \rangle \cdot \langle u_1, u_2 \rangle$. The vector $\langle f_x(a, b), f_y(a, b) \rangle$ that appears in the dot product is important in its own right and is called the *gradient* of f.

➤ Recall that the unit coordinate vectors in $\mathbb{R}^2$ are $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$. The gradient of f is also written grad f, read *grad f*.

DEFINITION Gradient (Two Dimensions)

Let f be differentiable at the point (x, y). The **gradient** of f at (x, y) is the vector-valued function

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = f_x(x, y)\,\mathbf{i} + f_y(x, y)\,\mathbf{j}.$$

With the definition of the gradient, the directional derivative of f at (a, b) in the direction of the unit vector **u** can be written

$$D_{\mathbf{u}}\, f(a, b) = \nabla f(a, b) \cdot \mathbf{u}.$$

The gradient satisfies sum, product, and quotient rules analogous to those for ordinary derivatives (Exercise 85).

EXAMPLE 2 **Computing gradients** Find $\nabla f(3, 2)$ for $f(x, y) = x^2 + 2xy - y^3$.

SOLUTION Computing $f_x = 2x + 2y$ and $f_y = 2x - 3y^2$, we have

$$\nabla f(x, y) = \langle 2(x + y), 2x - 3y^2 \rangle = 2(x + y)\,\mathbf{i} + (2x - 3y^2)\,\mathbf{j}.$$

Substituting $x = 3$ and $y = 2$ gives

$$\nabla f(3, 2) = \langle 10, -6 \rangle = 10\mathbf{i} - 6\mathbf{j}.$$

Related Exercises 13–15 ◄

EXAMPLE 3 **Computing directional derivatives with gradients** Let

$$f(x, y) = 3 - \frac{x^2}{10} + \frac{xy^2}{10}.$$

a. Compute $\nabla f(3, -1)$.

b. Compute $D_{\mathbf{u}}\, f(3, -1)$, where $\mathbf{u} = \left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle$.

c. Compute the directional derivative of f at $(3, -1)$ in the direction of the vector $\langle 3, 4 \rangle$.

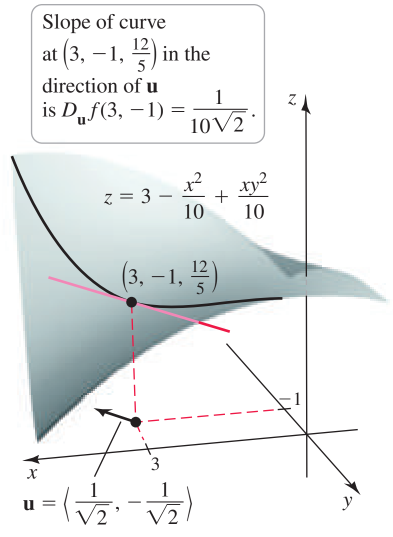


Figure 15.49

SOLUTION

a. Note that $f_x = -x/5 + y^2/10$ and $f_y = xy/5$. Therefore,

$$\nabla f(3, -1) = \left\langle -\frac{x}{5} + \frac{y^2}{10}, \frac{xy}{5} \right\rangle \bigg|_{(3,-1)} = \left\langle -\frac{1}{2}, -\frac{3}{5} \right\rangle.$$

b. Before computing the directional derivative, it is important to verify that **u** is a unit vector (in this case, it is). The required directional derivative is

$$D_{\mathbf{u}} f(3, -1) = \nabla f(3, -1) \cdot \mathbf{u} = \left\langle -\frac{1}{2}, -\frac{3}{5} \right\rangle \cdot \left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle = \frac{1}{10\sqrt{2}}.$$

Figure 15.49 shows the line tangent to the intersection curve in the plane corresponding to **u**; its slope is $D_{\mathbf{u}} f(3, -1)$.

c. In this case, the direction is given in terms of a nonunit vector. The vector $\langle 3, 4 \rangle$ has length 5, so the unit vector in the direction of $\langle 3, 4 \rangle$ is $\mathbf{u} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$. The directional derivative at $(3, -1)$ in the direction of **u** is

$$D_{\mathbf{u}} f(3, -1) = \nabla f(3, -1) \cdot \mathbf{u} = \left\langle -\frac{1}{2}, -\frac{3}{5} \right\rangle \cdot \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle = -\frac{39}{50},$$

which gives the slope of the surface in the direction of $\langle 3, 4 \rangle$ at $(3, -1)$.

Related Exercises 22, 27 ◄

Interpretations of the Gradient

The gradient is important not only in calculating directional derivatives; it plays many other roles in multivariable calculus. Our present goal is to develop some intuition about the meaning of the gradient.

➤ Recall that $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos \theta$, where θ is the angle between **u** and **v**.

We have seen that the directional derivative of f at (a, b) in the direction of the unit vector **u** is $D_{\mathbf{u}} f(a, b) = \nabla f(a, b) \cdot \mathbf{u}$. Using properties of the dot product, we have

$$\begin{aligned} D_{\mathbf{u}} f(a, b) &= \nabla f(a, b) \cdot \mathbf{u} \\ &= |\nabla f(a, b)||\mathbf{u}| \cos \theta \\ &= |\nabla f(a, b)| \cos \theta, \qquad |\mathbf{u}| = 1 \end{aligned}$$

➤ It is important to remember (but easy to forget) that $\nabla f(a, b)$ lies in the same plane as the domain of f.

where θ is the angle between $\nabla f(a, b)$ and **u**. It follows that $D_{\mathbf{u}} f(a, b)$ has its maximum value when $\cos \theta = 1$, which corresponds to $\theta = 0$. Therefore, $D_{\mathbf{u}} f(a, b)$ has its maximum value and f has its greatest rate of *increase* when $\nabla f(a, b)$ and **u** point in the same direction. Note that when $\cos \theta = 1$, the actual rate of increase is $D_{\mathbf{u}} f(a, b) = |\nabla f(a, b)|$ (Figure 15.50).

Similarly, when $\theta = \pi$, we have $\cos \theta = -1$, and f has its greatest rate of *decrease* when $\nabla f(a, b)$ and **u** point in opposite directions. The actual rate of decrease is $D_{\mathbf{u}} f(a, b) = -|\nabla f(a, b)|$. These observations are summarized as follows: The gradient $\nabla f(a, b)$ points in the *direction of steepest ascent* at (a, b), while $-\nabla f(a, b)$ points in the *direction of steepest descent*.

Notice that $D_{\mathbf{u}} f(a, b) = 0$ when the angle between $\nabla f(a, b)$ and **u** is $\pi/2$, which means $\nabla f(a, b)$ and **u** are orthogonal (Figure 15.50). These observations justify the following theorem.

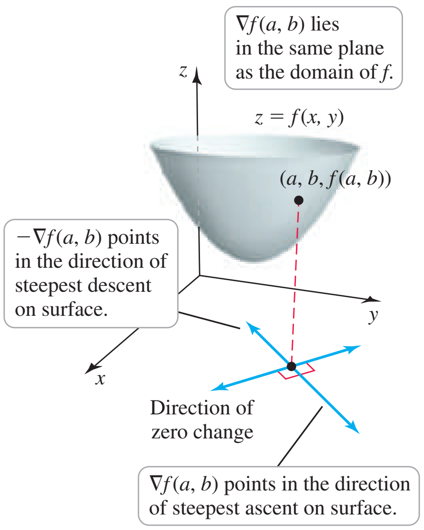


Figure 15.50

THEOREM 15.11 Directions of Change

Let f be differentiable at (a, b) with $\nabla f(a, b) \neq \mathbf{0}$.

1. f has its maximum rate of increase at (a, b) in the direction of the gradient $\nabla f(a, b)$. The rate of change in this direction is $|\nabla f(a, b)|$.
2. f has its maximum rate of decrease at (a, b) in the direction of $-\nabla f(a, b)$. The rate of change in this direction is $-|\nabla f(a, b)|$.
3. The directional derivative is zero in any direction orthogonal to $\nabla f(a, b)$.

EXAMPLE 4 **Steepest ascent and descent** Consider the bowl-shaped paraboloid $z = f(x, y) = 4 + x^2 + 3y^2$.

a. If you are located on the paraboloid at the point $\left(2, -\frac{1}{2}, \frac{35}{4}\right)$, in which direction should you move in order to *ascend* on the surface at the maximum rate? What is the rate of change?

b. If you are located at the point $\left(2, -\frac{1}{2}, \frac{35}{4}\right)$, in which direction should you move in order to *descend* on the surface at the maximum rate? What is the rate of change?

c. At the point $(3, 1, 16)$, in what direction(s) is there no change in the function values?

SOLUTION

a. At the point $\left(2, -\frac{1}{2}\right)$, the value of the gradient is

$$\nabla f\left(2, -\tfrac{1}{2}\right) = \langle 2x, 6y \rangle \big|_{(2,-1/2)} = \langle 4, -3 \rangle.$$

Therefore, the direction of steepest ascent in the xy-plane is in the direction of the gradient vector $\langle 4, -3 \rangle$ (or $\mathbf{u} = \frac{1}{5}\langle 4, -3 \rangle$, as a unit vector). The rate of change is $\left|\nabla f\left(2, -\frac{1}{2}\right)\right| = |\langle 4, -3 \rangle| = 5$ (Figure 15.51a).

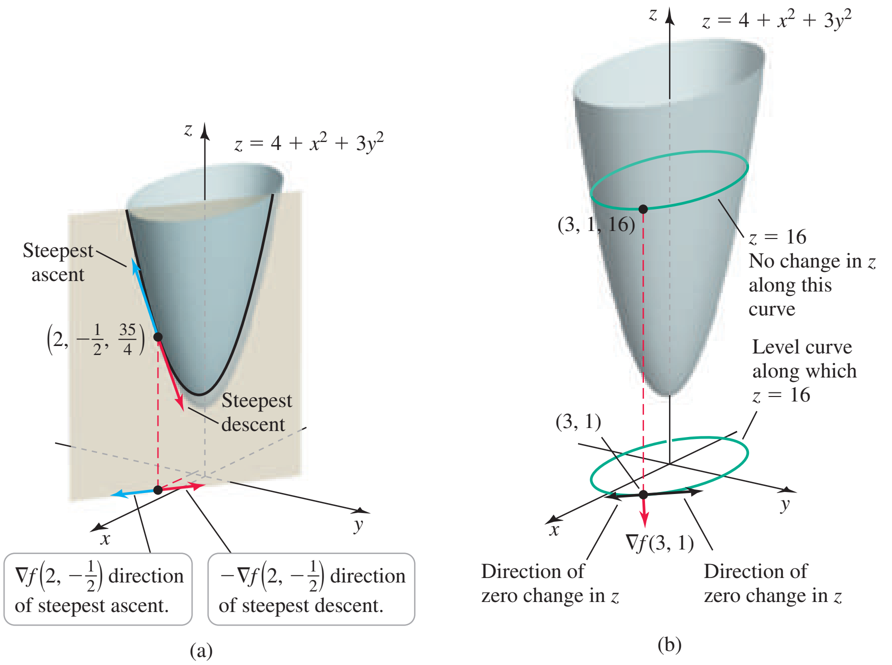


Figure 15.51

b. The direction of steepest *descent* is the direction of $-\nabla f\left(2, -\frac{1}{2}\right) = \langle -4, 3 \rangle$ (or $\mathbf{u} = \frac{1}{5}\langle -4, 3 \rangle$, as a unit vector). The rate of change is $-\left|\nabla f\left(2, -\frac{1}{2}\right)\right| = -5$.

c. At the point $(3, 1)$, the value of the gradient is $\nabla f(3, 1) = \langle 6, 6 \rangle$. The function has zero change if we move in either of the two directions orthogonal to $\langle 6, 6 \rangle$; these two directions are parallel to $\langle 6, -6 \rangle$. In terms of unit vectors, the directions of no change are $\mathbf{u} = \dfrac{1}{\sqrt{2}}\langle -1, 1 \rangle$ and $\mathbf{u} = \dfrac{1}{\sqrt{2}}\langle 1, -1 \rangle$ (Figure 15.51b).

➤ Note that $\langle 6, 6 \rangle$ and $\langle 6, -6 \rangle$ are orthogonal because $\langle 6, 6 \rangle \cdot \langle 6, -6 \rangle = 0$.

Related Exercises 31–32 ◄

EXAMPLE 5 **Interpreting directional derivatives** Consider the function $f(x, y) = 3x^2 - 2y^2$.

a. Compute $\nabla f(x, y)$ and $\nabla f(2, 3)$.

b. Let $\mathbf{u} = \langle \cos \theta, \sin \theta \rangle$ be a unit vector. At $(2, 3)$, for what values of θ (measured relative to the positive x-axis), with $0 \le \theta < 2\pi$, does the directional derivative have its maximum and minimum values? What are those values?

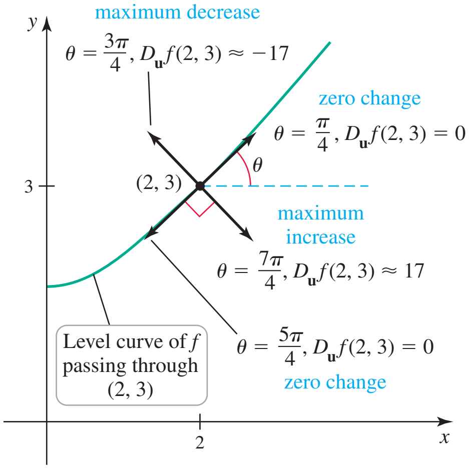


Figure 15.52

SOLUTION

a. The gradient is $\nabla f(x, y) = \langle f_x, f_y \rangle = \langle 6x, -4y \rangle$, and at $(2, 3)$ we have $\nabla f(2, 3) = \langle 12, -12 \rangle$.

b. The gradient $\nabla f(2, 3) = \langle 12, -12 \rangle$ makes an angle of $7\pi/4$ with the positive x-axis. So the maximum rate of change of f occurs in this direction, and that rate of change is $|\nabla f(2, 3)| = |\langle 12, -12 \rangle| = 12\sqrt{2} \approx 17$. The direction of maximum decrease is opposite the direction of the gradient, which corresponds to $\theta = 3\pi/4$. The maximum rate of decrease is the negative of the maximum rate of increase, or $-12\sqrt{2} \approx -17$. The function has zero change in the directions orthogonal to the gradient, which correspond to $\theta = \pi/4$ and $\theta = 5\pi/4$.

Figure 15.52 summarizes these conclusions. Notice that the gradient at $(2, 3)$ appears to be orthogonal to the level curve of f passing through $(2, 3)$. We next see that this is always the case.

Related Exercises 37–38 ◀

The Gradient and Level Curves

Theorem 15.11 states that in any direction orthogonal to the gradient $\nabla f(a, b)$, the function f does not change at (a, b). Recall from Section 15.1 that the curve $f(x, y) = z_0$, where z_0 is a constant, is a *level curve*, on which function values are constant. Combining these two observations, we conclude that the gradient $\nabla f(a, b)$ is orthogonal to the line tangent to the level curve through (a, b).

> **THEOREM 15.12 The Gradient and Level Curves**
> Given a function f differentiable at (a, b), the line tangent to the level curve of f at (a, b) is orthogonal to the gradient $\nabla f(a, b)$, provided $\nabla f(a, b) \neq \mathbf{0}$.

Proof: Consider the function $z = f(x, y)$ and its level curve $f(x, y) = z_0$, where the constant z_0 is chosen so that the curve passes through the point (a, b). Let $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ be a parameterization for the level curve near (a, b) (where it is smooth), and let $\mathbf{r}(t_0)$ correspond to the point (a, b). We now differentiate $f(x, y) = z_0$ with respect to t. The derivative of the right side is 0. Applying the Chain Rule to the left side results in

$$\begin{aligned}\frac{d}{dt}(f(x, y)) &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \\ &= \underbrace{\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle}_{\nabla f(x,y)} \cdot \underbrace{\left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle}_{\mathbf{r}'(t)} \\ &= \nabla f(x, y) \cdot \mathbf{r}'(t).\end{aligned}$$

Substituting $t = t_0$, we have $\nabla f(a, b) \cdot \mathbf{r}'(t_0) = 0$, which implies that $\mathbf{r}'(t_0)$ (the tangent vector at (a, b)) is orthogonal to $\nabla f(a, b)$. Figure 15.53 illustrates the geometry of the theorem. ◀

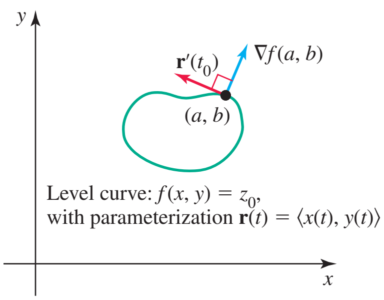


Figure 15.53

QUICK CHECK 4 Draw a circle in the xy-plane centered at the origin and regard it is as a level curve of the surface $z = x^2 + y^2$. At the point (a, a) of the level curve in the xy-plane, the slope of the tangent line is -1. Show that the gradient at (a, a) is orthogonal to the tangent line. ◀

An immediate consequence of Theorem 15.12 is an alternative equation of the tangent line. The curve described by $f(x, y) = z_0$ can be viewed as a level curve in the xy-plane for the surface $z = f(x, y)$. By Theorem 15.12, the line tangent to the curve at (a, b) is orthogonal to $\nabla f(a, b)$. Therefore, if (x, y) is a point on the tangent line, then $\nabla f(a, b) \cdot \langle x - a, y - b \rangle = 0$, which, when simplified, gives an equation of the line tangent to the curve $f(x, y) = z_0$ at (a, b):

$$f_x(a, b)(x - a) + f_y(a, b)(y - b) = 0.$$

EXAMPLE 6 **Gradients and level curves** Consider the upper sheet $z = f(x, y) = \sqrt{1 + 2x^2 + y^2}$ of a hyperboloid of two sheets.

a. Verify that the gradient at $(1, 1)$ is orthogonal to the corresponding level curve at that point.

b. Find an equation of the line tangent to the level curve at $(1, 1)$.

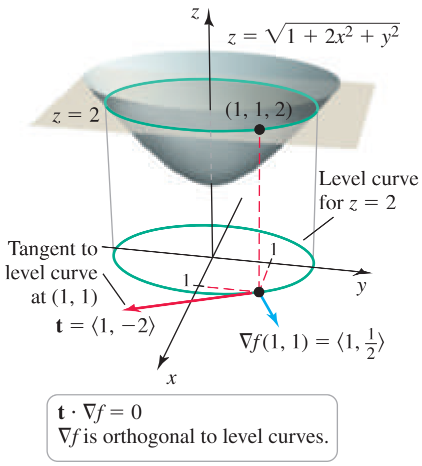


Figure 15.54

➤ The fact that $y' = -2x/y$ may also be obtained using Theorem 15.9: If $F(x, y) = 0$, then $y'(x) = -F_x/F_y$.

SOLUTION

a. You can verify that $(1, 1, 2)$ is on the surface; therefore, $(1, 1)$ is on the level curve corresponding to $z = 2$. Setting $z = 2$ in the equation of the surface and squaring both sides, the equation of the level curve is $4 = 1 + 2x^2 + y^2$, or $2x^2 + y^2 = 3$, which is the equation of an ellipse (Figure 15.54). Differentiating $2x^2 + y^2 = 3$ with respect to x gives $4x + 2yy'(x) = 0$, which implies that the slope of the level curve is $y'(x) = -\dfrac{2x}{y}$. Therefore, at the point $(1, 1)$, the slope of the tangent line is -2. Any vector proportional to $\mathbf{t} = \langle 1, -2 \rangle$ has slope -2 and points in the direction of the tangent line.

We now compute the gradient:

$$\nabla f(x, y) = \langle f_x, f_y \rangle = \left\langle \frac{2x}{\sqrt{1 + 2x^2 + y^2}}, \frac{y}{\sqrt{1 + 2x^2 + y^2}} \right\rangle.$$

It follows that $\nabla f(1, 1) = \left\langle 1, \frac{1}{2} \right\rangle$ (Figure 15.54). The tangent vector $\mathbf{t}$ and the gradient are orthogonal because

$$\mathbf{t} \cdot \nabla f(1, 1) = \langle 1, -2 \rangle \cdot \left\langle 1, \tfrac{1}{2} \right\rangle = 0.$$

b. An equation of the line tangent to the level curve at $(1, 1)$ is

$$\underbrace{f_x(1, 1)}_{1}(x - 1) + \underbrace{f_y(1, 1)}_{\frac{1}{2}}(y - 1) = 0,$$

or $y = -2x + 3$.

Related Exercises 49, 52 ◄

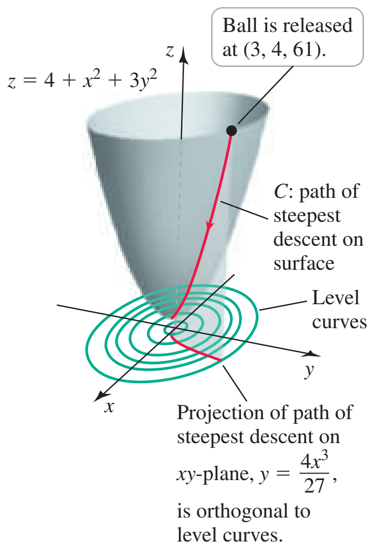


Figure 15.55

EXAMPLE 7 **Path of steepest descent** The paraboloid $z = f(x, y) = 4 + x^2 + 3y^2$ is shown in Figure 15.55. A ball is released at the point $(3, 4, 61)$ on the surface, and it follows the path of steepest descent C to the vertex of the paraboloid.

a. Find an equation of the projection of C in the xy-plane.

b. Find an equation of C on the paraboloid.

SOLUTION

a. The projection of C in the xy-plane points in the direction of $-\nabla f(x, y) = \langle -2x, -6y \rangle$, which means that at the point (x, y), the line tangent to the path has slope $y'(x) = (-6y)/(-2x) = 3y/x$. Therefore, the path in the xy-plane satisfies $y'(x) = 3y/x$ and passes through the initial point $(3, 4)$. You can verify that the solution to this differential equation is $y = 4x^3/27$. Therefore, the projection of the path of steepest descent in the xy-plane is the curve $y = 4x^3/27$. The descent ends at $(0, 0)$, which corresponds to the vertex of the paraboloid (Figure 15.55). At all points of the descent, the projection curve in the xy-plane is orthogonal to the level curves of the paraboloid.

b. To find a parametric description of C, it is easiest to define the parameter $t = x$. Using part (a), we find that

$$y = \frac{4x^3}{27} = \frac{4t^3}{27} \text{ and } z = 4 + x^2 + 3y^2 = 4 + t^2 + \frac{16}{243}t^6.$$

Because $0 \le x \le 3$, the parameter t varies over the interval $0 \le t \le 3$. A parametric description of C is

$$C: \mathbf{r}(t) = \left\langle t, \frac{4t^3}{27}, 4 + t^2 + \frac{16}{243}t^6 \right\rangle, \text{ for } 0 \le t \le 3.$$

With this parameterization, C is traced from $\mathbf{r}(0) = \langle 0, 0, 4 \rangle$ to $\mathbf{r}(3) = \langle 3, 4, 61 \rangle$—in the direction opposite to that of the ball's descent.

Related Exercise 57 ◄

QUICK CHECK 5 Verify that $y = 4x^3/27$ satisfies the equation $y'(x) = 3y/x$, with $y(3) = 4$. ◄

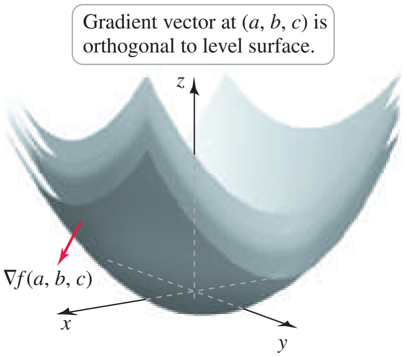


Figure 15.56

The Gradient in Three Dimensions

The directional derivative, the gradient, and the idea of a level curve extend immediately to functions of three variables of the form $w = f(x, y, z)$. The main differences are that the gradient is a vector in $\mathbb{R}^3$, and level curves become *level surfaces* (Section 15.1). Here is how the gradient looks when we step up one dimension.

The easiest way to visualize the surface $w = f(x, y, z)$ is to picture its level surfaces—the surfaces in $\mathbb{R}^3$ on which f has a constant value. The level surfaces are given by the equation $f(x, y, z) = C$, where C is a constant (**Figure 15.56**). The level surfaces *can* be graphed, and they may be viewed as layers of the full four-dimensional surface (like layers of an onion). With this image in mind, we now extend the concepts of directional derivative and gradient to three dimensions.

Given the function $w = f(x, y, z)$, we begin just as we did in the two-variable case and define the directional derivative and the gradient.

DEFINITION Directional Derivative and Gradient in Three Dimensions

Let f be differentiable at (a, b, c) and let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ be a unit vector. The **directional derivative of f at (a, b, c) in the direction of u** is

$$D_{\mathbf{u}} f(a, b, c) = \lim_{h \to 0} \frac{f(a + hu_1, b + hu_2, c + hu_3) - f(a, b, c)}{h},$$

provided this limit exists.

The **gradient** of f at the point (x, y, z) is the vector-valued function

$$\begin{aligned} \nabla f(x, y, z) &= \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle \\ &= f_x(x, y, z)\, \mathbf{i} + f_y(x, y, z)\, \mathbf{j} + f_z(x, y, z)\, \mathbf{k}. \end{aligned}$$

An argument similar to that given in two dimensions leads from the definition of the directional derivative to a computational formula. Given a unit vector $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, the directional derivative of f in the direction of $\mathbf{u}$ at the point (a, b, c) is

$$D_{\mathbf{u}} f(a, b, c) = f_x(a, b, c)\, u_1 + f_y(a, b, c)\, u_2 + f_z(a, b, c)\, u_3.$$

As before, we recognize this expression as a dot product of the vector $\mathbf{u}$ and the vector $\nabla f(a, b, c) = \langle f_x(a, b, c), f_y(a, b, c), f_z(a, b, c) \rangle$, which is the gradient evaluated at (a, b, c). These observations lead to Theorem 15.13, which mirrors Theorems 15.10 and 15.11.

➤ When we introduce the tangent plane in Section 15.6, we can also claim that $\nabla f(a, b, c)$ is orthogonal to the level surface that passes through (a, b, c).

THEOREM 15.13 Directional Derivative and Interpreting the Gradient

Let f be differentiable at (a, b, c) and let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ be a unit vector. The directional derivative of f at (a, b, c) in the direction of $\mathbf{u}$ is

$$\begin{aligned} D_{\mathbf{u}} f(a, b, c) &= \nabla f(a, b, c) \cdot \mathbf{u} \\ &= \langle f_x(a, b, c), f_y(a, b, c), f_z(a, b, c) \rangle \cdot \langle u_1, u_2, u_3 \rangle. \end{aligned}$$

Assuming $\nabla f(a, b, c) \neq \mathbf{0}$, the gradient in three dimensions has the following properties.

1. f has its maximum rate of increase at (a, b, c) in the direction of the gradient $\nabla f(a, b, c)$, and the rate of change in this direction is $|\nabla f(a, b, c)|$.
2. f has its maximum rate of decrease at (a, b, c) in the direction of $-\nabla f(a, b, c)$, and the rate of change in this direction is $-|\nabla f(a, b, c)|$.
3. The directional derivative is zero in any direction orthogonal to $\nabla f(a, b, c)$.

QUICK CHECK 6 Compute $\nabla f(-1, 2, 1)$, where $f(x, y, z) = xy/z$. ◂

EXAMPLE 8 Gradients in three dimensions Consider the function $f(x, y, z) = x^2 + 2y^2 + 4z^2 - 1$ and its level surface $f(x, y, z) = 3$.

a. Find and interpret the gradient at the points $P(2, 0, 0)$, $Q(0, \sqrt{2}, 0)$, $R(0, 0, 1)$, and $S\left(1, 1, \frac{1}{2}\right)$ on the level surface.

b. What are the actual rates of change of f in the directions of the gradients in part (a)?

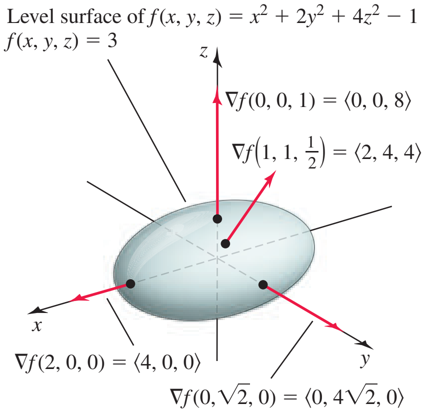


Figure 15.57

SOLUTION

a. The gradient is

$$\nabla f = \langle f_x, f_y, f_z \rangle = \langle 2x, 4y, 8z \rangle.$$

Evaluating the gradient at the four points, we find that

$$\nabla f(2, 0, 0) = \langle 4, 0, 0 \rangle, \qquad \nabla f(0, \sqrt{2}, 0) = \langle 0, 4\sqrt{2}, 0 \rangle,$$
$$\nabla f(0, 0, 1) = \langle 0, 0, 8 \rangle, \quad \text{and} \quad \nabla f\left(1, 1, \tfrac{1}{2}\right) = \langle 2, 4, 4 \rangle.$$

The level surface $f(x, y, z) = 3$ is an ellipsoid (**Figure 15.57**), which is one layer of a four-dimensional surface. The four points P, Q, R, and S are shown on the level surface with the respective gradient vectors. In each case, the gradient points in the direction that f has its maximum rate of increase. Of particular importance is the fact—to be made clear in the next section—that at each point, the gradient is orthogonal to the level surface.

b. The actual rate of increase of f at (a, b, c) in the direction of the gradient is $|\nabla f(a, b, c)|$. At P, the rate of increase of f in the direction of the gradient is $|\langle 4, 0, 0 \rangle| = 4$; at Q, the rate of increase is $|\langle 0, 4\sqrt{2}, 0 \rangle| = 4\sqrt{2}$; at R, the rate of increase is $|\langle 0, 0, 8 \rangle| = 8$; and at S, the rate of increase is $|\langle 2, 4, 4 \rangle| = 6$.

Related Exercises 59–60 ◀

SECTION 15.5 EXERCISES

Getting Started

1. Explain how a directional derivative is formed from the two partial derivatives f_x and f_y.

2. How do you compute the gradient of the functions $f(x, y)$ and $f(x, y, z)$?

3. Interpret the direction of the gradient vector at a point.

4. Interpret the magnitude of the gradient vector at a point.

5. Given a function f, explain the relationship between the gradient and the level curves of f.

6. The level curves of the surface $z = x^2 + y^2$ are circles in the xy-plane centered at the origin. Without computing the gradient, what is the direction of the gradient at $(1, 1)$ and $(-1, -1)$ (determined up to a scalar multiple)?

7. Suppose f is differentiable at $(3, 4)$, $\nabla f(3, 4) = \langle -\sqrt{3}, 1 \rangle$, and $\mathbf{u} = \left\langle \frac{\sqrt{3}}{2}, -\frac{1}{2} \right\rangle$. Compute $D_{\mathbf{u}} f(3, 4)$.

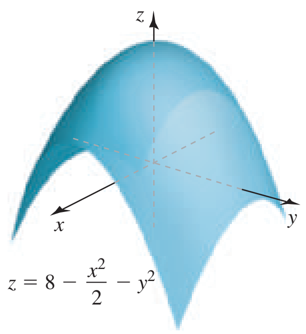
8. Suppose f is differentiable at $(9, 9)$, $\nabla f(9, 9) = \langle 3, 1 \rangle$, and $\mathbf{w} = \langle 1, -1 \rangle$. Compute the directional derivative of f at $(9, 9)$ in the direction of the vector $\mathbf{w}$.

9. Suppose f is differentiable at $(3, 4)$. Assume $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are unit vectors, $\mathbf{v} = -\mathbf{u}$, $\mathbf{w} \cdot \nabla f(3, 4) = 0$, and $D_{\mathbf{u}} f(3, 4) = 7$. Find $D_{\mathbf{v}} f(3, 4)$ and $D_{\mathbf{w}} f(3, 4)$.

10. Suppose f is differentiable at $(1, 2)$ and $\nabla f(1, 2) = \langle 3, 4 \rangle$. Find the slope of the line tangent to the level curve of f at $(1, 2)$.

Practice Exercises

11. Directional derivatives Consider the function $f(x, y) = 8 - \frac{x^2}{2} - y^2$, whose graph is a paraboloid (see figure).

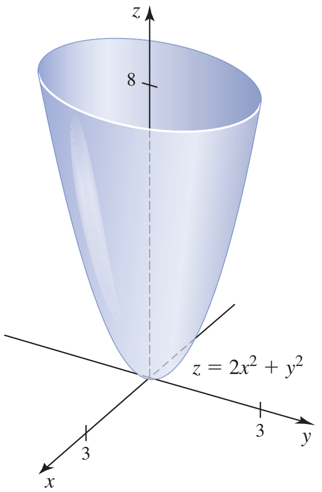


	$(a, b) = (2, 0)$	$(a, b) = (0, 2)$	$(a, b) = (1, 1)$
$\mathbf{u} = \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$			
$\mathbf{v} = \left\langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$			
$\mathbf{w} = \left\langle -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right\rangle$			

a. Fill in the table with the values of the directional derivative at the points (a, b) in the directions given by the unit vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

b. Interpret each of the directional derivatives computed in part (a) at the point $(2, 0)$.

12. Directional derivatives Consider the function $f(x, y) = 2x^2 + y^2$, whose graph is a paraboloid (see figure).



	$(a, b) = (1, 0)$	$(a, b) = (1, 1)$	$(a, b) = (1, 2)$
$\mathbf{u} = \langle 1, 0 \rangle$			
$\mathbf{v} = \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$			
$\mathbf{w} = \langle 0, 1 \rangle$			

a. Fill in the table with the values of the directional derivative at the points (a, b) in the directions given by the unit vectors **u**, **v**, and **w**.
b. Interpret each of the directional derivatives computed in part (a) at the point $(1, 0)$.

13–20. Computing gradients *Compute the gradient of the following functions and evaluate it at the given point P.*

13. $f(x, y) = 2 + 3x^2 - 5y^2$; $P(2, -1)$

14. $f(x, y) = 4x^2 - 2xy + y^2$; $P(-1, -5)$

15. $g(x, y) = x^2 - 4x^2y - 8xy^2$; $P(-1, 2)$

16. $p(x, y) = \sqrt{12 - 4x^2 - y^2}$; $P(-1, -1)$

17. $f(x, y) = xe^{2xy}$; $P(1, 0)$

18. $f(x, y) = \sin(3x + 2y)$; $P(\pi, 3\pi/2)$

19. $F(x, y) = e^{-x^2 - 2y^2}$; $P(-1, 2)$

20. $h(x, y) = \ln(1 + x^2 + 2y^2)$; $P(2, -3)$

21–30. Computing directional derivatives with the gradient *Compute the directional derivative of the following functions at the given point P in the direction of the given vector. Be sure to use a unit vector for the direction vector.*

21. $f(x, y) = x^2 - y^2$; $P(-1, -3)$; $\left\langle \frac{3}{5}, -\frac{4}{5} \right\rangle$

22. $f(x, y) = 3x^2 + y^3$; $P(3, 2)$; $\left\langle \frac{5}{13}, \frac{12}{13} \right\rangle$

23. $f(x, y) = 10 - 3x^2 + \frac{y^4}{4}$; $P(2, -3)$; $\left\langle \frac{\sqrt{3}}{2}, -\frac{1}{2} \right\rangle$

24. $g(x, y) = \sin \pi(2x - y)$; $P(-1, -1)$; $\left\langle \frac{5}{13}, -\frac{12}{13} \right\rangle$

25. $f(x, y) = \sqrt{4 - x^2 - 2y}$; $P(2, -2)$; $\left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$

26. $f(x, y) = 13e^{xy}$; $P(1, 0)$; $\langle 5, 12 \rangle$

27. $f(x, y) = 3x^2 + 2y + 5$; $P(1, 2)$; $\langle -3, 4 \rangle$

28. $h(x, y) = e^{-x-y}$; $P(\ln 2, \ln 3)$; $\langle 1, 1 \rangle$

29. $g(x, y) = \ln(4 + x^2 + y^2)$; $g(-1, 2)$; $\langle 2, 1 \rangle$

30. $f(x, y) = \frac{x}{x - y}$; $P(4, 1)$; $\langle -1, 2 \rangle$

31–36. Direction of steepest ascent and descent *Consider the following functions and points P.*

a. *Find the unit vectors that give the direction of steepest ascent and steepest descent at P.*
b. *Find a vector that points in a direction of no change in the function at P.*

31. $f(x, y) = x^2 - 4y^2 - 9$; $P(1, -2)$

32. $f(x, y) = x^2 + 4xy - y^2$; $P(2, 1)$

33. $f(x, y) = x^4 - x^2y + y^2 + 6$; $P(-1, 1)$

34. $p(x, y) = \sqrt{20 + x^2 + 2xy - y^2}$; $P(1, 2)$

35. $F(x, y) = e^{-x^2/2 - y^2/2}$; $P(-1, 1)$

36. $f(x, y) = 2\sin(2x - 3y)$; $P(0, \pi)$

37–42. Interpreting directional derivatives *A function f and a point P are given. Let θ correspond to the direction of the directional derivative.*

a. *Find the gradient and evaluate it at P.*
b. *Find the angles θ (with respect to the positive x-axis) associated with the directions of maximum increase, maximum decrease, and zero change.*
c. *Write the directional derivative at P as a function of θ; call this function g.*
d. *Find the value of θ that maximizes $g(\theta)$ and find the maximum value.*
e. *Verify that the value of θ that maximizes g corresponds to the direction of the gradient. Verify that the maximum value of g equals the magnitude of the gradient.*

37. $f(x, y) = 10 - 2x^2 - 3y^2$; $P(3, 2)$

38. $f(x, y) = 8 + x^2 + 3y^2$; $P(-3, -1)$

39. $f(x, y) = \sqrt{2 + x^2 + y^2}$; $P(\sqrt{3}, 1)$

40. $f(x, y) = \sqrt{12 - x^2 - y^2}$; $P(-1, -1/\sqrt{3})$

41. $f(x, y) = e^{-x^2 - 2y^2}$; $P(-1, 0)$

T **42.** $f(x, y) = \ln(1 + 2x^2 + 3y^2)$; $P(3/4, -\sqrt{3})$

43–46. Directions of change *Consider the following functions f and points P. Sketch the xy-plane showing P and the level curve through P. Indicate (as in Figure 15.52) the directions of maximum increase, maximum decrease, and no change for f.*

43. $f(x, y) = 8 + 4x^2 + 2y^2$; $P(2, -4)$

44. $f(x, y) = -4 + 6x^2 + 3y^2$; $P(-1, -2)$

T **45.** $f(x, y) = x^2 + xy + y^2 + 7$; $P(-3, 3)$

T **46.** $f(x, y) = \tan(2x + 2y)$; $P(\pi/16, \pi/16)$

47–50. Level curves *Consider the paraboloid* $f(x, y) = 16 - \dfrac{x^2}{4} - \dfrac{y^2}{16}$ *and the point P on the given level curve of f. Compute the slope of the line tangent to the level curve at P, and verify that the tangent line is orthogonal to the gradient at that point.*

47. $f(x, y) = 0; P(0, 16)$

48. $f(x, y) = 0; P(8, 0)$

49. $f(x, y) = 12; P(4, 0)$

50. $f(x, y) = 12; P(2\sqrt{3}, 4)$

51–54. Level curves *Consider the upper half of the ellipsoid* $f(x, y) = \sqrt{1 - \dfrac{x^2}{4} - \dfrac{y^2}{16}}$ *and the point P on the given level curve of f. Compute the slope of the line tangent to the level curve at P, and verify that the tangent line is orthogonal to the gradient at that point.*

51. $f(x, y) = \dfrac{\sqrt{3}}{2}; P\left(\dfrac{1}{2}, \sqrt{3}\right)$

52. $f(x, y) = \dfrac{1}{\sqrt{2}}; P(0, \sqrt{8})$

53. $f(x, y) = \dfrac{1}{\sqrt{2}}; P(\sqrt{2}, 0)$

54. $f(x, y) = \dfrac{1}{\sqrt{2}}; P(1, 2)$

55–58. Path of steepest descent *Consider each of the following surfaces and the point P on the surface.*

a. Find the gradient of f.
b. Let C′ be the path of steepest descent on the surface beginning at P, and let C be the projection of C′ on the xy-plane. Find an equation of C in the xy-plane.
c. Find parametric equations for the path C′ on the surface.

55. $f(x, y) = 4 + x$ (a plane); $P(4, 4, 8)$

56. $f(x, y) = y + x$ (a plane); $P(2, 2, 4)$

57. $f(x, y) = 4 - x^2 - 2y^2$ (a paraboloid); $P(1, 1, 1)$

58. $f(x, y) = y + x^{-1}; P(1, 2, 3)$

59–66. Gradients in three dimensions *Consider the following functions f, points P, and unit vectors* **u**.

a. Compute the gradient of f and evaluate it at P.
b. Find the unit vector in the direction of maximum increase of f at P.
c. Find the rate of change of the function in the direction of maximum increase at P.
d. Find the directional derivative at P in the direction of the given vector.

59. $f(x, y, z) = x^2 + 2y^2 + 4z^2 + 10; P(1, 0, 4); \left\langle \dfrac{1}{\sqrt{2}}, 0, \dfrac{1}{\sqrt{2}} \right\rangle$

60. $f(x, y, z) = 4 - x^2 + 3y^2 + \dfrac{z^2}{2}; P(0, 2, -1); \left\langle 0, \dfrac{1}{\sqrt{2}}, -\dfrac{1}{\sqrt{2}} \right\rangle$

61. $f(x, y, z) = 1 + 4xyz; P(1, -1, -1); \left\langle \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, -\dfrac{1}{\sqrt{3}} \right\rangle$

62. $f(x, y, z) = xy + yz + xz + 4; P(2, -2, 1); \left\langle 0, -\dfrac{1}{\sqrt{2}}, -\dfrac{1}{\sqrt{2}} \right\rangle$

63. $f(x, y, z) = 1 + \sin(x + 2y - z); P\left(\dfrac{\pi}{6}, \dfrac{\pi}{6}, -\dfrac{\pi}{6}\right); \left\langle \dfrac{1}{3}, \dfrac{2}{3}, \dfrac{2}{3} \right\rangle$

64. $f(x, y, z) = e^{xyz-1}; P(0, 1, -1); \left\langle -\dfrac{2}{3}, \dfrac{2}{3}, -\dfrac{1}{3} \right\rangle$

65. $f(x, y, z) = \ln(1 + x^2 + y^2 + z^2); P(1, 1, -1); \left\langle \dfrac{2}{3}, \dfrac{2}{3}, -\dfrac{1}{3} \right\rangle$

66. $f(x, y, z) = \dfrac{x - z}{y - z}; P(3, 2, -1); \left\langle \dfrac{1}{3}, \dfrac{2}{3}, -\dfrac{2}{3} \right\rangle$

67. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. If $f(x, y) = x^2 + y^2 - 10$, then $\nabla f(x, y) = 2x + 2y$.
b. Because the gradient gives the direction of maximum increase of a function, the gradient is always positive.
c. The gradient of $f(x, y, z) = 1 + xyz$ has four components.
d. If $f(x, y, z) = 4$, then $\nabla f = \mathbf{0}$.

68. Gradient of a composite function Consider the function $F(x, y, z) = e^{xyz}$.

a. Write F as a composite function $f \circ g$, where f is a function of one variable and g is a function of three variables.
b. Relate ∇F to ∇g.

69–72. Directions of zero change *Find the directions in the xy-plane in which the following functions have zero change at the given point. Express the directions in terms of unit vectors.*

69. $f(x, y) = 12 - 4x^2 - y^2; P(1, 2, 4)$

70. $f(x, y) = x^2 - 4y^2 - 8; P(4, 1, 4)$

71. $f(x, y) = \sqrt{3 + 2x^2 + y^2}; P(1, -2, 3)$

72. $f(x, y) = e^{1-xy}; P(1, 0, e)$

73. Steepest ascent on a plane Suppose a long sloping hillside is described by the plane $z = ax + by + c$, where a, b, and c are constants. Find the path in the xy-plane, beginning at (x_0, y_0), that corresponds to the path of steepest ascent on the hillside.

74. Gradient of a distance function Let (a, b) be a given point in $\mathbb{R}^2$, and let $d = f(x, y)$ be the distance between (a, b) and the variable point (x, y).

a. Show that the graph of f is a cone.
b. Show that the gradient of f at any point other than (a, b) is a unit vector.
c. Interpret the direction and magnitude of ∇f.

75–78. Looking ahead—tangent planes *Consider the following surfaces* $f(x, y, z) = 0$*, which may be regarded as a level surface of the function* $w = f(x, y, z)$*. A point* $P(a, b, c)$ *on the surface is also given.*

a. Find the (three-dimensional) gradient of f and evaluate it at P.
b. The set of all vectors orthogonal to the gradient with their tails at P form a plane. Find an equation of that plane (soon to be called the tangent plane).

75. $f(x, y, z) = x^2 + y^2 + z^2 - 3 = 0; P(1, 1, 1)$

76. $f(x, y, z) = 8 - xyz = 0; P(2, 2, 2)$

77. $f(x, y, z) = e^{x+y-z} - 1 = 0; P(1, 1, 2)$

78. $f(x, y, z) = xy + xz - yz - 1 = 0; P(1, 1, 1)$

T **79. A traveling wave** A snapshot (frozen in time) of a set of water waves is described by the function $z = 1 + \sin(x - y)$, where z gives the height of the waves and (x, y) are coordinates in the horizontal plane $z = 0$.

a. Use a graphing utility to graph $z = 1 + \sin(x - y)$.
b. The crests and the troughs of the waves are aligned in the direction in which the height function has zero change. Find the direction in which the crests and troughs are aligned.
c. If you were surfing on one of these waves and wanted the steepest descent from the crest to the trough, in which direction would you point your surfboard (given in terms of a unit vector in the xy-plane)?
d. Check that your answers to parts (b) and (c) are consistent with the graph of part (a).

80. Traveling waves in general Generalize Exercise 79 by considering a set of waves described by the function $z = A + \sin(ax - by)$, where a, b, and A are real numbers.

a. Find the direction in which the crests and troughs of the waves are aligned. Express your answer as a unit vector in terms of a and b.

b. Find the surfer's direction—that is, the direction of steepest descent from a crest to a trough. Express your answer as a unit vector in terms of a and b.

Explorations and Challenges

81–83. Potential functions *Potential functions arise frequently in physics and engineering. A potential function has the property that a field of interest (for example, an electric field, a gravitational field, or a velocity field) is the gradient of the potential (or sometimes the negative of the gradient of the potential). (Potential functions are considered in depth in Chapter 17.)*

81. Electric potential due to a point charge The electric field due to a point charge of strength Q at the origin has a potential function $\varphi = kQ/r$, where $r^2 = x^2 + y^2 + z^2$ is the square of the distance between a variable point $P(x, y, z)$ and the charge, and $k > 0$ is a physical constant. The electric field is given by $\mathbf{E} = -\nabla\varphi$, where $\nabla\varphi$ is the gradient in three dimensions.

a. Show that the three-dimensional electric field due to a point charge is given by

$$\mathbf{E}(x, y, z) = kQ\left\langle \frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3} \right\rangle.$$

b. Show that the electric field at a point has a magnitude $|\mathbf{E}| = \dfrac{kQ}{r^2}$. Explain why this relationship is called an inverse square law.

82. Gravitational potential The gravitational potential associated with two objects of mass M and m is $\varphi = -GMm/r$, where G is the gravitational constant. If one of the objects is at the origin and the other object is at $P(x, y, z)$, then $r^2 = x^2 + y^2 + z^2$ is the square of the distance between the objects. The gravitational field at P is given by $\mathbf{F} = -\nabla\varphi$, where $\nabla\varphi$ is the gradient in three dimensions. Show that the force has a magnitude $|\mathbf{F}| = GMm/r^2$. Explain why this relationship is called an inverse square law.

83. Velocity potential In two dimensions, the motion of an ideal fluid (an incompressible and irrotational fluid) is governed by a velocity potential φ. The velocity components of the fluid, u in the x-direction and v in the y-direction, are given by $\langle u, v \rangle = \nabla\varphi$. Find the velocity components associated with the velocity potential $\varphi(x, y) = \sin \pi x \sin 2\pi y$.

84. Gradients for planes Prove that for the plane described by $f(x, y) = Ax + By$, where A and B are nonzero constants, the gradient is constant (independent of (x, y)). Interpret this result.

85. Rules for gradients Use the definition of the gradient (in two or three dimensions), assume f and g are differentiable functions on $\mathbb{R}^2$ or $\mathbb{R}^3$, and let c be a constant. Prove the following gradient rules.

a. Constants Rule: $\nabla(cf) = c\nabla f$

b. Sum Rule: $\nabla(f + g) = \nabla f + \nabla g$

c. Product Rule: $\nabla(fg) = (\nabla f)g + f\nabla g$

d. Quotient Rule: $\nabla\left(\dfrac{f}{g}\right) = \dfrac{g\nabla f - f\nabla g}{g^2}$

e. Chain Rule: $\nabla(f \circ g) = f'(g)\nabla g$, where f is a function of one variable

86–91. Using gradient rules *Use the gradient rules of Exercise 85 to find the gradient of the following functions.*

86. $f(x, y) = xy\cos(xy)$

87. $f(x, y) = \dfrac{x + y}{x^2 + y^2}$

88. $f(x, y) = \ln(1 + x^2 + y^2)$

89. $f(x, y, z) = \sqrt{25 - x^2 - y^2 - z^2}$

90. $f(x, y, z) = (x + y + z)e^{xyz}$

91. $f(x, y, z) = \dfrac{x + yz}{y + xz}$

QUICK CHECK ANSWERS

1. If $\mathbf{u} = \langle u_1, u_2 \rangle = \langle 1, 0 \rangle$ then

$$D_{\mathbf{u}} f(a, b) = \lim_{h \to 0} \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h}$$
$$= \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h} = f_x(a, b).$$

Similarly, when $\mathbf{u} = \langle 0, 1 \rangle$, the partial derivative $f_y(a, b)$ results. **2.** The vector from (a, b) to $(a + \Delta s u_1, b + \Delta s u_2)$ is $\langle \Delta s u_1, \Delta s u_2 \rangle = \Delta s \langle u_1, u_2 \rangle = \Delta s\mathbf{u}$. Its length is $|\Delta s\mathbf{u}| = \Delta s|\mathbf{u}| = \Delta s$. Therefore, s measures arc length. **3.** Reversing (negating) the direction vector negates the directional derivative, so the respective values are approximately -2.47 and 0.98. **4.** The gradient is $\langle 2x, 2y \rangle$, which, evaluated at (a, a), is $\langle 2a, 2a \rangle$. Taking the dot product of the gradient and the vector $\langle -1, 1 \rangle$ (a vector parallel to a line of slope -1), we see that $\langle 2a, 2a \rangle \cdot \langle -1, 1 \rangle = 0$. **6.** $\langle 2, -1, 2 \rangle$ ➤

15.6 Tangent Planes and Linear Approximation

In Section 4.6, we saw that if we zoom in on a point on a smooth curve (one described by a differentiable function), the curve looks more and more like the tangent line at that point. Once we have the tangent line at a point, it can be used to approximate function values and to estimate changes in the dependent variable. In this section, the analogous story is developed in three dimensions. Now we see that differentiability at a point (as discussed in Section 15.3) implies the existence of a tangent *plane* at that point (**Figure 15.58**).

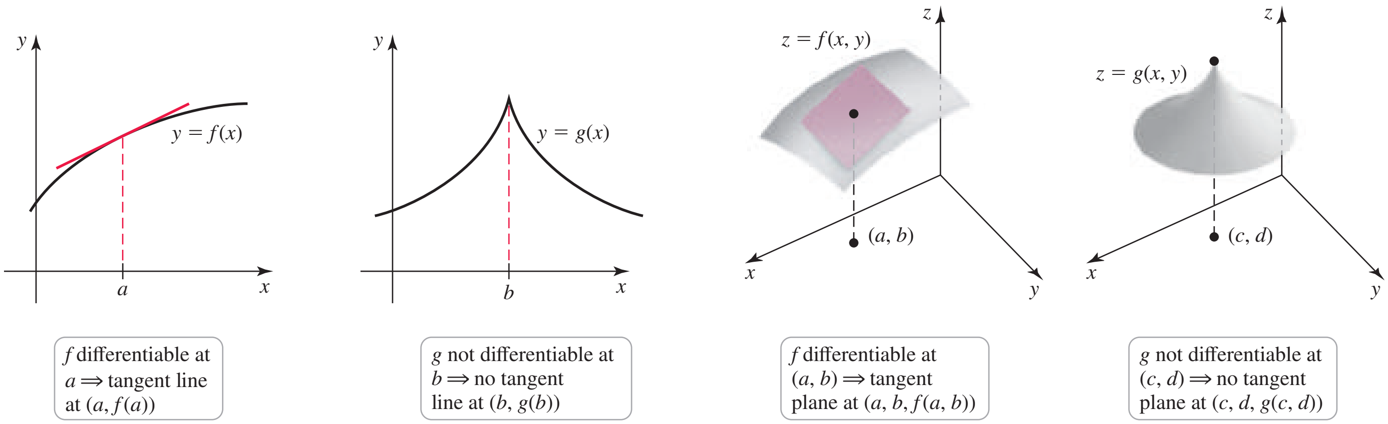


Figure 15.58

Consider a smooth surface described by a differentiable function f, and focus on a single point on the surface. As we zoom in on that point (**Figure 15.59**), the surface appears more and more like a plane. The first step is to define this plane carefully; it is called the *tangent plane*. Once we have the tangent plane, we can use it to approximate function values and to estimate changes in the dependent variable.

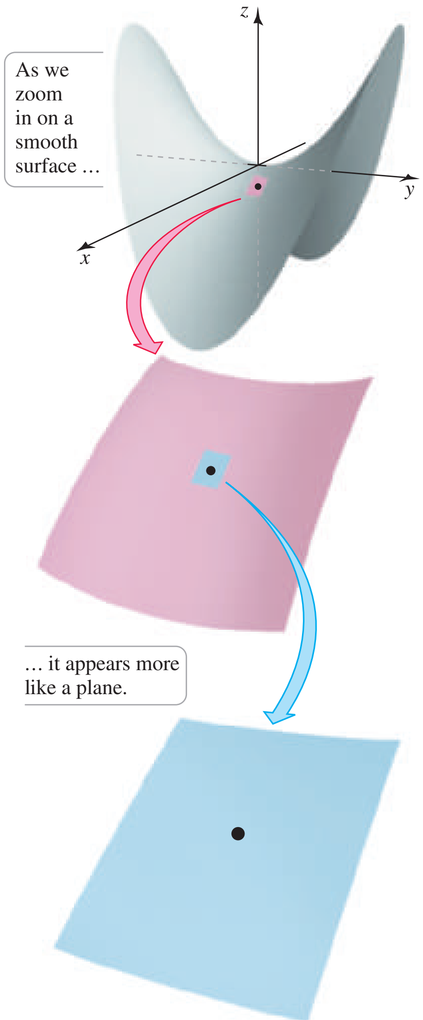


Figure 15.59

Tangent Planes

Recall that a surface in $\mathbb{R}^3$ may be defined in at least two different ways:

- **Explicitly** in the form $z = f(x, y)$ or
- **Implicitly** in the form $F(x, y, z) = 0$.

It is easiest to begin by considering a surface defined implicitly by $F(x, y, z) = 0$, where F is differentiable at a particular point. Such a surface may be viewed as a level surface of a function $w = F(x, y, z)$; it is the level surface for $w = 0$.

QUICK CHECK 1 Write the function $z = xy + x - y$ in the form $F(x, y, z) = 0$. ◀

Tangent Planes for $F(x, y, z) = 0$ To find an equation of the tangent plane, consider a smooth curve C: $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ that lies on the surface $F(x, y, z) = 0$ (**Figure 15.60a**). Because the points of C lie on the surface, we have $F(x(t), y(t), z(t)) = 0$.

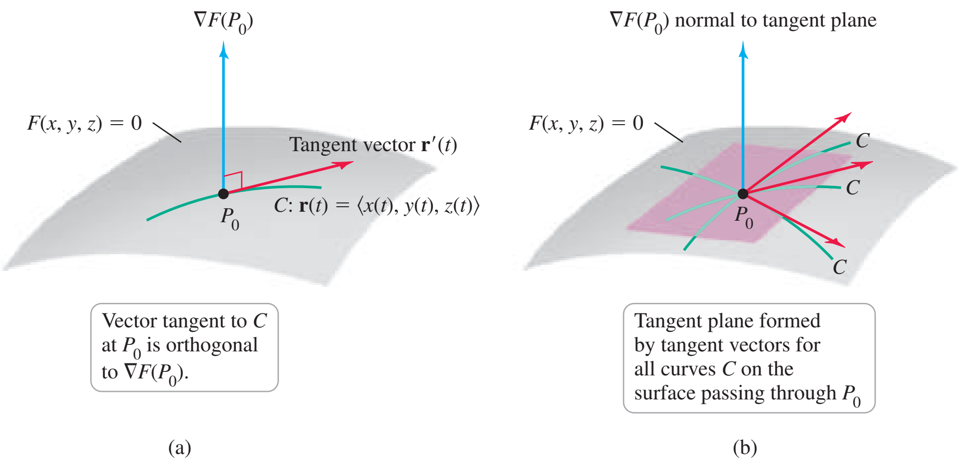


Figure 15.60

Differentiating both sides of this equation with respect to t reveals a useful relationship. The derivative of the right side is 0. The Chain Rule applied to the left side yields

$$\begin{aligned}\frac{d}{dt}\big(F(x(t), y(t), z(t))\big) &= \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} \\ &= \underbrace{\left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right\rangle}_{\nabla F(x,y,z)} \cdot \underbrace{\left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle}_{\mathbf{r}'(t)} \\ &= \nabla F(x, y, z) \cdot \mathbf{r}'(t).\end{aligned}$$

Therefore, $\nabla F(x, y, z) \cdot \mathbf{r}'(t) = 0$ and at any point on the curve, the tangent vector $\mathbf{r}'(t)$ is orthogonal to the gradient.

Now fix a point $P_0(a, b, c)$ on the surface, assume $\nabla F(a, b, c) \neq \mathbf{0}$, and let C be any smooth curve on the surface passing through P_0. We have shown that any vector tangent to C is orthogonal to $\nabla F(a, b, c)$ at P_0. Because this argument applies to *all* smooth curves on the surface passing through P_0, the tangent vectors for all these curves (with their tails at P_0) are orthogonal to $\nabla F(a, b, c)$; therefore, they all lie in the same plane (**Figure 15.60b**). This plane is called the *tangent plane* at P_0. We can easily find an equation of the tangent plane because we know both a point on the plane $P_0(a, b, c)$ and a normal vector $\nabla F(a, b, c)$; an equation is

$$\nabla F(a, b, c) \cdot \langle x - a, y - b, z - c \rangle = 0.$$

➤ Recall that an equation of the plane passing though (a, b, c) with a normal vector $\mathbf{n} = \langle n_1, n_2, n_3 \rangle$ is $n_1(x - a) + n_2(y - b) + n_3(z - c) = 0$.

➤ If $\mathbf{r}$ is a position vector corresponding to an arbitrary point on the tangent plane, and $\mathbf{r}_0$ is a position vector corresponding to a fixed point (a, b, c) on the plane, then an equation of the tangent plane may be written concisely as

$$\nabla F(a, b, c) \cdot (\mathbf{r} - \mathbf{r}_0) = 0.$$

Notice the analogy with tangent lines and level curves (Section 15.5). An equation of the line tangent to $f(x, y) = 0$ at (a, b) is

$$\nabla f(a, b) \cdot \langle x - a, y - b \rangle = 0.$$

DEFINITION Equation of the Tangent Plane for $F(x, y, z) = 0$

Let F be differentiable at the point $P_0(a, b, c)$ with $\nabla F(a, b, c) \neq \mathbf{0}$. The plane tangent to the surface $F(x, y, z) = 0$ at P_0, called the **tangent plane**, is the plane passing through P_0 orthogonal to $\nabla F(a, b, c)$. An equation of the tangent plane is

$$F_x(a, b, c)(x - a) + F_y(a, b, c)(y - b) + F_z(a, b, c)(z - c) = 0.$$

EXAMPLE 1 Equation of a tangent plane Consider the ellipsoid

$$F(x, y, z) = \frac{x^2}{9} + \frac{y^2}{25} + z^2 - 1 = 0.$$

a. Find an equation of the plane tangent to the ellipsoid at $\left(0, 4, \frac{3}{5}\right)$.

b. At what points on the ellipsoid is the tangent plane horizontal?

SOLUTION

a. Notice that we have written the equation of the ellipsoid in the implicit form $F(x, y, z) = 0$. The gradient of F is $\nabla F(x, y, z) = \left\langle \frac{2x}{9}, \frac{2y}{25}, 2z \right\rangle$. Evaluating at $\left(0, 4, \frac{3}{5}\right)$, we have

$$\nabla F\left(0, 4, \frac{3}{5}\right) = \left\langle 0, \frac{8}{25}, \frac{6}{5} \right\rangle.$$

An equation of the tangent plane at this point is

$$0 \cdot (x - 0) + \frac{8}{25}(y - 4) + \frac{6}{5}\left(z - \frac{3}{5}\right) = 0,$$

or $4y + 15z = 25$. The equation does not involve x, so the tangent plane is parallel to (does not intersect) the x-axis (**Figure 15.61**).

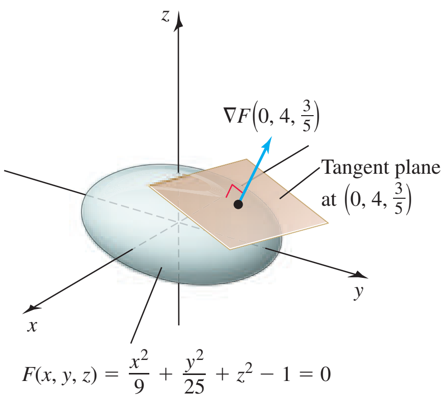


Figure 15.61

b. A horizontal plane has a normal vector of the form $\langle 0, 0, c \rangle$, where $c \neq 0$. A plane tangent to the ellipsoid has a normal vector $\nabla F(x, y, z) = \left\langle \frac{2x}{9}, \frac{2y}{25}, 2z \right\rangle$. Therefore, the ellipsoid has a horizontal tangent plane when $F_x = \frac{2x}{9} = 0$ and $F_y = \frac{2y}{25} = 0$, or when $x = 0$ and $y = 0$. Substituting these values into the original equation for the ellipsoid, we find that horizontal planes occur at $(0, 0, 1)$ and $(0, 0, -1)$.

Related Exercises 14, 16 ◀

➤ This result extends Theorem 15.12, which states that for functions $f(x, y) = 0$, the gradient at a point is orthogonal to the level curve that passes through that point.

The preceding discussion allows us to confirm a claim made in Section 15.5. The surface $F(x, y, z) = 0$ is a level surface of the function $w = F(x, y, z)$ (corresponding to $w = 0$). At any point on that surface, the tangent plane has a normal vector $\nabla F(x, y, z)$. Therefore, the gradient $\nabla F(x, y, z)$ is orthogonal to the level surface $F(x, y, z) = 0$ at all points of the domain at which F is differentiable.

Tangent Planes for $z = f(x, y)$ Surfaces in $\mathbb{R}^3$ are often defined explicitly in the form $z = f(x, y)$. In this situation, the equation of the tangent plane is a special case of the general equation just derived. The equation $z = f(x, y)$ is written as $F(x, y, z) = z - f(x, y) = 0$, and the gradient of F at the point $(a, b, f(a, b))$ is

➤ To be clear, when $F(x, y, z) = z - f(x, y)$, we have $F_x = -f_x$, $F_y = -f_y$, and $F_z = 1$.

$$\begin{aligned}\nabla F(a, b, f(a, b)) &= \langle F_x(a, b, f(a, b)), F_y(a, b, f(a, b)), F_z(a, b, f(a, b)) \rangle \\ &= \langle -f_x(a, b), -f_y(a, b), 1 \rangle.\end{aligned}$$

Using the tangent plane definition, an equation of the plane tangent to the surface $z = f(x, y)$ at the point $(a, b, f(a, b))$ is

$$-f_x(a, b)(x - a) - f_y(a, b)(y - b) + 1(z - f(a, b)) = 0.$$

After some rearranging, we obtain an equation of the tangent plane.

> **Tangent Plane for $z = f(x, y)$**
>
> Let f be differentiable at the point (a, b). An equation of the plane tangent to the surface $z = f(x, y)$ at the point $(a, b, f(a, b))$ is
>
> $$z = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b).$$

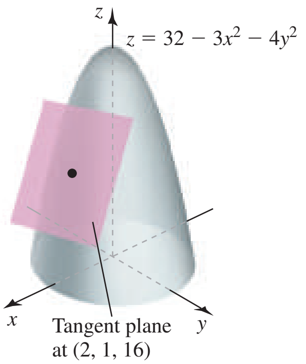


Figure 15.62

EXAMPLE 2 Tangent plane for $z = f(x, y)$ Find an equation of the plane tangent to the paraboloid $z = f(x, y) = 32 - 3x^2 - 4y^2$ at $(2, 1, 16)$.

SOLUTION The partial derivatives are $f_x = -6x$ and $f_y = -8y$. Evaluating the partial derivatives at $(2, 1)$, we have $f_x(2, 1) = -12$ and $f_y(2, 1) = -8$. Therefore, an equation of the tangent plane (**Figure 15.62**) is

$$\begin{aligned}z &= f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b) \\ &= -12(x - 2) - 8(y - 1) + 16 \\ &= -12x - 8y + 48.\end{aligned}$$

Related Exercises 17–18 ◀

Linear Approximation

With a function of the form $y = f(x)$, the tangent line at a point often gives good approximations to the function near that point. A straightforward extension of this idea applies to approximating functions of two variables with tangent planes. As before, the method is called *linear approximation.*

Figure 15.63 shows the details of linear approximation in the one- and two-variable cases. In the one-variable case (Section 4.6), if f is differentiable at a, the equation of the line tangent to the curve $y = f(x)$ at the point $(a, f(a))$ is

$$L(x) = f(a) + f'(a)(x - a).$$

The tangent line gives an approximation to the function. At points near a, we have $f(x) \approx L(x)$.

The two-variable case is analogous. If f is differentiable at (a, b), an equation of the plane tangent to the surface $z = f(x, y)$ at the point $(a, b, f(a, b))$ is

$$L(x, y) = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b).$$

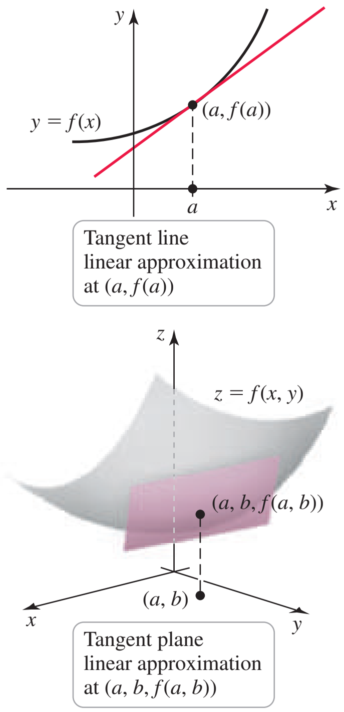


Figure 15.63

This tangent plane is the linear approximation to f at (a, b). At points near (a, b), we have $f(x, y) \approx L(x, y)$. The pattern established here continues for linear approximations in higher dimensions: For each additional variable, a new term is added to the approximation formula.

➤ The term *linear approximation* applies in $\mathbb{R}^2$, in $\mathbb{R}^3$, and in higher dimensions. Recall that lines in $\mathbb{R}^2$ and planes in $\mathbb{R}^3$ are described by linear functions of the independent variables. In both cases, we call the linear approximation L.

DEFINITION Linear Approximation

Let f be differentiable at (a, b). The linear approximation to the surface $z = f(x, y)$ at the point $(a, b, f(a, b))$ is the tangent plane at that point, given by the equation

$$L(x, y) = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b).$$

For a function of three variables, the linear approximation to $w = f(x, y, z)$ at the point $(a, b, c, f(a, b, c))$ is given by

$$\begin{aligned} L(x, y, z) = {} & f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) \\ & + f_z(a, b, c)(z - c) + f(a, b, c). \end{aligned}$$

EXAMPLE 3 **Linear approximation** Let $f(x, y) = \dfrac{5}{x^2 + y^2}$.

a. Find the linear approximation to the function at the point $(-1, 2, 1)$.

b. Use the linear approximation to estimate the value of $f(-1.05, 2.1)$.

SOLUTION

a. The partial derivatives of f are

$$f_x = -\frac{10x}{(x^2 + y^2)^2} \quad \text{and} \quad f_y = -\frac{10y}{(x^2 + y^2)^2}.$$

Evaluated at $(-1, 2)$, we have $f_x(-1, 2) = \frac{2}{5} = 0.4$ and $f_y(-1, 2) = -\frac{4}{5} = -0.8$. Therefore, the linear approximation to the function at $(-1, 2, 1)$ is

$$\begin{aligned} L(x, y) &= f_x(-1, 2)(x - (-1)) + f_y(-1, 2)(y - 2) + f(-1, 2) \\ &= 0.4(x + 1) - 0.8(y - 2) + 1 \\ &= 0.4x - 0.8y + 3. \end{aligned}$$

The surface and the tangent plane are shown in Figure 15.64.

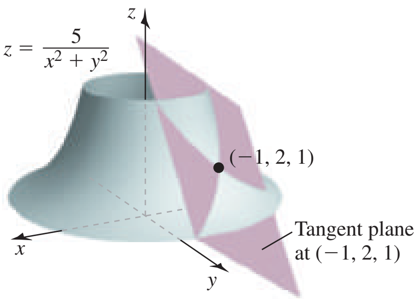


Figure 15.64

b. The value of the function at the point $(-1.05, 2.1)$ is approximated by the value of the linear approximation at that point, which is

$$L(-1.05, 2.1) = 0.4(-1.05) - 0.8(2.1) + 3 = 0.90.$$

In this case, we can easily evaluate $f(-1.05, 2.1) \approx 0.907$ and compare the linear approximation with the exact value; the approximation has a relative error of about 0.8%.

➤ Relative error $= \dfrac{|\text{approximation} - \text{exact value}|}{|\text{exact value}|}$

Related Exercise 36 ◄

QUICK CHECK 2 Look at the graph of the surface in Example 3 (Figure 15.64) and explain why $f_x(-1, 2) > 0$ and $f_y(-1, 2) < 0$. ◄

Differentials and Change

Recall that for a function of the form $y = f(x)$, if the independent variable changes from x to $x + dx$, the corresponding change Δy in the dependent variable is approximated by the differential $dy = f'(x)\,dx$, which is the change in the linear approximation. Therefore, $\Delta y \approx dy$, with the approximation improving as dx approaches 0.

For functions of the form $z = f(x, y)$, we start with the linear approximation to the surface

$$f(x, y) \approx L(x, y) = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b).$$

The exact change in the function between the points (a, b) and (x, y) is

$$\Delta z = f(x, y) - f(a, b).$$

Replacing $f(x, y)$ with its linear approximation, the change Δz is approximated by

$$\Delta z \approx \underbrace{L(x, y) - f(a, b)}_{dz} = f_x(a, b)\underbrace{(x - a)}_{dx} + f_y(a, b)\underbrace{(y - b)}_{dy}.$$

➤ Alternative notation for the differential at (a, b) is $dz|_{(a, b)}$ or $df|_{(a, b)}$.

The change in the x-coordinate is $dx = x - a$ and the change in the y-coordinate is $dy = y - b$ (**Figure 15.65**). As before, we let the differential dz denote the change in the linear approximation. Therefore, the approximate change in the z-coordinate is

$$\Delta z \approx dz = \underbrace{f_x(a, b)\,dx}_{\substack{\text{change in } z \text{ due} \\ \text{to change in } x}} + \underbrace{f_y(a, b)\,dy}_{\substack{\text{change in } z \text{ due} \\ \text{to change in } y}}.$$

This expression says that if we move the independent variables from (a, b) to $(x, y) = (a + dx, b + dy)$, the corresponding change in the dependent variable Δz has two contributions—one due to the change in x and one due to the change in y. If dx and dy are small in magnitude, then so is Δz. The approximation $\Delta z \approx dz$ improves as dx and dy approach 0. The relationships among the differentials are illustrated in Figure 15.65.

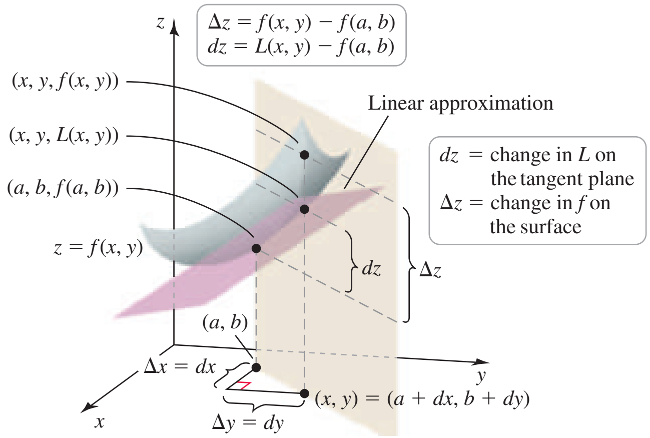


Figure 15.65

QUICK CHECK 3 Explain why, if $dx = 0$ or $dy = 0$ in the change formula for Δz, the result is the change formula for one variable. ◄

More generally, we replace the fixed point (a, b) in the previous discussion with the variable point (x, y) to arrive at the following definition.

> **DEFINITION The differential dz**
>
> Let f be differentiable at the point (x, y). The change in $z = f(x, y)$ as the independent variables change from (x, y) to $(x + dx, y + dy)$ is denoted Δz and is approximated by the differential dz:
>
> $$\Delta z \approx dz = f_x(x, y)\,dx + f_y(x, y)\,dy.$$

EXAMPLE 4 Approximating function change Let $z = f(x, y) = \dfrac{5}{x^2 + y^2}$. Approximate the change in z when the independent variables change from $(-1, 2)$ to $(-0.93, 1.94)$.

SOLUTION If the independent variables change from $(-1, 2)$ to $(-0.93, 1.94)$, then $dx = 0.07$ (an increase) and $dy = -0.06$ (a decrease). Using the values of the partial derivatives evaluated in Example 3, the corresponding change in z is approximately

$$\begin{aligned} dz &= f_x(-1, 2)\,dx + f_y(-1, 2)\,dy \\ &= 0.4(0.07) + (-0.8)(-0.06) \\ &= 0.076. \end{aligned}$$

Again, we can check the accuracy of the approximation. The actual change is $f(-0.93, 1.94) - f(-1, 2) \approx 0.080$, so the approximation has a 5% error.

Related Exercise 40 ◄

EXAMPLE 5 **Body mass index** The body mass index (BMI) for an adult human is given by the function $B(w, h) = w/h^2$, where w is weight measured in kilograms and h is height measured in meters.

a. Use differentials to approximate the change in the BMI when weight increases from 55 to 56.5 kg and height increases from 1.65 to 1.66 m.

b. Which produces a greater *percentage* change in the BMI, a 1% change in the weight (at a constant height) or a 1% change in the height (at a constant weight)?

SOLUTION

a. The approximate change in the BMI is $dB = B_w\,dw + B_h\,dh$, where the derivatives are evaluated at $w = 55$ and $h = 1.65$, and the changes in the independent variables are $dw = 1.5$ and $dh = 0.01$. Evaluating the partial derivatives, we find that

$$B_w(w, h) = \frac{1}{h^2}, \qquad B_w(55, 1.65) \approx 0.37,$$

$$B_h(w, h) = -\frac{2w}{h^3}, \quad \text{and} \quad B_h(55, 1.65) \approx -24.49.$$

Therefore, the approximate change in the BMI is

$$\begin{aligned} dB &= B_w(55, 1.65)\,dw + B_h(55, 1.65)\,dh \\ &\approx (0.37)(1.5) + (-24.49)(0.01) \\ &\approx 0.56 - 0.25 \\ &= 0.31. \end{aligned}$$

As expected, an increase in weight *increases* the BMI, while an increase in height *decreases* the BMI. In this case, the two contributions combine for a net increase in the BMI.

b. The changes dw, dh, and dB that appear in the differential change formula in part (a) are *absolute changes*. The corresponding *relative*, or *percentage*, changes are $\frac{dw}{w}$, $\frac{dh}{h}$, and $\frac{dB}{B}$. To introduce relative changes into the change formula, we divide both sides of $dB = B_w\,dw + B_h\,dh$ by $B = w/h^2 = wh^{-2}$. The result is

$$\begin{aligned} \frac{dB}{B} &= B_w \frac{dw}{wh^{-2}} + B_h \frac{dh}{wh^{-2}} \\ &= \frac{1}{h^2}\frac{dw}{wh^{-2}} - \frac{2w}{h^3}\frac{dh}{wh^{-2}} && \text{Substitute for } B_w \text{ and } B_h. \\ &= \underbrace{\frac{dw}{w}}_{\substack{\text{relative}\\ \text{change}\\ \text{in } w}} - 2\underbrace{\frac{dh}{h}}_{\substack{\text{relative}\\ \text{change}\\ \text{in } h}}. && \text{Simplify.} \end{aligned}$$

This expression relates the relative changes in w, h, and B. With h constant $(dh = 0)$, a 1% change in w $(dw/w = 0.01)$ produces approximately a 1% change of the same sign in B. With w constant $(dw = 0)$, a 1% change in h $(dh/h = 0.01)$ produces approximately a 2% change in B of the opposite sign. We see that the BMI formula is more sensitive to small changes in h than in w.

Related Exercise 44 ◄

➤ See Exercises 68–69 for general results about relative or percentage changes in functions.

QUICK CHECK 4 In Example 5, interpret the facts that $B_w > 0$ and $B_h < 0$, for $w, h > 0$. ◄

The differential for functions of two variables extends naturally to more variables. For example, if f is differentiable at (x, y, z) with $w = f(x, y, z)$, then

$$dw = f_x(x, y, z)\,dx + f_y(x, y, z)\,dy + f_z(x, y, z)\,dz.$$

The differential dw (or df) gives the approximate change in f at the point (x, y, z) due to changes of dx, dy, and dz in the independent variables.

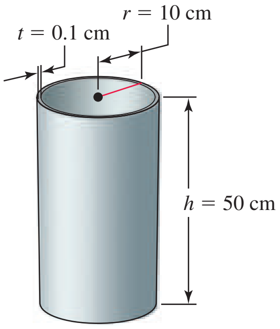


Figure 15.66

EXAMPLE 6 Manufacturing errors A company manufactures cylindrical aluminum tubes to rigid specifications. The tubes are designed to have an outside radius of $r = 10$ cm, a height of $h = 50$ cm, and a thickness of $t = 0.1$ cm (Figure 15.66). The manufacturing process produces tubes with a maximum error of ± 0.05 cm in the radius and height, and a maximum error of ± 0.0005 cm in the thickness. The volume of the cylindrical tube is $V(r, h, t) = \pi h t(2r - t)$. Use differentials to estimate the maximum error in the volume of a tube.

SOLUTION The approximate change in the volume of a tube due to changes dr, dh, and dt in the radius, height, and thickness, respectively, is

$$dV = V_r\,dr + V_h\,dh + V_t\,dt.$$

The partial derivatives evaluated at $r = 10$, $h = 50$, and $t = 0.1$ are

$$\begin{aligned} V_r(r, h, t) &= 2\pi h t, & V_r(10, 50, 0.1) &= 10\pi, \\ V_h(r, h, t) &= \pi t(2r - t), & V_h(10, 50, 0.1) &= 1.99\pi, \\ V_t(r, h, t) &= 2\pi h(r - t), & V_t(10, 50, 0.1) &= 990\pi. \end{aligned}$$

We let $dr = dh = 0.05$ and $dt = 0.0005$ be the maximum errors in the radius, height, and thickness, respectively. The maximum error in the volume is approximately

$$\begin{aligned} dV &= V_r(10, 50, 0.1)\,dr + V_h(10, 50, 0.1)\,dh + V_t(10, 50, 0.1)\,dt \\ &= 10\pi(0.05) + 1.99\pi(0.05) + 990\pi(0.0005) \\ &\approx 1.57 + 0.31 + 1.56 \\ &= 3.44. \end{aligned}$$

The maximum error in the volume is approximately 3.44 cm^3. Notice that the "magnification factor" for the thickness (990π) is roughly 100 and 500 times greater than the magnification factors for the radius and height, respectively. This means that for the same errors in r, h, and t, the volume is far more sensitive to errors in the thickness. The partial derivatives allow us to do a sensitivity analysis to determine which independent (input) variables are most critical in producing change in the dependent (output) variable.

Related Exercise 52 ◄

SECTION 15.6 EXERCISES

Getting Started

1. Suppose **n** is a vector normal to the tangent plane of the surface $F(x, y, z) = 0$ at a point. How is **n** related to the gradient of F at that point?

2. Write the explicit function $z = xy^2 + x^2y - 10$ in the implicit form $F(x, y, z) = 0$.

3. Write an equation for the plane tangent to the surface $F(x, y, z) = 0$ at the point (a, b, c).

4. Write an equation for the plane tangent to the surface $z = f(x, y)$ at the point $(a, b, f(a, b))$.

5. Explain how to approximate a function f at a point near (a, b), where the values of f, f_x, and f_y are known at (a, b).

6. Explain how to approximate the change in a function f when the independent variables change from (a, b) to $(a + \Delta x, b + \Delta y)$.

7. Write the approximate change formula for a function $z = f(x, y)$ at the point (x, y) in terms of differentials.

8. Write the differential dw for the function $w = f(x, y, z)$.

9–10. *Suppose $f(1, 2) = 4$, $f_x(1, 2) = 5$, and $f_y(1, 2) = -3$.*

9. Find an equation of the plane tangent to the surface $z = f(x, y)$ at the point $P_0(1, 2, 4)$.

10. Find the linear approximation to f at $P_0(1, 2, 4)$, and use it to estimate $f(1.01, 1.99)$.

11–12. *Suppose $F(0, 2, 1) = 0$, $F_x(0, 2, 1) = 3$, $F_y(0, 2, 1) = -1$, and $F_z(0, 2, 1) = 6$.*

11. Find an equation of the plane tangent to the surface $F(x, y, z) = 0$ at the point $P_0(0, 2, 1)$.

12. Find the linear approximation to the function $w = F(x, y, z)$ at the point $P_0(0, 2, 1)$ and use it to estimate $F(0.1, 2, 0.99)$.

Practice Exercises

13–28. Tangent planes *Find an equation of the plane tangent to the following surfaces at the given points (two planes and two equations).*

13. $x^2 + y + z = 3$; $(1, 1, 1)$ and $(2, 0, -1)$

14. $x^2 + y^3 + z^4 = 2$; $(1, 0, 1)$ and $(-1, 0, 1)$

15. $xy + xz + yz - 12 = 0$; $(2, 2, 2)$ and $(2, 0, 6)$

16. $x^2 + y^2 - z^2 = 0$; $(3, 4, 5)$ and $(-4, -3, 5)$

17. $z = 4 - 2x^2 - y^2$; $(2, 2, -8)$ and $(-1, -1, 1)$

18. $z = 2 + 2x^2 + \dfrac{y^2}{2}$; $\left(-\dfrac{1}{2}, 1, 3\right)$ and $(3, -2, 22)$

19. $z = e^{xy}$; $(1, 0, 1)$ and $(0, 1, 1)$

20. $z = \sin xy + 2$; $(1, 0, 2)$ and $(0, 5, 2)$

21. $xy \sin z = 1$; $(1, 2, \pi/6)$ and $(-2, -1, 5\pi/6)$

22. $yze^{xz} - 8 = 0$; $(0, 2, 4)$ and $(0, -8, -1)$

23. $z^2 - x^2/16 - y^2/9 - 1 = 0$; $(4, 3, -\sqrt{3})$ and $(-8, 9, \sqrt{14})$

24. $2x + y^2 - z^2 = 0$; $(0, 1, 1)$ and $(4, 1, -3)$

25. $z = x^2 e^{x-y}$; $(2, 2, 4)$ and $(-1, -1, 1)$

26. $z = \ln(1 + xy)$; $(1, 2, \ln 3)$ and $(-2, -1, \ln 3)$

27. $z = \dfrac{x - y}{x^2 + y^2}$; $\left(1, 2, -\dfrac{1}{5}\right)$ and $\left(2, -1, \dfrac{3}{5}\right)$

28. $z = 2\cos(x - y) + 2$; $\left(\dfrac{\pi}{6}, -\dfrac{\pi}{6}, 3\right)$ and $\left(\dfrac{\pi}{3}, \dfrac{\pi}{3}, 4\right)$

29–32. Tangent planes *Find an equation of the plane tangent to the following surfaces at the given point.*

29. $z = \tan^{-1}(xy)$; $\left(1, 1, \dfrac{\pi}{4}\right)$

30. $z = \tan^{-1}(x + y)$; $(0, 0, 0)$

31. $\sin xyz = \dfrac{1}{2}$; $\left(\pi, 1, \dfrac{1}{6}\right)$

32. $\dfrac{x + z}{y - z} = 2$; $(4, 2, 0)$

33–38. Linear approximation

a. *Find the linear approximation to the function f at the given point.*
b. *Use part (a) to estimate the given function value.*

33. $f(x, y) = xy + x - y$; $(2, 3)$; estimate $f(2.1, 2.99)$.

34. $f(x, y) = 12 - 4x^2 - 8y^2$; $(-1, 4)$; estimate $f(-1.05, 3.95)$.

35. $f(x, y) = -x^2 + 2y^2$; $(3, -1)$; estimate $f(3.1, -1.04)$.

36. $f(x, y) = \sqrt{x^2 + y^2}$; $(3, -4)$; estimate $f(3.06, -3.92)$.

37. $f(x, y, z) = \ln(1 + x + y + 2z)$; $(0, 0, 0)$; estimate $f(0.1, -0.2, 0.2)$.

38. $f(x, y, z) = \dfrac{x + y}{x - z}$; $(3, 2, 4)$; estimate $f(2.95, 2.05, 4.02)$.

39–42. Approximate function change *Use differentials to approximate the change in z for the given changes in the independent variables.*

39. $z = 2x - 3y - 2xy$ when (x, y) changes from $(1, 4)$ to $(1.1, 3.9)$

40. $z = -x^2 + 3y^2 + 2$ when (x, y) changes from $(-1, 2)$ to $(-1.05, 1.9)$

41. $z = e^{x+y}$ when (x, y) changes from $(0, 0)$ to $(0.1, -0.05)$

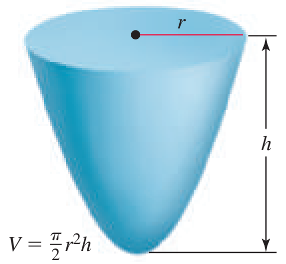
42. $z = \ln(1 + x + y)$ when (x, y) changes from $(0, 0)$ to $(-0.1, 0.03)$

43. **Changes in torus surface area** The surface area of a torus with an inner radius r and an outer radius $R > r$ is $S = 4\pi^2(R^2 - r^2)$.
   a. If r increases and R decreases, does S increase or decrease, or is it impossible to say?
   b. If r increases and R increases, does S increase or decrease, or is it impossible to say?
   c. Estimate the change in the surface area of the torus when r changes from $r = 3.00$ to $r = 3.05$ and R changes from $R = 5.50$ to $R = 5.65$.
   d. Estimate the change in the surface area of the torus when r changes from $r = 3.00$ to $r = 2.95$ and R changes from $R = 7.00$ to $R = 7.04$.
   e. Find the relationship between the changes in r and R that leaves the surface area (approximately) unchanged.

44. **Changes in cone volume** The volume of a right circular cone with radius r and height h is $V = \pi r^2 h/3$.
   a. Approximate the change in the volume of the cone when the radius changes from $r = 6.5$ to $r = 6.6$ and the height changes from $h = 4.20$ to $h = 4.15$.
   b. Approximate the change in the volume of the cone when the radius changes from $r = 5.40$ to $r = 5.37$ and the height changes from $h = 12.0$ to $h = 11.96$.

45. **Area of an ellipse** The area of an ellipse with axes of length $2a$ and $2b$ is $A = \pi ab$. Approximate the percent change in the area when a increases by 2% and b increases by 1.5%.

46. **Volume of a paraboloid** The volume of a segment of a circular paraboloid (see figure) with radius r and height h is $V = \pi r^2 h/2$. Approximate the percent change in the volume when the radius decreases by 1.5% and the height increases by 2.2%.



47–50. Differentials with more than two variables *Write the differential dw in terms of the differentials of the independent variables.*

47. $w = f(x, y, z) = xy^2 + x^2 z + yz^2$

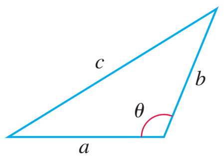
48. $w = f(x, y, z) = \sin(x + y - z)$

49. $w = f(u, x, y, z) = \dfrac{u + x}{y + z}$

50. $w = f(p, q, r, s) = \dfrac{pq}{rs}$

T 51. **Law of Cosines** The side lengths of any triangle are related by the Law of Cosines,

$$c^2 = a^2 + b^2 - 2ab\cos\theta.$$



   a. Estimate the change in the side length c when a changes from $a = 2$ to $a = 2.03$, b changes from $b = 4.00$ to $b = 3.96$, and θ changes from $\theta = \dfrac{\pi}{3}$ to $\theta = \dfrac{\pi}{3} + \dfrac{\pi}{90}$.
   b. If a changes from $a = 2$ to $a = 2.03$ and b changes from $b = 4.00$ to $b = 3.96$, is the resulting change in c greater in magnitude when $\theta = \dfrac{\pi}{20}$ (small angle) or when $\theta = \dfrac{9\pi}{20}$ (close to a right angle)?

52. Travel cost The cost of a trip that is L miles long, driving a car that gets m miles per gallon, with gas costs of $\$p/\text{gal}$ is $C = Lp/m$ dollars. Suppose you plan a trip of $L = 1500$ mi in a car that gets $m = 32$ mi/gal, with gas costs of $p = \$3.80/\text{gal}$.

a. Explain how the cost function is derived.
b. Compute the partial derivatives C_L, C_m, and C_p. Explain the meaning of the signs of the derivatives in the context of this problem.
c. Estimate the change in the total cost of the trip if L changes from $L = 1500$ to $L = 1520$, m changes from $m = 32$ to $m = 31$, and p changes from $p = \$3.80$ to $p = \$3.85$.
d. Is the total cost of the trip (with $L = 1500$ mi, $m = 32$ mi/gal, and $p = \$3.80$) more sensitive to a 1% change in L, in m, or in p (assuming the other two variables are fixed)? Explain.

53. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. The planes tangent to the cylinder $x^2 + y^2 = 1$ in $\mathbb{R}^3$ all have the form $ax + bz + c = 0$.
b. Suppose $w = xy/z$, for $x > 0$, $y > 0$, and $z > 0$. A decrease in z with x and y fixed results in an increase in w.
c. The gradient $\nabla F(a, b, c)$ lies in the plane tangent to the surface $F(x, y, z) = 0$ at (a, b, c).

54–57. Horizontal tangent planes *Find the points at which the following surfaces have horizontal tangent planes.*

54. $x^2 + 2y^2 + z^2 - 2x - 2z - 2 = 0$

55. $x^2 + y^2 - z^2 - 2x + 2y + 3 = 0$

56. $z = \sin(x - y)$ in the region $-2\pi \le x \le 2\pi$, $-2\pi \le y \le 2\pi$

57. $z = \cos 2x \sin y$ in the region $-\pi \le x \le \pi$, $-\pi \le y \le \pi$

58. Heron's formula The area of a triangle with sides of length a, b, and c is given by a formula from antiquity called Heron's formula:

$$A = \sqrt{s(s - a)(s - b)(s - c)},$$

where $s = \frac{1}{2}(a + b + c)$ is the *semiperimeter* of the triangle.

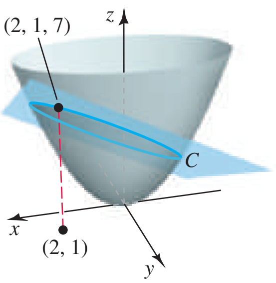
a. Find the partial derivatives A_a, A_b, and A_c.
b. A triangle has sides of length $a = 2$, $b = 4$, $c = 5$. Estimate the change in the area when a increases by 0.03, b decreases by 0.08, and c increases by 0.6.
c. For an equilateral triangle with $a = b = c$, estimate the percent change in the area when all sides increase in length by $p\%$.

59. Surface area of a cone A cone with height h and radius r has a lateral surface area (the curved surface only, excluding the base) of $S = \pi r\sqrt{r^2 + h^2}$.

a. Estimate the change in the surface area when r increases from $r = 2.50$ to $r = 2.55$ and h decreases from $h = 0.60$ to $h = 0.58$.
b. When $r = 100$ and $h = 200$, is the surface area more sensitive to a small change in r or a small change in h? Explain.

60. Line tangent to an intersection curve Consider the paraboloid $z = x^2 + 3y^2$ and the plane $z = x + y + 4$, which intersects the paraboloid in a curve C at $(2, 1, 7)$ (see figure). Find the equation of the line tangent to C at the point $(2, 1, 7)$. Proceed as follows.

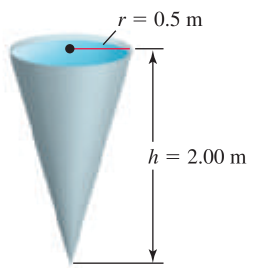
a. Find a vector normal to the plane at $(2, 1, 7)$.
b. Find a vector normal to the plane tangent to the paraboloid at $(2, 1, 7)$.
c. Argue that the line tangent to C at $(2, 1, 7)$ is orthogonal to both normal vectors found in parts (a) and (b). Use this fact to find a direction vector for the tangent line.
d. Knowing a point on the tangent line and the direction of the tangent line, write an equation of the tangent line in parametric form.



61. Batting averages Batting averages in baseball are defined by $A = x/y$, where $x \ge 0$ is the total number of hits and $y > 0$ is the total number of at-bats. Treat x and y as positive real numbers and note that $0 \le A \le 1$.

a. Use differentials to estimate the change in the batting average if the number of hits increases from 60 to 62 and the number of at-bats increases from 175 to 180.
b. If a batter currently has a batting average of $A = 0.350$, does the average decrease more if the batter fails to get a hit than it increases if the batter gets a hit?
c. Does the answer to part (b) depend on the current batting average? Explain.

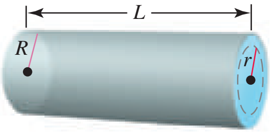
62. Water-level changes A conical tank with radius 0.50 m and height 2.00 m is filled with water (see figure). Water is released from the tank, and the water level drops by 0.05 m (from 2.00 m to 1.95 m). Approximate the change in the volume of water in the tank. (*Hint:* When the water level drops, both the radius and height of the cone of water change.)



63. Flow in a cylinder Poiseuille's Law is a fundamental law of fluid dynamics that describes the flow velocity of a viscous incompressible fluid in a cylinder (it is used to model blood flow through veins and arteries). It says that in a cylinder of radius R and length L, the velocity of the fluid $r \le R$ units from the centerline of the cylinder is $V = \frac{P}{4Lv}(R^2 - r^2)$, where P is the difference in the pressure between the ends of the cylinder, and v is the viscosity of the fluid (see figure). Assuming P and v are constant, the velocity V along the centerline of the cylinder $(r = 0)$ is $V = \frac{kR^2}{L}$, where k is a constant that we will take to be $k = 1$.

a. Estimate the change in the centerline velocity $(r = 0)$ if the radius of the flow cylinder increases from $R = 3$ cm to $R = 3.05$ cm and the length increases from $L = 50$ cm to $L = 50.5$ cm.
b. Estimate the percent change in the centerline velocity if the radius of the flow cylinder R decreases by 1% and its length L increases by 2%.

c. Complete the following sentence: If the radius of the cylinder increases by $p\%$, then the length of the cylinder must increase by approximately ___% in order for the velocity to remain constant.



Explorations and Challenges

64. Floating-point operations In general, real numbers (with infinite decimal expansions) cannot be represented exactly in a computer by floating-point numbers (with finite decimal expansions). Suppose floating-point numbers on a particular computer carry an error of at most 10^{-16}. Estimate the maximum error that is committed in evaluating the following functions. Express the error in absolute and relative (percent) terms.

a. $f(x, y) = xy$ **b.** $f(x, y) = \dfrac{x}{y}$

c. $F(x, y, z) = xyz$ **d.** $F(x, y, z) = \dfrac{x/y}{z}$

65. Probability of at least one encounter Suppose in a large group of people, a fraction $0 \le r \le 1$ of the people have flu. The probability that in n random encounters you will meet at least one person with flu is $P = f(n, r) = 1 - (1 - r)^n$. Although n is a positive integer, regard it as a positive real number.

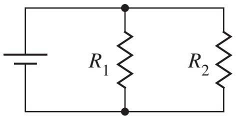
a. Compute f_r and f_n.

b. How sensitive is the probability P to the flu rate r? Suppose you meet $n = 20$ people. Approximately how much does the probability P increase if the flu rate increases from $r = 0.1$ to $r = 0.11$ (with n fixed)?

c. Approximately how much does the probability P increase if the flu rate increases from $r = 0.9$ to $r = 0.91$ with $n = 20$?

d. Interpret the results of parts (b) and (c).

66. Two electrical resistors When two electrical resistors with resistance $R_1 > 0$ and $R_2 > 0$ are wired in parallel in a circuit (see figure), the combined resistance R, measured in ohms (Ω), is given by $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2}$.



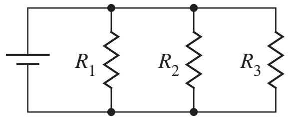
a. Estimate the change in R if R_1 increases from $2\ \Omega$ to $2.05\ \Omega$ and R_2 decreases from $3\ \Omega$ to $2.95\ \Omega$.

b. Is it true that if $R_1 = R_2$ and R_1 increases by the same small amount as R_2 decreases, then R is approximately unchanged? Explain.

c. Is it true that if R_1 and R_2 increase, then R increases? Explain.

d. Suppose $R_1 > R_2$ and R_1 increases by the same small amount as R_2 decreases. Does R increase or decrease?

67. Three electrical resistors Extending Exercise 66, when three electrical resistors with resistances $R_1 > 0$, $R_2 > 0$, and $R_3 > 0$ are wired in parallel in a circuit (see figure), the combined resistance R, measured in ohms (Ω), is given by $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3}$. Estimate the change in R if R_1 increases from $2\ \Omega$ to $2.05\ \Omega$, R_2 decreases from $3\ \Omega$ to $2.95\ \Omega$, and R_3 increases from $1.5\ \Omega$ to $1.55\ \Omega$.



68. Power functions and percent change Suppose $z = f(x, y) = x^a y^b$, where a and b are real numbers. Let $\dfrac{dx}{x}$, $\dfrac{dy}{y}$, and $\dfrac{dz}{z}$ be the approximate relative (percent) changes in x, y, and z, respectively. Show that $\dfrac{dz}{z} = \dfrac{a(dx)}{x} + \dfrac{b(dy)}{y}$; that is, the relative changes are additive when weighted by the exponents a and b.

69. Logarithmic differentials Let f be a differentiable function of one or more variables that is positive on its domain.

a. Show that $d(\ln f) = \dfrac{df}{f}$.

b. Use part (a) to explain the statement that the absolute change in $\ln f$ is approximately equal to the relative change in f.

c. Let $f(x, y) = xy$, note that $\ln f = \ln x + \ln y$, and show that relative changes add; that is, $\dfrac{df}{f} = \dfrac{dx}{x} + \dfrac{dy}{y}$.

d. Let $f(x, y) = \dfrac{x}{y}$, note that $\ln f = \ln x - \ln y$, and show that relative changes subtract; that is, $\dfrac{df}{f} = \dfrac{dx}{x} - \dfrac{dy}{y}$.

e. Show that in a product of n numbers, $f = x_1 x_2 \cdots x_n$, the relative change in f is approximately equal to the sum of the relative changes in the variables.

70. Distance from a plane to an ellipsoid (Adapted from 1938 Putnam Exam) Consider the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ and the plane P given by $Ax + By + Cz + 1 = 0$. Let $h = (A^2 + B^2 + C^2)^{-1/2}$ and $m = (a^2A^2 + b^2B^2 + c^2C^2)^{1/2}$.

a. Find the equation of the plane tangent to the ellipsoid at the point (p, q, r).

b. Find the two points on the ellipsoid at which the tangent plane is parallel to P, and find equations of the tangent planes.

c. Show that the distance between the origin and the plane P is h.

d. Show that the distance between the origin and the tangent planes is hm.

e. Find a condition that guarantees the plane P does not intersect the ellipsoid.

QUICK CHECK ANSWERS

1. $F(x, y, z) = z - xy - x + y = 0$ **2.** If you walk in the positive x-direction from $(-1, 2, 1)$, then you walk uphill. If you walk in the positive y-direction from $(-1, 2, 1)$, then you walk downhill. **3.** If $\Delta x = 0$, then the change formula becomes $\Delta z \approx f_y(a, b)\,\Delta y$, which is the change formula for the single variable y. If $\Delta y = 0$, then the change formula becomes $\Delta z \approx f_x(a, b)\,\Delta x$, which is the change formula for the single variable x. **4.** The BMI increases with weight w and decreases with height h. ◀

15.7 Maximum/Minimum Problems

In Chapter 4, we showed how to use derivatives to find maximum and minimum values of functions of a single variable. When those techniques are extended to functions of two variables, we discover both similarities and differences. The landscape of a surface is far more complicated than the profile of a curve in the plane, so we see more interesting features when working with several variables. In addition to peaks (maximum values) and hollows (minimum values), we encounter winding ridges, long valleys, and mountain passes. Yet despite these complications, many of the ideas used for single-variable functions reappear in higher dimensions. For example, the Second Derivative Test, suitably adapted for two variables, plays a central role. As with single-variable functions, the techniques developed here are useful for solving practical optimization problems.

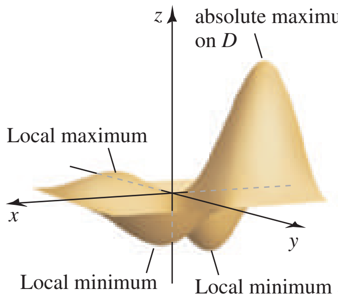


Figure 15.67

Local Maximum/Minimum Values

The concepts of local maximum and minimum values encountered in Chapter 4 extend readily to functions of two variables of the form $z = f(x, y)$. Figure 15.67 shows a general surface defined on a domain D, which is a subset of $\mathbb{R}^2$. The surface has peaks (local high points) and hollows (local low points) at points in the interior of D. The goal is to locate and classify these extreme points.

➤ We maintain the convention adopted in Chapter 4 that local maxima or minima occur at interior points of the domain. Recall that an open disk centered at (a, b) is the set of points within a circle centered at (a, b).

DEFINITION Local Maximum/Minimum Values

Suppose (a, b) is a point in a region R on which f is defined. If $f(x, y) \le f(a, b)$ for all (x, y) in the domain of f and in some open disk centered at (a, b), then $f(a, b)$ is a **local maximum value** of f. If $f(x, y) \ge f(a, b)$ for all (x, y) in the domain of f and in some open disk centered at (a, b), then $f(a, b)$ is a **local minimum value** of f. Local maximum and local minimum values are also called **local extreme values** or **local extrema**.

In familiar terms, a local maximum is a point on a surface from which you cannot walk uphill. A local minimum is a point from which you cannot walk downhill. The following theorem is the analog of Theorem 4.2.

THEOREM 15.14 Derivatives and Local Maximum/Minimum Values

If f has a local maximum or minimum value at (a, b) and the partial derivatives f_x and f_y exist at (a, b), then $f_x(a, b) = f_y(a, b) = 0$.

Proof: Suppose f has a local maximum value at (a, b). The function of one variable $g(x) = f(x, b)$, obtained by holding $y = b$ fixed, also has a local maximum at (a, b). By Theorem 4.2, $g'(a) = 0$. However, $g'(a) = f_x(a, b)$; therefore, $f_x(a, b) = 0$. Similarly, the function $h(y) = f(a, y)$, obtained by holding $x = a$ fixed, has a local maximum at (a, b), which implies that $f_y(a, b) = h'(b) = 0$. An analogous argument is used for the local minimum case. ◄

QUICK CHECK 1 The paraboloid $z = x^2 + y^2 - 4x + 2y + 5$ has a local minimum at $(2, -1)$. Verify the conclusion of Theorem 15.14 for this function. ◄

Suppose f is differentiable at (a, b) (ensuring the existence of a tangent plane) and f has a local extremum at (a, b). Then $f_x(a, b) = f_y(a, b) = 0$, which, when substituted into the equation of the tangent plane, gives the equation $z = f(a, b)$ (a constant). Therefore, if the tangent plane exists at a local extremum, then it is horizontal there.

Recall that for a function of one variable, the condition $f'(a) = 0$ does not guarantee a local extremum at a. A similar precaution must be taken with Theorem 15.14. The conditions $f_x(a, b) = f_y(a, b) = 0$ do not imply that f has a local extremum at (a, b), as we show momentarily. Theorem 15.14 provides *candidates* for local extrema. We call these candidates *critical points*, as we did for functions of one variable. Therefore, the procedure for locating local maximum and minimum values is to find the critical points and then determine whether these candidates correspond to genuine local maximum and minimum values.

DEFINITION Critical Point

An interior point (a, b) in the domain of f is a **critical point** of f if either

1. $f_x(a, b) = f_y(a, b) = 0$, or
2. at least one of the partial derivatives f_x and f_y does not exist at (a, b).

EXAMPLE 1 Finding critical points Find the critical points of $f(x, y) = xy(x - 2)(y + 3)$.

SOLUTION This function is differentiable at all points of $\mathbb{R}^2$, so the critical points occur only at points where $f_x(x, y) = f_y(x, y) = 0$. Computing and simplifying the partial derivatives, these conditions become

$$f_x(x, y) = 2y(x - 1)(y + 3) = 0$$
$$f_y(x, y) = x(x - 2)(2y + 3) = 0.$$

We must now identify all (x, y) pairs that satisfy both equations. The first equation is satisfied if and only if $y = 0$, $x = 1$, or $y = -3$. We consider each of these cases.

- Substituting $y = 0$, the second equation is $3x(x - 2) = 0$, which has solutions $x = 0$ and $x = 2$. So $(0, 0)$ and $(2, 0)$ are critical points.
- Substituting $x = 1$, the second equation is $-(2y + 3) = 0$, which has the solution $y = -\frac{3}{2}$. So $\left(1, -\frac{3}{2}\right)$ is a critical point.
- Substituting $y = -3$, the second equation is $-3x(x - 2) = 0$, which has roots $x = 0$ and $x = 2$. So $(0, -3)$ and $(2, -3)$ are critical points.

We find that there are five critical points: $(0, 0)$, $(2, 0)$, $\left(1, -\frac{3}{2}\right)$, $(0, -3)$, and $(2, -3)$. Some of these critical points may correspond to local maximum or minimum values. We will return to this example and a complete analysis shortly.

Related Exercises 15, 18 ◀

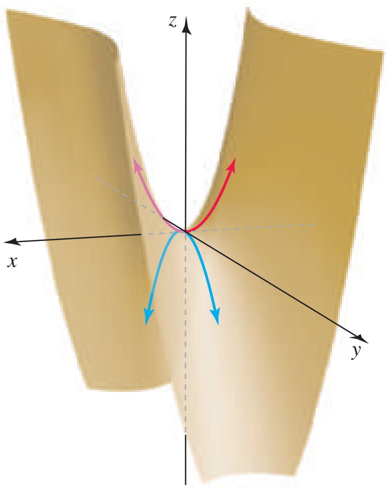
➤ The usual image of a saddle point is that of a mountain pass (or a horse saddle), where you can walk upward in some directions and downward in other directions. The definition of a saddle point given here includes other less common situations. For example, with this definition, the cylinder $z = x^3$ has a line of saddle points along the y-axis.

Second Derivative Test

Critical points are candidates for local extreme values. With functions of one variable, the Second Derivative Test is used to determine whether critical points correspond to local maxima or minima (the test can also be inconclusive). The analogous test for functions of two variables not only detects local maxima and minima, but also identifies another type of point known as a *saddle point*.

DEFINITION Saddle Point

Consider a function f that is differentiable at a critical point (a, b). Then f has a **saddle point** at (a, b) if, in every open disk centered at (a, b), there are points (x, y) for which $f(x, y) > f(a, b)$ and points for which $f(x, y) < f(a, b)$.



The hyperbolic paraboloid $z = x^2 - y^2$ has a saddle point at $(0, 0)$.

Figure 15.68

If (a, b) is a critical point of f and f has a saddle point at (a, b), then from the point $(a, b, f(a, b))$, it is possible to walk uphill in some directions and downhill in other directions. The function $f(x, y) = x^2 - y^2$ (a hyperbolic paraboloid) is a good example to remember. The surface *rises* from the critical point $(0, 0)$ along the x-axis and *falls* from $(0, 0)$ along the y-axis (**Figure 15.68**). We can easily check that $f_x(0, 0) = f_y(0, 0) = 0$, demonstrating that critical points do not necessarily correspond to local maxima or minima.

QUICK CHECK 2 Consider the plane tangent to a surface at a saddle point. In what direction does the normal to the plane point? ◀

➤ The Second Derivative Test for functions of a single variable states that if a is a critical point with $f'(a) = 0$, then $f''(a) > 0$ implies that f has a local minimum at a and $f''(a) < 0$ implies that f has a local maximum at a; if $f''(a) = 0$, the test is inconclusive. Theorem 15.15 is easier to remember if you notice the parallels between the two second derivative tests.

THEOREM 15.15 Second Derivative Test
Suppose the second partial derivatives of f are continuous throughout an open disk centered at the point (a, b), where $f_x(a, b) = f_y(a, b) = 0$. Let $D(x, y) = f_{xx}(x, y)\, f_{yy}(x, y) - (f_{xy}(x, y))^2$.

1. If $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then f has a local maximum value at (a, b).
2. If $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then f has a local minimum value at (a, b).
3. If $D(a, b) < 0$, then f has a saddle point at (a, b).
4. If $D(a, b) = 0$, then the test is inconclusive.

The proof of this theorem is given in Appendix A, but a few comments are in order. The test relies on the quantity $D(x, y) = f_{xx} f_{yy} - (f_{xy})^2$, which is called the **discriminant** of f. It can be remembered as the 2×2 determinant of the **Hessian** matrix $\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$, where $f_{xy} = f_{yx}$, provided these derivatives are continuous (Theorem 15.4). The condition $D(x, y) > 0$ means that the surface has the same general behavior in all directions near (a, b); either the surface rises in all directions or it falls in all directions. In the case that $D(a, b) = 0$, the test is inconclusive: (a, b) could correspond to a local maximum, a local minimum, or a saddle point.

Finally, another useful characterization of a saddle point can be derived from Theorem 15.15: The tangent plane at a saddle point lies both above and below the surface.

QUICK CHECK 3 Compute the discriminant $D(x, y)$ of $f(x, y) = x^2y^2$. ◄

EXAMPLE 2 Analyzing critical points Use the Second Derivative Test to classify the critical points of $f(x, y) = x^2 + 2y^2 - 4x + 4y + 6$.

SOLUTION We begin with the following derivative calculations:

$$f_x = 2x - 4, \qquad f_y = 4y + 4,$$
$$f_{xx} = 2, \qquad f_{xy} = f_{yx} = 0, \quad \text{and} \quad f_{yy} = 4.$$

Setting both f_x and f_y equal to zero yields the single critical point $(2, -1)$. The value of the discriminant at the critical point is $D(2, -1) = f_{xx} f_{yy} - (f_{xy})^2 = 8 > 0$. Furthermore, $f_{xx}(2, -1) = 2 > 0$. By the Second Derivative Test, f has a local minimum at $(2, -1)$; the value of the function at that point is $f(2, -1) = 0$ (Figure 15.69).

Related Exercise 24 ◄

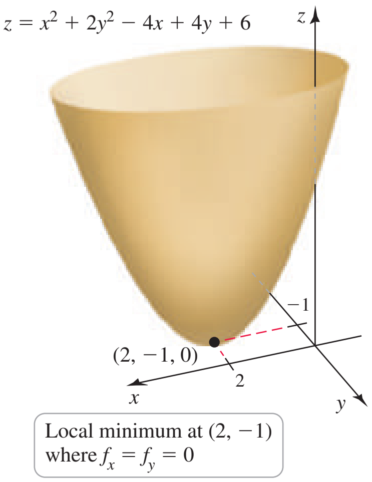


Figure 15.69

EXAMPLE 3 Analyzing critical points Use the Second Derivative Test to classify the critical points of $f(x, y) = xy(x - 2)(y + 3)$.

SOLUTION In Example 1, we determined that the critical points of f are $(0, 0)$, $(2, 0)$, $\left(1, -\frac{3}{2}\right)$, $(0, -3)$, and $(2, -3)$. The derivatives needed to evaluate the discriminant are

$$f_x = 2y(x - 1)(y + 3), \qquad f_y = x(x - 2)(2y + 3),$$
$$f_{xx} = 2y(y + 3), \qquad f_{xy} = 2(2y + 3)(x - 1), \quad \text{and} \quad f_{yy} = 2x(x - 2).$$

The values of the discriminant at the critical points and the conclusions of the Second Derivative Test are shown in Table 15.4.

Table 15.4

(x, y)	$D(x, y)$	f_{xx}	Conclusion
$(0, 0)$	-36	0	Saddle point
$(2, 0)$	-36	0	Saddle point
$\left(1, -\frac{3}{2}\right)$	9	$-\frac{9}{2}$	Local maximum
$(0, -3)$	-36	0	Saddle point
$(2, -3)$	-36	0	Saddle point

The surface described by f has one local maximum at $\left(1, -\frac{3}{2}\right)$, surrounded by four saddle points (**Figure 15.70a**). The structure of the surface may also be visualized by plotting the level curves of f (**Figure 15.70b**).

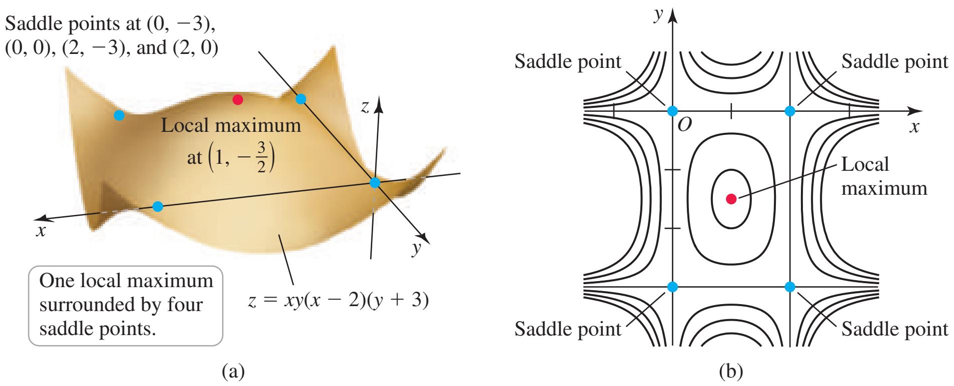


Figure 15.70

Related Exercise 27

EXAMPLE 4 **Inconclusive tests** Apply the Second Derivative Test to the following functions and interpret the results.

a. $f(x, y) = 2x^4 + y^4$ **b.** $f(x, y) = 2 - xy^2$

SOLUTION

a. The critical points of f satisfy the conditions

$$f_x = 8x^3 = 0 \quad \text{and} \quad f_y = 4y^3 = 0,$$

so the sole critical point is $(0, 0)$. The second partial derivatives evaluated at $(0, 0)$ are

$$f_{xx}(0, 0) = f_{xy}(0, 0) = f_{yy}(0, 0) = 0.$$

We see that $D(0, 0) = 0$, and the Second Derivative Test is inconclusive. While the bowl-shaped surface (**Figure 15.71**) described by f has a local minimum at $(0, 0)$, the surface also has a broad flat bottom, which makes the local minimum "invisible" to the Second Derivative Test.

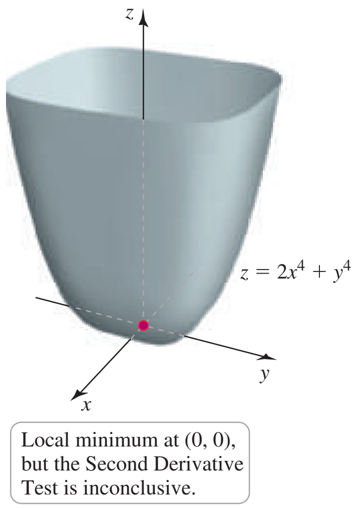


Figure 15.71

➤ The same "flat" behavior occurs with functions of one variable, such as $f(x) = x^4$. Although f has a local minimum at $x = 0$, the Second Derivative Test is inconclusive.

b. The critical points of this function satisfy

$$f_x(x, y) = -y^2 = 0 \quad \text{and} \quad f_y(x, y) = -2xy = 0.$$

The solutions of these equations have the form $(a, 0)$, where a is a real number. It is easy to check that the second partial derivatives evaluated at $(a, 0)$ are

$$f_{xx}(a, 0) = f_{xy}(a, 0) = 0 \quad \text{and} \quad f_{yy}(a, 0) = -2a.$$

Therefore, the discriminant is $D(a, 0) = 0$, and the Second Derivative Test is inconclusive. **Figure 15.72** shows that f has a flat ridge above the x-axis that the Second Derivative Test is unable to classify.

Related Exercises 29–30

➤ It is not surprising that the Second Derivative Test is inconclusive in Example 4b. The function has a line of local maxima at $(a, 0)$ for $a > 0$, a line of local minima at $(a, 0)$ for $a < 0$, and a saddle point at $(0, 0)$.

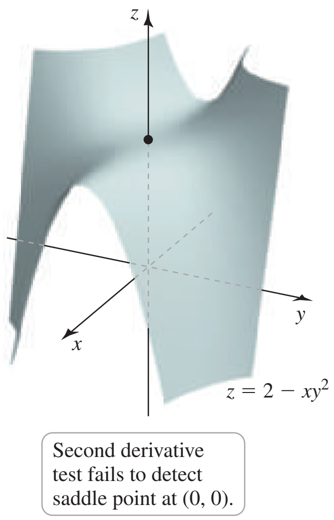


Figure 15.72

Absolute Maximum and Minimum Values

As in the one-variable case, we are often interested in knowing where a function of two or more variables attains its extreme values over its domain (or a subset of its domain).

DEFINITION **Absolute Maximum/Minimum Values**

Let f be defined on a set R in $\mathbb{R}^2$ containing the point (a, b). If $f(a, b) \geq f(x, y)$ for every (x, y) in R, then $f(a, b)$ is an **absolute maximum value** of f on R. If $f(a, b) \leq f(x, y)$ for every (x, y) in R, then $f(a, b)$ is an **absolute minimum value** of f on R.

It should be noted that the Extreme Value Theorem of Chapter 4 has an analog in $\mathbb{R}^2$ (or in higher dimensions): A function that is continuous on a closed bounded set in $\mathbb{R}^2$ attains its absolute maximum and absolute minimum values on that set. Absolute maximum and minimum values on a closed bounded set R occur in two ways.

➤ Recall that a *closed set* in $\mathbb{R}^2$ is a set that includes its boundary. A *bounded set* in $\mathbb{R}^2$ is a set that may be enclosed by a circle of finite radius.

- They may be local maximum or minimum values at interior points of R, where they are associated with critical points.
- They may occur on the boundary of R.

Therefore, the search for absolute maximum and minimum values on a closed bounded set amounts to examining the behavior of the function on the boundary of R and at the interior points of R.

➤ Example 5 is a *constrained optimization problem*, in which the goal is to maximize the volume subject to an additional condition called a *constraint*. We return to such problems in the next section and present another method of solution.

EXAMPLE 5 **Shipping regulations** A shipping company handles rectangular boxes provided the sum of the length, width, and height of the box does not exceed 96 in. Find the dimensions of the box that meets this condition and has the largest volume.

SOLUTION Let x, y, and z be the dimensions of the box; its volume is $V = xyz$. The box with the maximum volume must also satisfy the condition $x + y + z = 96$, which is used to eliminate any one of the variables from the volume function. Noting that $z = 96 - x - y$, the volume function becomes

$$V(x, y) = xy(96 - x - y).$$

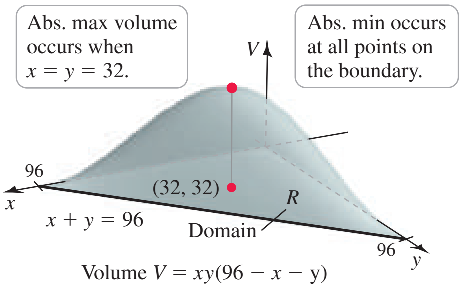


Figure 15.73

Notice that because x, y, and $96 - x - y$ are dimensions of the box, they must be nonnegative. The condition $96 - x - y \geq 0$ implies that $x + y \leq 96$. Therefore, among points in the xy-plane, the constraint is met only if (x, y) lies in the triangle bounded by the lines $x = 0$, $y = 0$, and $x + y = 96$ (Figure 15.73).

At this stage, we have reduced the original problem to a related problem: Find the absolute maximum value of $V(x, y) = xy(96 - x - y)$ over the triangular region

$$R = \{(x, y): 0 \leq x \leq 96, 0 \leq y \leq 96 - x\}.$$

The boundaries of R consist of the line segments $x = 0, 0 \leq y \leq 96$; $y = 0, 0 \leq x \leq 96$; and $x + y = 96, 0 \leq x \leq 96$. We find that on these boundary segments, $V = 0$. To determine the behavior of V at interior points of R, we need to find critical points. The critical points of V satisfy

$$V_x = 96y - 2xy - y^2 = y(96 - 2x - y) = 0$$
$$V_y = 96x - 2xy - x^2 = x(96 - 2y - x) = 0.$$

You can check that these two equations have four solutions: $(0, 0)$, $(96, 0)$, $(0, 96)$, and $(32, 32)$. The first three solutions lie on the boundary of the domain, where $V = 0$. At the fourth point, we have $V(32, 32) = 32{,}768 \text{ in}^3$, which is the absolute maximum volume of the box. The dimensions of the box with maximum volume are $x = 32$, $y = 32$, and $z = 96 - x - y = 32$ (it is a cube). We also found that V has an absolute minimum of 0 at every point on the boundary of R.

Related Exercise 43 ◀

We summarize the method of solution given in Example 5 in the following procedure box.

PROCEDURE **Finding Absolute Maximum/Minimum Values on Closed Bounded Sets**

Let f be continuous on a closed bounded set R in $\mathbb{R}^2$. To find the absolute maximum and minimum values of f on R:

1. Determine the values of f at all critical points in R.
2. Find the maximum and minimum values of f on the boundary of R.
3. The greatest function value found in Steps 1 and 2 is the absolute maximum value of f on R, and the least function value found in Steps 1 and 2 is the absolute minimum value of f on R.

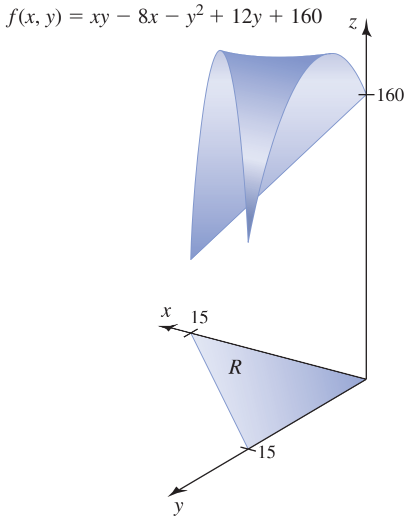


Figure 15.74

The techniques for carrying out Step 1 of this process have been presented. The challenge often lies in locating extreme values on the boundary. Examples 6 and 7 illustrate two approaches to handling the boundary of R. The first expresses the boundary using functions of a single variable, and the second describes the boundary parametrically. In both cases, finding extreme values on the boundary becomes a one-variable problem. In the next section, we discuss an alternative method for finding extreme values on boundaries.

EXAMPLE 6 **Extreme values over a region** Find the absolute maximum and minimum values of $f(x, y) = xy - 8x - y^2 + 12y + 160$ over the triangular region $R = \{(x, y): 0 \le x \le 15, 0 \le y \le 15 - x\}$.

SOLUTION Figure 15.74 shows the graph of f over the region R. The goal is to determine the absolute maximum and minimum values of f over R—including the boundary of R. We begin by finding the critical points of f on the interior of R. The partial derivatives of f are

$$f_x(x, y) = y - 8 \quad \text{and} \quad f_y(x, y) = x - 2y + 12.$$

The conditions $f_x(x, y) = f_y(x, y) = 0$ are satisfied only when $(x, y) = (4, 8)$, which is a point in the interior of R. This critical point is a candidate for the location of an extreme value of f, and the value of the function at this point is $f(4, 8) = 192$.

To search for extrema on the boundary of R, we consider each edge of R separately. Let C_1 be the line segment $\{(x, y): y = 0, \text{ for } 0 \le x \le 15\}$ on the x-axis, and define the single-variable function g_1 to equal f at all points along C_1 (Figure 15.75). We substitute $y = 0$ and find that g_1 has the form

$$g_1(x) = f(x, 0) = 160 - 8x.$$

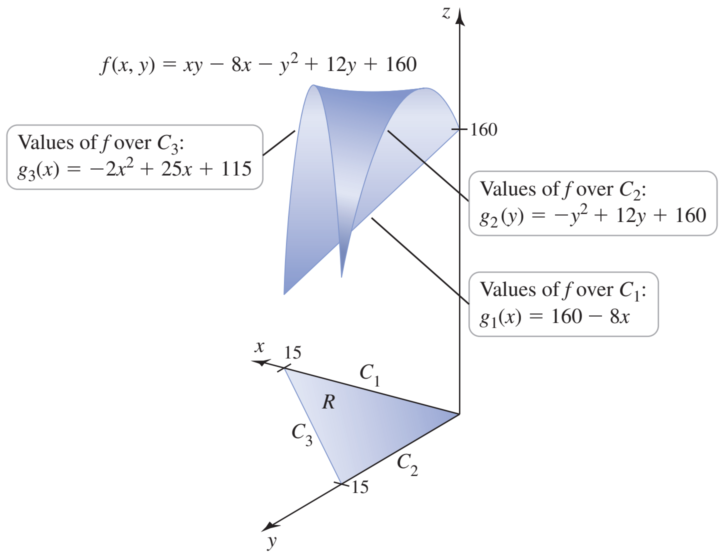


Figure 15.75

Using what we learned in Chapter 4, the candidates for absolute extreme values of g_1 on $0 \le x \le 15$ occur at critical points and endpoints. Specifically, the critical points of g_1 correspond to values where its derivative is zero, but in this case $g_1'(x) = -8$. So there is no critical point, which implies that the extreme values of g_1 occur at the endpoints of the interval $[0, 15]$. At the endpoints, we find that

$$g_1(0) = f(0, 0) = 160 \quad \text{and} \quad g_1(15) = f(15, 0) = 40.$$

Let's set aside this information while we do a similar analysis on the other two edges of the boundary of R.

Let C_2 be the line segment $\{(x, y): x = 0, \text{ for } 0 \le y \le 15\}$ and define g_2 to equal f on C_2 (Figure 15.75). Substituting $x = 0$, we see that

$$g_2(y) = f(0, y) = -y^2 + 12y + 160.$$

The critical points of g_2 satisfy

$$g_2'(y) = -2y + 12 = 0,$$

which has the single root $y = 6$. Evaluating g_2 at this point and the endpoints, we have

$$g_2(6) = f(0, 6) = 196, \quad g_2(0) = f(0, 0) = 160, \quad \text{and} \quad g_2(15) = f(0, 15) = 115.$$

Observe that $g_1(0) = g_2(0)$ because C_1 and C_2 intersect at the origin.

Finally, we let C_3 be the line segment $\{(x, y): y = 15 - x, 0 \le x \le 15\}$ and define g_3 to equal f on C_3 (Figure 15.75). Substituting $y = 15 - x$ and simplifying, we find that

$$g_3(x) = f(x, 15 - x) = -2x^2 + 25x + 115.$$

The critical points of g_3 satisfy

$$g_3'(x) = -4x + 25,$$

whose only root on the interval $0 \le x \le 15$ is $x = 6.25$. Evaluating g_3 at this critical point and the endpoints, we have

$$g_3(6.25) = f(6.25, 8.75) = 193.125, \quad g_3(15) = f(15, 0) = 40, \quad \text{and}$$
$$g_3(0) = f(0, 15) = 115.$$

Observe that $g_3(15) = g_1(15)$ and $g_3(0) = g_2(15)$; the only new candidate for the location of an extreme value is the point $(6.25, 8.75)$.

Collecting and summarizing our work, we have 6 candidates for absolute extreme values:

$$f(4, 8) = 192, \quad f(0, 0) = 160, \quad f(15, 0) = 40, \quad f(0, 6) = 196,$$
$$f(0, 15) = 115, \quad \text{and} \quad f(6.25, 8.75) = 193.125.$$

We see that f has an absolute minimum value of 40 at $(15, 0)$ and an absolute maximum value of 196 at $(0, 6)$. These findings are illustrated in Figure 15.76.

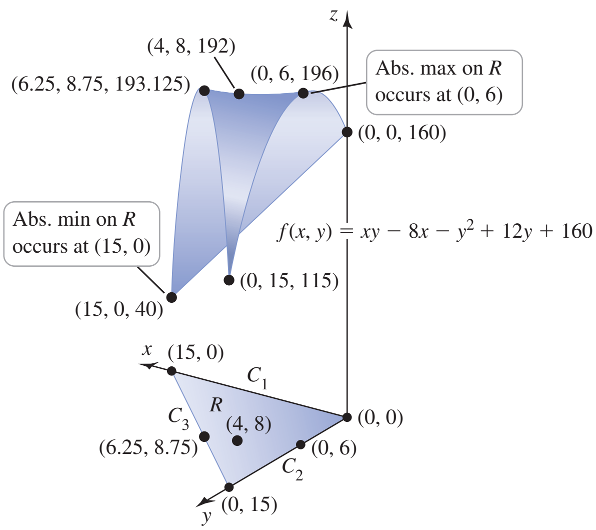


Figure 15.76

Related Exercise 52 ◄

➤ Finding absolute extrema on a closed set does not require using the Second Derivative Test to classify the critical points. In Example 7, the test *could* be used to show that $\left(\frac{1}{\sqrt{3}}, 0\right)$ and $\left(-\frac{1}{\sqrt{3}}, 0\right)$ correspond to a saddle point and a local maximum, respectively, but that information isn't needed.

EXAMPLE 7 Absolute maximum and minimum values Find the absolute maximum and minimum values of $f(x, y) = \frac{1}{2}(x^3 - x - y^2) + 3$ on the region $R = \{(x, y): x^2 + y^2 \le 1\}$ (the closed disk centered at $(0, 0)$ with radius 1).

SOLUTION We begin by locating the critical points of f on the interior of R. The critical points satisfy the equations

$$f_x(x, y) = \frac{1}{2}(3x^2 - 1) = 0 \quad \text{and} \quad f_y(x, y) = -y = 0,$$

which have the solutions $x = \pm\dfrac{1}{\sqrt{3}}$ and $y = 0$. The values of the function at these points are $f\left(\dfrac{1}{\sqrt{3}}, 0\right) = 3 - \dfrac{1}{3\sqrt{3}}$ and $f\left(-\dfrac{1}{\sqrt{3}}, 0\right) = 3 + \dfrac{1}{3\sqrt{3}}$.

➤ Recall that a parametric description of a circle of radius $|a|$ centered at the origin is $x = a\cos\theta$, $y = a\sin\theta$, for $0 \le \theta \le 2\pi$.

We now determine the maximum and minimum values of f on the boundary of R, which is a circle of radius 1 described by the parametric equations

$$x = \cos\theta, \quad y = \sin\theta, \quad \text{for} \quad 0 \le \theta \le 2\pi.$$

Substituting x and y in terms of θ into the function f, we obtain a new function $g(\theta)$ that gives the values of f on the boundary of R:

$$g(\theta) = \frac{1}{2}(\cos^3\theta - \cos\theta - \sin^2\theta) + 3.$$

Finding the maximum and minimum boundary values is now a one-variable problem. The critical points of g satisfy

$$\begin{aligned} g'(\theta) &= \frac{1}{2}(-3\cos^2\theta\sin\theta + \sin\theta - 2\sin\theta\cos\theta) \\ &= -\frac{1}{2}\sin\theta(3\cos^2\theta + 2\cos\theta - 1) \\ &= -\frac{1}{2}\sin\theta(3\cos\theta - 1)(\cos\theta + 1) = 0. \end{aligned}$$

➤ There are two solutions to the equation $\cos\theta = \frac{1}{3}$ on the interval $(0, 2\pi)$. Recall, however, that using the inverse cosine to solve the equation reveals only the solution $\theta = \cos^{-1}\frac{1}{3}$ because the range of $\cos^{-1}x$ is $[0, \pi]$. The other solution, $\theta = 2\pi - \cos^{-1}\frac{1}{3}$, is found using symmetry.

This condition is satisfied when $\sin\theta = 0$, $\cos\theta = \dfrac{1}{3}$, or $\cos\theta = -1$. The solutions of these equations on the interval $(0, 2\pi)$ are $\theta = \pi$, $\theta = \cos^{-1}\dfrac{1}{3}$, and $\theta = 2\pi - \cos^{-1}\dfrac{1}{3}$, which correspond to the points $(-1, 0)$, $\left(\dfrac{1}{3}, \dfrac{2\sqrt{2}}{3}\right)$, and $\left(\dfrac{1}{3}, -\dfrac{2\sqrt{2}}{3}\right)$ in the xy-plane, respectively. Notice that the endpoints of the interval ($\theta = 0$ and $\theta = 2\pi$) correspond to the same point on the boundary of R, namely $(1, 0)$.

Having completed the first two steps of the procedure, we have six function values to consider:

- $f\left(\dfrac{1}{\sqrt{3}}, 0\right) = 3 - \dfrac{1}{3\sqrt{3}} \approx 2.81$ and $f\left(-\dfrac{1}{\sqrt{3}}, 0\right) = 3 + \dfrac{1}{3\sqrt{3}} \approx 3.19$ (critical points),
- $f(-1, 0) = 3$ (boundary point),
- $f\left(\dfrac{1}{3}, \dfrac{2\sqrt{2}}{3}\right) = f\left(\dfrac{1}{3}, -\dfrac{2\sqrt{2}}{3}\right) = \dfrac{65}{27}$ (boundary points), and
- $f(1, 0) = 3$ (boundary point).

➤ Observe that the level curves of f in Figure 15.77b appear to be tangent to the blue curve $x^2 + y^2 = 1$ (the boundary of the region R) at the points corresponding to the maximum and minimum values of f on this boundary. The significance of this observation is explained in Section 15.8.

The greatest value of f on R, $f\left(-\dfrac{1}{\sqrt{3}}, 0\right) = 3 + \dfrac{1}{3\sqrt{3}}$, is the absolute maximum value, and it occurs at an interior point (**Figure 15.77a**). The least value, $f\left(\dfrac{1}{3}, \dfrac{2\sqrt{2}}{3}\right) = f\left(\dfrac{1}{3}, -\dfrac{2\sqrt{2}}{3}\right) = \dfrac{65}{27}$, is the absolute minimum value, and it occurs at two symmetric boundary points. Also revealing is the plot of the level curves of the surface with the boundary of R superimposed (**Figure 15.77b**). As the boundary of R is traversed, the values of f vary, reaching a maximum value of 3 at $(1, 0)$ and $(-1, 0)$, and a minimum value of $\dfrac{65}{27}$ at $\left(\dfrac{1}{3}, \dfrac{2\sqrt{2}}{3}\right)$ and $\left(\dfrac{1}{3}, -\dfrac{2\sqrt{2}}{3}\right)$.

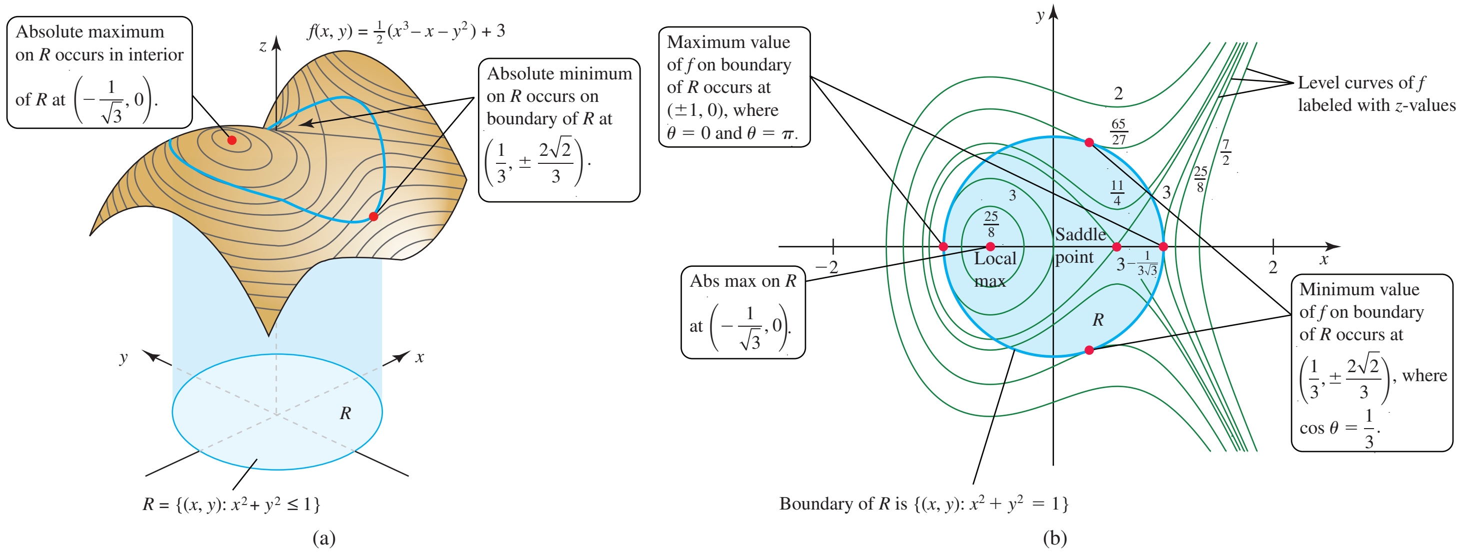


Figure 15.77

Related Exercises 47–48 ◂

Open and/or Unbounded Regions Finding absolute maximum and minimum values of a function on an open region (for example, $R = \{(x, y) = x^2 + y^2 < 9\}$) or an unbounded domain (for example, $R = \{(x, y): x > 0, y > 0\}$) presents additional challenges. Because there is no systematic procedure for dealing with such problems, some ingenuity is generally needed. Notice that absolute extrema may not exist on such regions.

EXAMPLE 8 Absolute extreme values on an open region Find the absolute maximum and minimum values of $f(x, y) = 4 - x^2 - y^2$ on the open disk $R = \{(x, y): x^2 + y^2 < 1\}$ (if they exist).

SOLUTION You should verify that f has a critical point at $(0, 0)$ and it corresponds to a local maximum (on an inverted paraboloid). Moving away from $(0, 0)$ in all directions, the function values decrease, so f also has an absolute maximum value of 4 at $(0, 0)$. The boundary of R is the unit circle $\{(x, y): x^2 + y^2 = 1\}$, which is not contained in R. As (x, y) approaches any point on the unit circle along any path in R, the function values $f(x, y) = 4 - (x^2 + y^2)$ decrease and approach 3 but never reach 3. Therefore, f does not have an absolute minimum on R.

Related Exercise 59 ◂

QUICK CHECK 4 Does the linear function $f(x, y) = 2x + 3y$ have an absolute maximum or minimum value on the open unit square $\{(x, y): 0 < x < 1, 0 < y < 1\}$? ◂

➤ Notice that $\frac{\partial}{\partial x}(d^2) = 2d\frac{\partial d}{\partial x}$ and $\frac{\partial}{\partial y}(d^2) = 2d\frac{\partial d}{\partial y}$. Because $d \ge 0$, d^2 and d have the same critical points.

EXAMPLE 9 Absolute extreme values on an open region Find the point(s) on the plane $x + 2y + z = 2$ closest to the point $P(2, 0, 4)$.

SOLUTION Suppose (x, y, z) is a point on the plane, which means that $z = 2 - x - 2y$. The distance between $P(2, 0, 4)$ and (x, y, z) that we seek to minimize is

$$d(x, y, z) = \sqrt{(x - 2)^2 + y^2 + (z - 4)^2}.$$

It is easier to minimize d^2, which has the same critical points as d. Squaring d and eliminating z using $z = 2 - x - 2y$, we have

$$\begin{aligned} f(x, y) = (d(x, y, z))^2 &= (x - 2)^2 + y^2 + \underbrace{(-x - 2y - 2)}_{z - 4}{}^2 \\ &= 2x^2 + 5y^2 + 4xy + 8y + 8. \end{aligned}$$

The critical points of f satisfy the equations

$$f_x = 4x + 4y = 0 \quad \text{and} \quad f_y = 4x + 10y + 8 = 0,$$

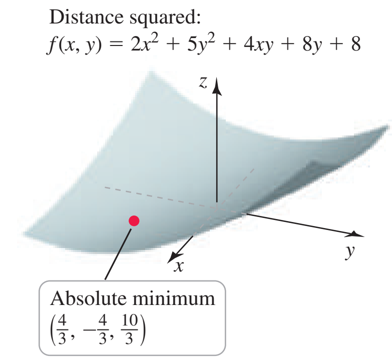


Figure 15.78

whose only solution is $x = \frac{4}{3}, y = -\frac{4}{3}$. The Second Derivative Test confirms that this point corresponds to a local minimum of f. We now ask: Does $\left(\frac{4}{3}, -\frac{4}{3}\right)$ correspond to the *absolute* minimum value of f over the entire domain of the problem, which is $\mathbb{R}^2$? Because the domain has no boundary, we cannot check values of f on the boundary. Instead, we argue geometrically that there is exactly one point on the plane that is closest to P. We have found a point that is closest to P among nearby points on the plane. As we move away from this point, the values of f increase without bound. Therefore, $\left(\frac{4}{3}, -\frac{4}{3}\right)$ corresponds to the absolute minimum value of f. A graph of f (**Figure 15.78**) confirms this reasoning, and we conclude that the point $\left(\frac{4}{3}, -\frac{4}{3}, \frac{10}{3}\right)$ is the point on the plane nearest P.

Related Exercises 62–63 ◄

SECTION 15.7 EXERCISES

Getting Started

1. Describe the appearance of a smooth surface with a local maximum at a point.
2. Describe the usual appearance of a smooth surface at a saddle point.
3. What are the conditions for a critical point of a function f?
4. If $f_x(a, b) = f_y(a, b) = 0$, does it follow that f has a local maximum or local minimum at (a, b)? Explain.
5. Consider the function $z = f(x, y)$. What is the discriminant of f, and how do you compute it?
6. Explain how the Second Derivative Test is used.
7. What is an absolute minimum value of a function f on a set R in $\mathbb{R}^2$?
8. What is the procedure for locating absolute maximum and minimum values on a closed bounded domain?

9–12. *Assume the second derivatives of f are continuous throughout the xy-plane and $f_x(0, 0) = f_y(0, 0) = 0$. Use the given information and the Second Derivative Test to determine whether f has a local minimum, a local maximum, or a saddle point at $(0, 0)$, or state that the test is inconclusive.*

9. $f_{xx}(0, 0) = 5$, $f_{yy}(0, 0) = 3$, and $f_{xy}(0, 0) = -4$
10. $f_{xx}(0, 0) = -6$, $f_{yy}(0, 0) = -3$, and $f_{xy}(0, 0) = 4$
11. $f_{xx}(0, 0) = 8$, $f_{yy}(0, 0) = 5$, and $f_{xy}(0, 0) = -6$
12. $f_{xx}(0, 0) = -9$, $f_{yy}(0, 0) = -4$, and $f_{xy}(0, 0) = -6$

Practice Exercises

13–22. Critical points *Find all critical points of the following functions.*

13. $f(x, y) = 3x^2 - 4y^2$
14. $f(x, y) = x^2 - 6x + y^2 + 8y$
15. $f(x, y) = 3x^2 + 3y - y^3$
16. $f(x, y) = x^3 - 12x + 6y^2$
17. $f(x, y) = x^4 + y^4 - 16xy$
18. $f(x, y) = \dfrac{x^3}{3} - \dfrac{y^3}{3} + 3xy$
19. $f(x, y) = x^4 - 2x^2 + y^2 - 4y + 5$
20. $f(x, y) = x^3 + 6xy - 6x + y^2 - 2y$
21. $f(x, y) = y^3 + 6xy + x^2 - 18y - 6x$
22. $f(x, y) = e^{8x^2y^2 - 24x^2 - 8xy^4}$

23–40. Analyzing critical points *Find the critical points of the following functions. Use the Second Derivative Test to determine (if possible) whether each critical point corresponds to a local maximum, a local minimum, or a saddle point. If the Second Derivative Test is inconclusive, determine the behavior of the function at the critical points.*

23. $f(x, y) = -4x^2 + 8y^2 - 3$
24. $f(x, y) = x^4 + y^4 - 4x - 32y + 10$
25. $f(x, y) = 4 + 2x^2 + 3y^2$
26. $f(x, y) = xye^{-x-y}$
27. $f(x, y) = x^4 + 2y^2 - 4xy$
28. $f(x, y) = (4x - 1)^2 + (2y + 4)^2 + 1$
29. $f(x, y) = 4 + x^4 + 3y^4$
30. $f(x, y) = x^4y^2$
31. $f(x, y) = \sqrt{x^2 + y^2 - 4x + 5}$
32. $f(x, y) = \tan^{-1} xy$
33. $f(x, y) = 2xye^{-x^2-y^2}$
34. $f(x, y) = x^2 + xy^2 - 2x + 1$
35. $f(x, y) = \dfrac{x}{1 + x^2 + y^2}$
36. $f(x, y) = \dfrac{x - 1}{x^2 + y^2}$
37. $f(x, y) = x^4 + 4x^2(y - 2) + 8(y - 1)^2$
38. $f(x, y) = xe^{-x-y} \sin y$, for $|x| \le 2, 0 \le y \le \pi$
39. $f(x, y) = ye^x - e^y$
40. $f(x, y) = \sin(2\pi x)\cos(\pi y)$, for $|x| \le \frac{1}{2}$ and $|y| \le \frac{1}{2}$

41–42. Inconclusive tests *Show that the Second Derivative Test is inconclusive when applied to the following functions at* $(0, 0)$. *Describe the behavior of the function at* $(0, 0)$.

41. $f(x, y) = x^2y - 3$
42. $f(x, y) = \sin(x^2y^2)$

43. **Shipping regulations** A shipping company handles rectangular boxes provided the sum of the height and the girth of the box does not exceed 96 in. (The girth is the perimeter of the smallest side of the box.) Find the dimensions of the box that meets this condition and has the largest volume.
44. **Cardboard boxes** A lidless box is to be made using 2 m² of cardboard. Find the dimensions of the box with the largest possible volume.

45. Cardboard boxes A lidless cardboard box is to be made with a volume of 4 m^3. Find the dimensions of the box that requires the least amount of cardboard.

46. Optimal box Find the dimensions of the largest rectangular box in the first octant of the xyz-coordinate system that has one vertex at the origin and the opposite vertex on the plane $x + 2y + 3z = 6$.

47–56. Absolute maxima and minima *Find the absolute maximum and minimum values of the following functions on the given region R.*

47. $f(x, y) = x^2 + y^2 - 2y + 1$; $R = \{(x, y): x^2 + y^2 \leq 4\}$

48. $f(x, y) = 2x^2 + y^2$; $R = \{(x, y): x^2 + y^2 \leq 16\}$

49. $f(x, y) = 4 + 2x^2 + y^2$;
$R = \{(x, y): -1 \leq x \leq 1, -1 \leq y \leq 1\}$

50. $f(x, y) = 6 - x^2 - 4y^2$;
$R = \{(x, y): -2 \leq x \leq 2, -1 \leq y \leq 1\}$

51. $f(x, y) = 2x^2 - 4x + 3y^2 + 2$;
$R = \{(x, y): (x - 1)^2 + y^2 \leq 1\}$

52. $f(x, y) = x^2 + y^2 - 2x - 2y$; R is the closed region bounded by the triangle with vertices $(0, 0)$, $(2, 0)$, and $(0, 2)$.

53. $f(x, y) = -2x^2 + 4x - 3y^2 - 6y - 1$;
$R = \{(x, y): (x - 1)^2 + (y + 1)^2 \leq 1\}$

54. $f(x, y) = \sqrt{x^2 + y^2 - 2x + 2}$;
$R = \{(x, y): x^2 + y^2 \leq 4, y \geq 0\}$

55. $f(x, y) = \dfrac{2y^2 - x^2}{2 + 2x^2y^2}$; R is the closed region bounded by the lines $y = x$, $y = 2x$, and $y = 2$.

56. $f(x, y) = \sqrt{x^2 + y^2}$; R is the closed region bounded by the ellipse $\dfrac{x^2}{4} + y^2 = 1$.

T **57. Pectin Extraction** An increase in world production of processed fruit has led to an increase in fruit waste. One way of reducing this waste is to find useful waste byproducts. For example, waste from pineapples is reduced by extracting pectin from pineapple peels (pectin is commonly used as a thickening agent in jam and jellies, and it is also widely used in the pharmaceutical industry). Pectin extraction involves heating and drying the peels, then grinding the peels into a fine powder. The powder is next placed in a solution with a particular pH level H, for $1.5 \leq H \leq 2.5$, and heated to a temperature T (in degrees Celsius), for $70 \leq T \leq 90$. The percentage of the powder $F(H, T)$ that becomes extracted pectin is

$$F(H, T) = -0.042T^2 - 0.213TH - 11.219H^2 + 7.327T + 58.729H - 342.684.$$

a. It can be shown that F attains its absolute maximum in the interior of the domain $D = \{(H, T): 1.5 \leq H \leq 2.5, 70 \leq T \leq 90\}$. Find the pH level H and temperature T that together maximize the amount of pectin extracted from the powder.

b. What is the maximum percentage of pectin that can be extracted from the powder? Round your answer to the nearest whole number. (*Source: Carpathian Journal of Food Science and Technology*, Dec 2014)

58–61. Absolute extrema on open and/or unbounded regions *If possible, find the absolute maximum and minimum values of the following functions on the region R.*

58. $f(x, y) = x + 3y$; $R = \{(x, y): |x| < 1, |y| < 2\}$

59. $f(x, y) = x^2 + y^2 - 4$; $R = \{(x, y): x^2 + y^2 < 4\}$

60. $f(x, y) = x^2 - y^2$; $R = \{(x, y): |x| < 1, |y| < 1\}$

61. $f(x, y) = 2e^{-x-y}$; $R = \{(x, y): x \geq 0, y \geq 0\}$

62–66. Absolute extrema on open and/or unbounded regions

62. Find the point on the plane $x + y + z = 4$ nearest the point $P(5, 4, 4)$.

63. Find the point on the plane $x - y + z = 2$ nearest the point $P(1, 1, 1)$.

64. Find the point on the paraboloid $z = x^2 + y^2$ nearest the point $P(3, 3, 1)$.

65. Find the points on the cone $z^2 = x^2 + y^2$ nearest the point $P(6, 8, 0)$.

66. Rectangular boxes with a volume of 10 m^3 are made of two materials. The material for the top and bottom of the box costs $\$10/m^2$ and the material for the sides of the box costs $\$1/m^2$. What are the dimensions of the box that minimize the cost of the box?

67. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample. Assume f is differentiable at the points in question.

a. The fact that $f_x(2, 2) = f_y(2, 2) = 0$ implies that f has a local maximum, local minimum, or saddle point at $(2, 2)$.

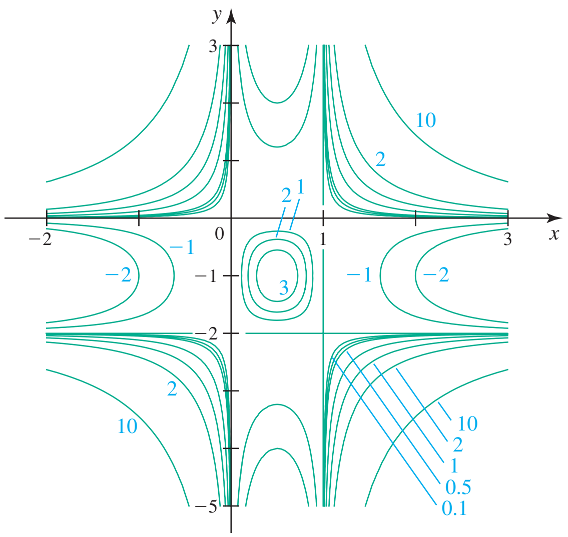
b. The function f could have a local maximum at (a, b) where $f_y(a, b) \neq 0$.

c. The function f could have both an absolute maximum and an absolute minimum at two different points that are not critical points.

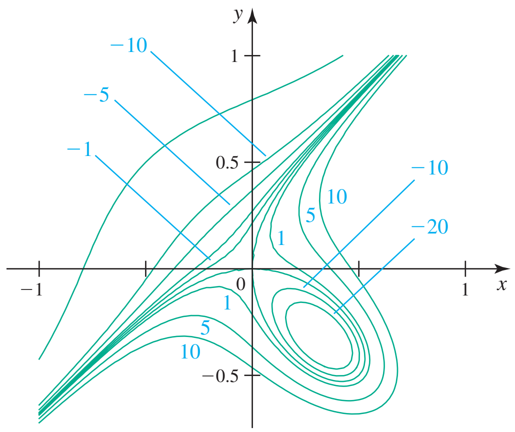
d. The tangent plane is horizontal at a point on a smooth surface corresponding to a critical point.

68–69. Extreme points from contour plots *Based on the level curves that are visible in the following graphs, identify the approximate locations of the local maxima, local minima, and saddle points.*

68.



69.



70. Optimal box Find the dimensions of the rectangular box with maximum volume in the first octant with one vertex at the origin and the opposite vertex on the ellipsoid $36x^2 + 4y^2 + 9z^2 = 36$.

Explorations and Challenges

71. Magic triples Let x, y, and z be nonnegative numbers with $x + y + z = 200$.

a. Find the values of x, y, and z that minimize $x^2 + y^2 + z^2$.

b. Find the values of x, y, and z that minimize $\sqrt{x^2 + y^2 + z^2}$.

c. Find the values of x, y, and z that maximize xyz.

d. Find the values of x, y, and z that maximize $x^2y^2z^2$.

72. Maximum/minimum of linear functions Let R be a closed bounded region in $\mathbb{R}^2$ and let $f(x, y) = ax + by + c$, where a, b, and c are real numbers, with a and b not both zero. Give a geometric argument explaining why the absolute maximum and minimum values of f over R occur on the boundaries of R.

T 73. Optimal locations Suppose n houses are located at the distinct points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$. A power substation must be located at a point such that the *sum of the squares* of the distances between the houses and the substation is minimized.

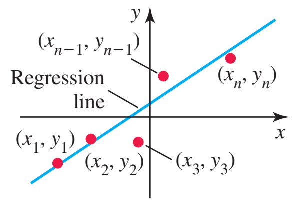
a. Find the optimal location of the substation in the case that $n = 3$ and the houses are located at $(0, 0)$, $(2, 0)$, and $(1, 1)$.

b. Find the optimal location of the substation in the case that $n = 3$ and the houses are located at distinct points (x_1, y_1), (x_2, y_2), and (x_3, y_3).

c. Find the optimal location of the substation in the general case of n houses located at distinct points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$.

d. You might argue that the locations found in parts (a), (b) and (c) are not optimal because they result from minimizing the sum of the *squares* of the distances, not the sum of the distances themselves. Use the locations in part (a) and write the function that gives the sum of the distances. Note that minimizing this function is much more difficult than in part (a). Then use a graphing utility to determine whether the optimal location is the same in the two cases. (Also see Exercise 81 about Steiner's problem.)

74–75. Least squares approximation *In its many guises, least squares approximation arises in numerous areas of mathematics and statistics. Suppose you collect data for two variables (for example, height and shoe size) in the form of pairs $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$. The data may be plotted as a scatterplot in the xy-plane, as shown in the figure. The technique known as* linear regression *asks the question: What is the equation of the line that "best fits" the data? The least squares criterion for best fit requires that the sum of the squares of the vertical distances between the line and the data points be a minimum.*



74. Let the equation of the best-fit line be $y = mx + b$, where the slope m and the y-intercept b must be determined using the least squares condition. First assume there are three data points $(1, 2)$, $(3, 5)$, and $(4, 6)$. Show that the function of m and b that gives the sum of the squares of the vertical distances between the line and the three data points is

$$E(m, b) = ((m + b) - 2)^2 + ((3m + b) - 5)^2 + ((4m + b) - 6)^2.$$

Find the critical points of E and find the values of m and b that minimize E. Graph the three data points and the best-fit line.

T 75. Generalize the procedure in Exercise 74 by assuming n data points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ are given. Write the function $E(m, b)$ (summation notation allows for a more compact calculation). Show that the coefficients of the best-fit line are

$$m = \frac{\left(\sum x_k\right)\left(\sum y_k\right) - n\sum x_k y_k}{\left(\sum x_k\right)^2 - n\sum x_k^2},$$

$$b = \frac{1}{n}\left(\sum y_k - m\sum x_k\right),$$

where all sums run from $k = 1$ to $k = n$.

T 76–77. Least squares practice *Use the results of Exercise 75 to find the best-fit line for the following data sets. Plot the points and the best-fit line.*

76. $(0, 0), (2, 3), (4, 5)$

77. $(-1, 0), (0, 6), (3, 8)$

78. Second Derivative Test Suppose the conditions of the Second Derivative Test are satisfied on an open disk containing the point (a, b). Use the test to prove that if (a, b) is a critical point of f at which $f_x(a, b) = f_y(a, b) = 0$ and $f_{xx}(a, b) < 0 < f_{yy}(a, b)$ or $f_{yy}(a, b) < 0 < f_{xx}(a, b)$, then f has a saddle point at (a, b).

79. Maximum area triangle Among all triangles with a perimeter of 9 units, find the dimensions of the triangle with the maximum area. It may be easiest to use Heron's formula, which states that the area of a triangle with side length a, b, and c is $A = \sqrt{s(s - a)(s - b)(s - c)}$, where $2s$ is the perimeter of the triangle.

80. Slicing plane Find an equation of the plane passing through the point $(3, 2, 1)$ that slices off the solid in the first octant with the least volume.

T 81. Steiner's problem for three points Given three distinct noncollinear points A, B, and C in the plane, find the point P in the plane such that the sum of the distances $|AP| + |BP| + |CP|$ is a minimum. Here is how to proceed with three points, assuming the triangle formed by the three points has no angle greater than $2\pi/3$ $(120°)$.

a. Assume the coordinates of the three given points are $A(x_1, y_1)$, $B(x_2, y_2)$, and $C(x_3, y_3)$. Let $d_1(x, y)$ be the distance between $A(x_1, y_1)$ and a variable point $P(x, y)$. Compute the gradient of d_1 and show that it is a unit vector pointing along the line between the two points.

b. Define d_2 and d_3 in a similar way and show that ∇d_2 and ∇d_3 are also unit vectors in the direction of the line between the two points.

c. The goal is to minimize $f(x, y) = d_1 + d_2 + d_3$. Show that the condition $f_x = f_y = 0$ implies that $\nabla d_1 + \nabla d_2 + \nabla d_3 = 0$.

d. Explain why part (c) implies that the optimal point P has the property that the three line segments AP, BP, and CP all intersect symmetrically in angles of $2\pi/3$.

e. What is the optimal solution if one of the angles in the triangle is greater than $2\pi/3$ (just draw a picture)?

f. Estimate the Steiner point for the three points $(0, 0)$, $(0, 1)$, and $(2, 0)$.

T 82. Solitary critical points A function of *one* variable has the property that a local maximum (or minimum) occurring at the only critical point is also the absolute maximum (or minimum) (for example, $f(x) = x^2$). Does the same result hold for a function of *two* variables? Show that the following functions have the property that they have a single local maximum (or minimum), occurring at the only critical point, but the local maximum (or minimum) is not an absolute maximum (or minimum) on $\mathbb{R}^2$.

a. $f(x, y) = 3xe^y - x^3 - e^{3y}$

b. $f(x, y) = (2y^2 - y^4)\left(e^x + \dfrac{1}{1 + x^2}\right) - \dfrac{1}{1 + x^2}$

This property has the following interpretation. Suppose a surface has a single local minimum that is not the absolute minimum. Then water can be poured into the basin around the local minimum and the surface never overflows, even though there are points on the surface below the local minimum.
(*Source: Mathematics Magazine*, May 1985, and *Calculus and Analytical Geometry*, 2nd ed., Philip Gillett, 1984)

T 83. Two mountains without a saddle Show that the following functions have two local maxima but no other extreme points (therefore, no saddle or basin between the mountains).

a. $f(x, y) = -(x^2 - 1)^2 - (x^2 - e^y)^2$

b. $f(x, y) = 4x^2e^y - 2x^4 - e^{4y}$

(*Source:* Ira Rosenholtz, *Mathematics Magazine*, Feb 1987)

84. Powers and roots Assume $x + y + z = 1$ with $x \ge 0$, $y \ge 0$, and $z \ge 0$.

a. Find the maximum and minimum values of $(1 + x^2)(1 + y^2)(1 + z^2)$.

b. Find the maximum and minimum values of $(1 + \sqrt{x})(1 + \sqrt{y})(1 + \sqrt{z})$.

(*Source: Math Horizons*, Apr 2004)

85. Ellipsoid inside a tetrahedron (1946 Putnam Exam) Let P be a plane tangent to the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ at a point in the first octant. Let T be the tetrahedron in the first octant bounded by P and the coordinate planes $x = 0$, $y = 0$, and $z = 0$. Find the minimum volume of T. (The volume of a tetrahedron is one-third the area of the base times the height.)

QUICK CHECK ANSWERS

1. $f_x(2, -1) = f_y(2, -1) = 0$ **2.** Vertically, in the directions $\langle 0, 0, \pm 1 \rangle$ **3.** $D(x, y) = -12x^2y^2$
4. It has neither an absolute maximum nor an absolute minimum value on this set. ◄

15.8 Lagrange Multipliers

One of many challenges in economics and marketing is predicting the behavior of consumers. Basic models of consumer behavior often involve a *utility function* that expresses consumers' combined preference for several different amenities. For example, a simple utility function might have the form $U = f(\ell, g)$, where ℓ represents the amount of leisure time and g represents the number of consumable goods. The model assumes consumers try to maximize their utility function, but they do so under certain constraints on the variables of the problem. For example, increasing leisure time may increase utility, but leisure time produces no income for consumable goods. Similarly, consumable goods may also increase utility, but they require income, which reduces leisure time. We first develop a general method for solving such constrained optimization problems and then return to economics problems later in the section.

The Basic Idea

We start with a typical constrained optimization problem with two independent variables and give its method of solution; a generalization to more variables then follows. We seek maximum and/or minimum values of a differentiable **objective function** f with the restriction that x and y must lie on a **constraint** curve C in the xy-plane given by $g(x, y) = 0$ (Figure 15.79).

Find the maximum and minimum values of z as (x, y) varies over C.

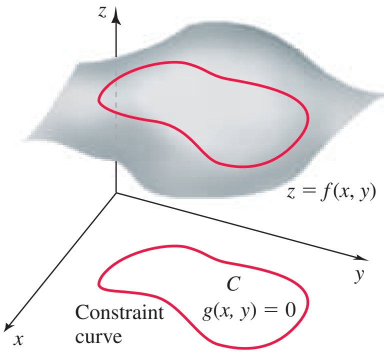


Figure 15.79

Figure 15.80 shows the details of a typical situation in which we assume the (green) level curves of f have increasing z-values moving away from the origin. Now imagine moving along the (red) constraint curve C: $g(x, y) = 0$ toward the point $P(a, b)$. As we approach P (from either side), the values of f evaluated on C increase, and as we move past P along C, the values of f decrease.

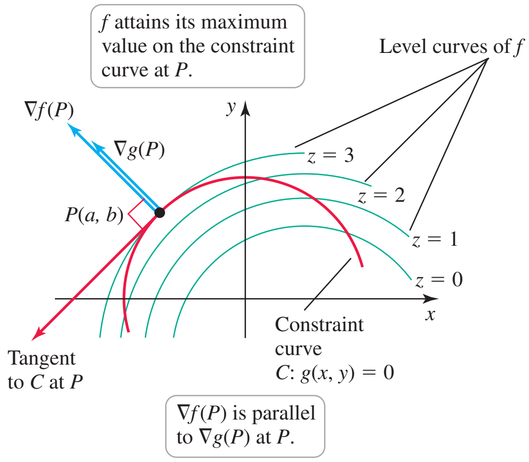


Figure 15.80

What is special about the point P at which f appears to have a local maximum value on C? From Theorem 15.12, we know that at any point $P(a, b)$ on a level curve of f, the line tangent to the level curve at P is orthogonal to $\nabla f(a, b)$. Figure 15.80 also suggests that the line tangent to the level curve of f at P is tangent to the constraint curve C at P. We prove this fact shortly. This observation implies that $\nabla f(a, b)$ is also orthogonal to the line tangent to C at $P(a, b)$.

We need one more observation. The constraint curve C is just one level curve of the function $z = g(x, y)$. Using Theorem 15.12 again, the line tangent to C at $P(a, b)$ is orthogonal to $\nabla g(a, b)$. We have now found two vectors $\nabla f(a, b)$ and $\nabla g(a, b)$ that are both orthogonal to the line tangent to the level curve C at $P(a, b)$. Therefore, these two gradient vectors are parallel. These properties characterize the point P at which f has a local extremum on the constraint curve. They are the basis of the method of *Lagrange multipliers* that we now formalize.

Lagrange Multipliers with Two Independent Variables

The major step in establishing the method of Lagrange multipliers is to prove that Figure 15.80 is drawn correctly; that is, at the point on the constraint curve C where f has a local extreme value, the line tangent to C is orthogonal to $\nabla f(a, b)$ and $\nabla g(a, b)$.

➤ The Greek lowercase ℓ is λ; it is read *lambda*.

THEOREM 15.16 Parallel Gradients
Let f be a differentiable function in a region of $\mathbb{R}^2$ that contains the smooth curve C given by $g(x, y) = 0$. Assume f has a local extreme value on C at a point $P(a, b)$. Then $\nabla f(a, b)$ is orthogonal to the line tangent to C at P. Assuming $\nabla g(a, b) \neq \mathbf{0}$, it follows that there is a real number λ (called a **Lagrange multiplier**) such that $\nabla f(a, b) = \lambda \nabla g(a, b)$.

Proof: Because C is smooth, it can be expressed parametrically in the form C: $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, where x and y are differentiable functions on an interval in t that contains t_0 with $P(a, b) = (x(t_0), y(t_0))$. As we vary t and follow C, the rate of change of f is given by the Chain Rule:

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \nabla f \cdot \mathbf{r}'(t).$$

At the point $(x(t_0), y(t_0)) = (a, b)$ at which f has a local extreme value, we have $\left.\dfrac{df}{dt}\right|_{t=t_0} = 0$, which implies that $\nabla f(a, b) \cdot \mathbf{r}'(t_0) = 0$. Because $\mathbf{r}'(t)$ is tangent to C, the gradient $\nabla f(a, b)$ is orthogonal to the line tangent to C at P.

To prove the second assertion, note that the constraint curve C given by $g(x, y) = 0$ is also a level curve of the surface $z = g(x, y)$. Recall that gradients are orthogonal to level curves. Therefore, at the point $P(a, b)$, $\nabla g(a, b)$ is orthogonal to C at (a, b). Because both $\nabla f(a, b)$ and $\nabla g(a, b)$ are orthogonal to C, the two gradients are parallel, so there is a real number λ such that $\nabla f(a, b) = \lambda \nabla g(a, b)$. ◄

Theorem 15.16 leads directly to the method of Lagrange Multipliers, which produces candidates for *local* maxima and minima of f on the constraint curve. In many problems, however, the goal is to find *absolute* maxima and minima of f on the constraint curve. Much as we did with optimization problems in one variable, we find absolute extrema by examining both local extrema and endpoints. Several different cases arise:

- If the constraint curve is bounded (it lies within a circle of finite radius) and it closes on itself (for example, an ellipse), then we know that the absolute extrema of f exist. In this case, there are no endpoints to consider, and the absolute extrema are found among the local extrema.
- If the constraint curve is bounded and includes its endpoints but does not close on itself (for example, a closed line segment), then the absolute extrema of f exist, and we find them by examining the local extrema and the endpoints.
- In the case that the constraint curve is unbounded (for example, a line or a parabola) or the curve excludes one or both of its endpoints, we have no guarantee that absolute extrema exist. We can find local extrema, but they must be examined carefully to determine whether they are, in fact, absolute extrema (see Example 2 and Exercise 65).

We deal first with the case of finding absolute extrema on closed and bounded constraint curves.

QUICK CHECK 1 It can be shown that the function $f(x, y) = x^2 + y^2$ attains its absolute minimum value on the curve

$$C\colon g(x, y) = \frac{1}{4}(x - 3)^2 - y = 0$$

at the point $(1, 1)$. Verify that $\nabla f(1, 1)$ and $\nabla g(1, 1)$ are parallel, and that both vectors are orthogonal to the line tangent to C at $(1, 1)$, thereby confirming Theorem 15.16. ◄

PROCEDURE Lagrange Multipliers: Absolute Extrema on Closed and Bounded Constraint Curves

Let the objective function f and the constraint function g be differentiable on a region of $\mathbb{R}^2$ with $\nabla g(x, y) \neq \mathbf{0}$ on the curve $g(x, y) = 0$. To locate the absolute maximum and minimum values of f subject to the constraint $g(x, y) = 0$, carry out the following steps.

1. Find the values of x, y, and λ (if they exist) that satisfy the equations
$$\nabla f(x, y) = \lambda \nabla g(x, y) \quad \text{and} \quad g(x, y) = 0.$$
2. Evaluate f at the values (x, y) found in Step 1 and at the endpoints of the constraint curve (if they exist). Select the largest and smallest corresponding function values. These values are the absolute maximum and minimum values of f subject to the constraint.

➤ *In principle*, it is possible to solve a constrained optimization problem by solving the constraint equation for one of the variables and eliminating that variable in the objective function. In practice, this method is often impractical, particularly with three or more variables or two or more constraints.

Notice that $\nabla f = \lambda \nabla g$ is a vector equation: $\langle f_x, f_y \rangle = \lambda \langle g_x, g_y \rangle$. It is satisfied provided $f_x = \lambda g_x$ and $f_y = \lambda g_y$. Therefore, the crux of the method is solving the three equations

$$f_x = \lambda g_x, \qquad f_y = \lambda g_y, \qquad \text{and} \qquad g(x, y) = 0,$$

for the three variables x, y, and λ.

EXAMPLE 1 Lagrange multipliers with two variables Find the absolute maximum and minimum values of the objective function $f(x, y) = x^2 + y^2 + 2$, where x and y lie on the ellipse C given by $g(x, y) = x^2 + xy + y^2 - 4 = 0$.

SOLUTION Because C is closed and bounded, the absolute maximum and minimum values of f exist. **Figure 15.81a** shows the paraboloid $z = f(x, y)$ above the ellipse C in the xy-plane. As the ellipse is traversed, the corresponding function values on the surface vary. The goal is to find the maximum and minimum of these function values. An alternative view is given in **Figure 15.81b**, where we see the level curves of f and the constraint curve C. As the ellipse is traversed, the values of f vary, reaching maximum and minimum values along the way.

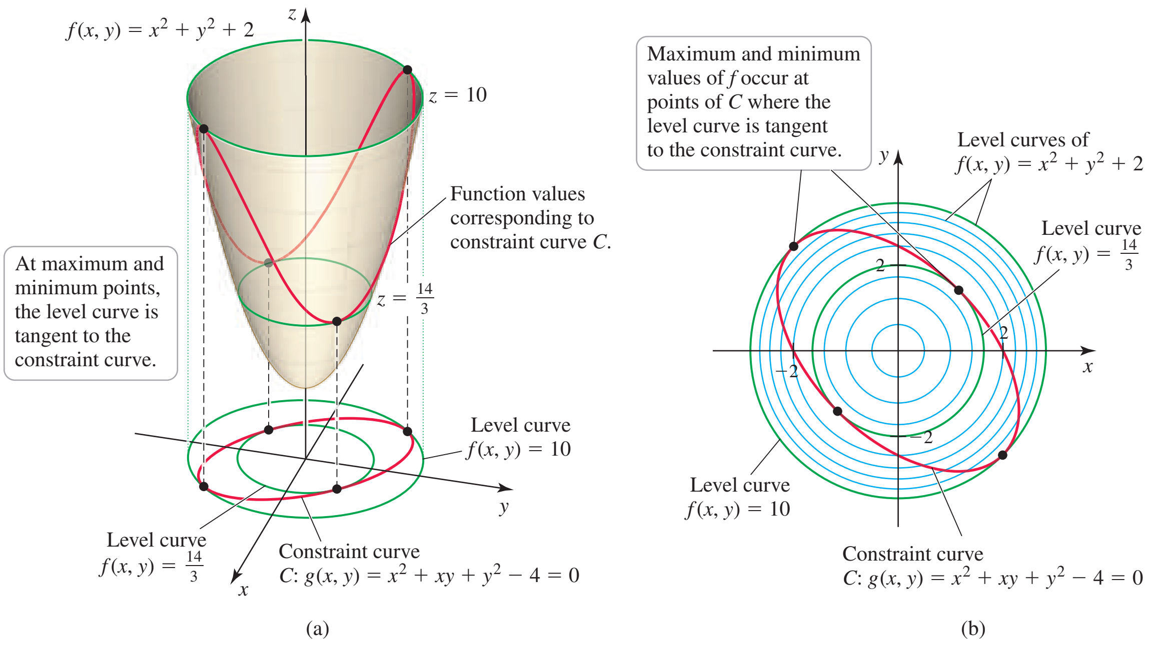


Figure 15.81

Noting that $\nabla f(x, y) = \langle 2x, 2y \rangle$ and $\nabla g(x, y) = \langle 2x + y, x + 2y \rangle$, the equations that result from $\nabla f = \lambda \nabla g$ and the constraint are

$$\underbrace{2x = \lambda(2x + y),}_{f_x = \lambda g_x} \quad \underbrace{2y = \lambda(x + 2y),}_{f_y = \lambda g_y} \quad \text{and} \quad \underbrace{x^2 + xy + y^2 - 4 = 0.}_{\text{constraint } g(x,y) = 0}$$

Subtracting the second equation from the first leads to

$$(x - y)(2 - \lambda) = 0,$$

which implies that $y = x$, or $\lambda = 2$. In the case that $y = x$, the constraint equation simplifies to $3x^2 - 4 = 0$, or $x = \pm\frac{2}{\sqrt{3}}$. Therefore, two candidates for locations of extreme values are $\left(\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$ and $\left(-\frac{2}{\sqrt{3}}, -\frac{2}{\sqrt{3}}\right)$.

Substituting $\lambda = 2$ into the first equation leads to $y = -x$, and then the constraint equation simplifies to $x^2 - 4 = 0$, or $x = \pm 2$. These values give two additional points of interest, $(2, -2)$ and $(-2, 2)$. Evaluating f at each of these points, we find that $f\left(\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right) = f\left(-\frac{2}{\sqrt{3}}, -\frac{2}{\sqrt{3}}\right) = \frac{14}{3}$ and $f(2, -2) = f(-2, 2) = 10$. Therefore, the absolute maximum value of f on C is 10, which occurs at $(2, -2)$ and $(-2, 2)$, and the absolute minimum value of f on C is $14/3$, which occurs at $\left(\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$ and $\left(-\frac{2}{\sqrt{3}}, -\frac{2}{\sqrt{3}}\right)$. Notice that the value of λ is not used in the final result.

Related Exercises 9–10 ◄

QUICK CHECK 2 Choose any point on the constraint curve in Figure 15.81b other than a solution point. Draw ∇f and ∇g at that point and show that they are not parallel. ◄

Lagrange Multipliers with Three Independent Variables

The technique just outlined extends to three or more independent variables. With three variables, suppose an objective function $w = f(x, y, z)$ is given; its level surfaces are surfaces in $\mathbb{R}^3$ (**Figure 15.82a**). The constraint equation takes the form $g(x, y, z) = 0$, which is another surface S in $\mathbb{R}^3$ (**Figure 15.82b**). To find the local maximum and minimum values of f on S (assuming they exist), we must find the points (a, b, c) on S at which $\nabla f(a, b, c)$ is parallel to $\nabla g(a, b, c)$, assuming $\nabla g(a, b, c) \neq \mathbf{0}$ (**Figure 15.82c, d**). In the case where the surface $g(x, y, z) = 0$ is closed and bounded, the procedure for finding the absolute maximum and minimum values of $f(x, y, z)$, where the point (x, y, z) is constrained to lie on S, is similar to the procedure for two variables.

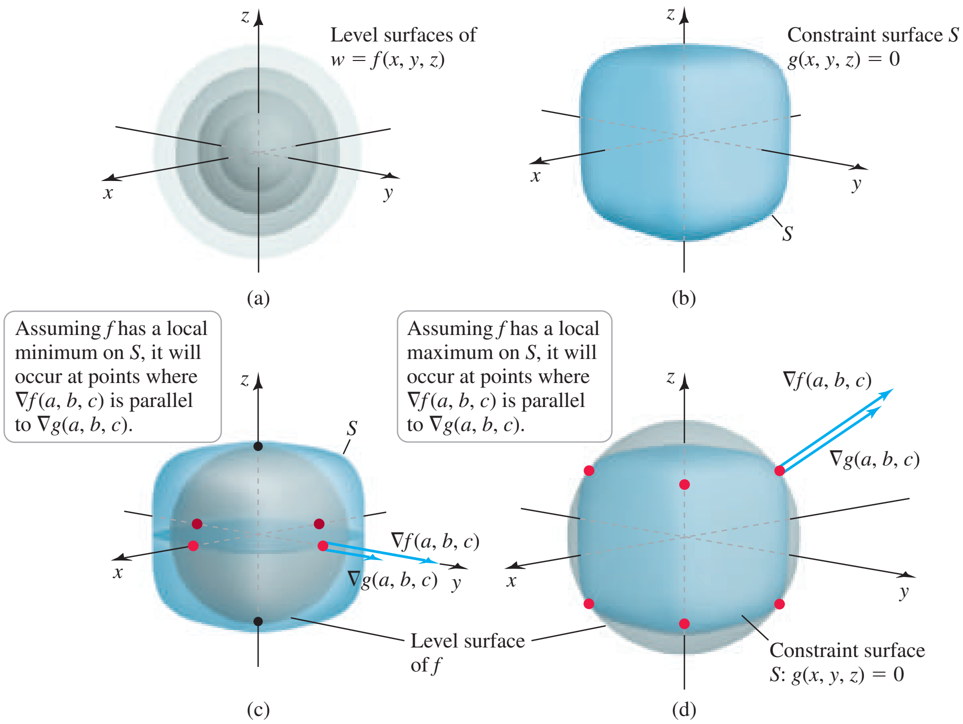
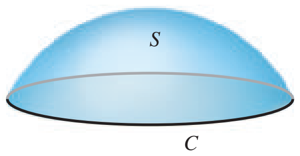


Figure 15.82

➤ If the constraint surface S: $g(x, y, z) = 0$ has a boundary curve C (see figure), then each point on C is a candidate for the location of an absolute maximum or minimum value of f, and these points must be analyzed in Step 2 of the procedure. We avoid this case in the exercise set.



➤ Problems similar to Example 2 were solved in Section 15.7 using ordinary optimization techniques. These methods may or may not be easier to apply than Lagrange multipliers.

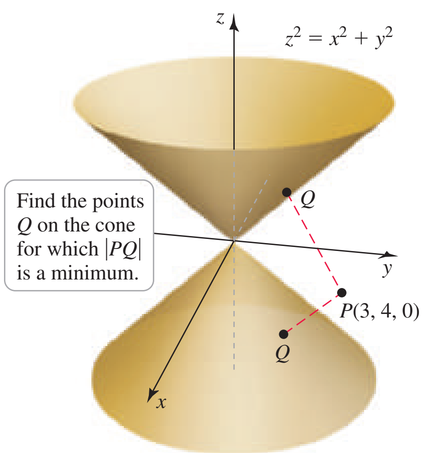


Figure 15.83

PROCEDURE Lagrange Multipliers: Absolute Extrema on Closed and Bounded Constraint Surfaces

Let f and g be differentiable on a region of $\mathbb{R}^3$ with $\nabla g(x, y, z) \neq \mathbf{0}$ on the surface $g(x, y, z) = 0$. To locate the absolute maximum and minimum values of f subject to the constraint $g(x, y, z) = 0$, carry out the following steps.

1. Find the values of x, y, z, and λ that satisfy the equations
$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) \quad \text{and} \quad g(x, y, z) = 0.$$
2. Among the points (x, y, z) found in Step 1, select the largest and smallest corresponding function values. These values are the absolute maximum and minimum values of f subject to the constraint.

Now there are four equations to be solved for x, y, z, and λ:

$$f_x(x, y, z) = \lambda g_x(x, y, z), \qquad f_y(x, y, z) = \lambda g_y(x, y, z),$$
$$f_z(x, y, z) = \lambda g_z(x, y, z), \quad \text{and} \quad g(x, y, z) = 0.$$

As in the two-variable case, special care must be given to constraint surfaces that are not closed and bounded. We examine one such case in Example 2.

EXAMPLE 2 A geometry problem Find the least distance between the point $P(3, 4, 0)$ and the surface of the cone $z^2 = x^2 + y^2$.

SOLUTION The cone is not bounded, so we begin our calculations recognizing that solutions are only candidates for local extrema. Figure 15.83 shows both sheets of the cone and the point $P(3, 4, 0)$. Because P is in the xy-plane, we anticipate two solutions, one for each sheet of the cone. The distance between P and any point $Q(x, y, z)$ on the cone is

$$d(x, y, z) = \sqrt{(x - 3)^2 + (y - 4)^2 + z^2}.$$

In many distance problems, it is easier to work with the *square* of the distance to avoid dealing with square roots. This maneuver is allowable because if a point minimizes $(d(x, y, z))^2$, it also minimizes $d(x, y, z)$. Therefore, we define

$$f(x, y, z) = (d(x, y, z))^2 = (x - 3)^2 + (y - 4)^2 + z^2.$$

The constraint is the condition that the point (x, y, z) must lie on the cone, which implies $z^2 = x^2 + y^2$, or $g(x, y, z) = z^2 - x^2 - y^2 = 0$.

Now we proceed with Lagrange multipliers; the conditions are

$$f_x(x, y, z) = \lambda g_x(x, y, z), \quad \text{or } 2(x - 3) = \lambda(-2x), \quad \text{or } x(1 + \lambda) = 3, \qquad (1)$$
$$f_y(x, y, z) = \lambda g_y(x, y, z), \quad \text{or } 2(y - 4) = \lambda(-2y), \quad \text{or } y(1 + \lambda) = 4, \qquad (2)$$
$$f_z(x, y, z) = \lambda g_z(x, y, z), \quad \text{or } 2z = \lambda(2z), \quad \text{or } z = \lambda z, \text{ and} \qquad (3)$$
$$g(x, y, z) = z^2 - x^2 - y^2 = 0. \qquad (4)$$

The solutions of equation (3) (the simplest of the four equations) are either $z = 0$, or $\lambda = 1$ and $z \neq 0$. In the first case, if $z = 0$, then by equation (4), $x = y = 0$; however, $x = 0$ and $y = 0$ do not satisfy (1) and (2). So no solution results from this case.

On the other hand, if $\lambda = 1$ in equation (3), then by (1) and (2), we find that $x = \frac{3}{2}$ and $y = 2$. Using (4), the corresponding values of z are $\pm\frac{5}{2}$. Therefore, the two solutions and the values of f are

$$x = \tfrac{3}{2}, \quad y = 2, \quad z = \tfrac{5}{2}, \qquad \text{with } f\left(\tfrac{3}{2}, 2, \tfrac{5}{2}\right) = \tfrac{25}{2}, \text{ and}$$
$$x = \tfrac{3}{2}, \quad y = 2, \quad z = -\tfrac{5}{2}, \qquad \text{with } f\left(\tfrac{3}{2}, 2, -\tfrac{5}{2}\right) = \tfrac{25}{2}.$$

➤ With three independent variables, it is possible to impose two constraints. These problems are explored in Exercises 61–64.

You can check that moving away from $\left(\frac{3}{2}, 2, \pm\frac{5}{2}\right)$ in any direction on the cone has the effect of increasing the values of f. Therefore, the points correspond to *local* minima of f. Do these points also correspond to *absolute* minima? The domain of this problem is unbounded; however, one can argue geometrically that f increases without bound moving away from $\left(\frac{3}{2}, 2, \pm\frac{5}{2}\right)$ on the cone with $|x| \to \infty$ and $|y| \to \infty$. Therefore, these points correspond to absolute minimum values and the points on the cone nearest to $(3, 4, 0)$ are $\left(\frac{3}{2}, 2, \pm\frac{5}{2}\right)$, at a distance of $\sqrt{\frac{25}{2}} = \frac{5}{\sqrt{2}}$. (Recall that $f = d^2$.)

Related Exercises 32–34 ◀

QUICK CHECK 3 In Example 2, is there a point that *maximizes* the distance between $(3, 4, 0)$ and the cone? If the point $(3, 4, 0)$ were replaced by $(3, 4, 1)$, how many minimizing solutions would there be? ◀

Economic Models In the opening of this section, we briefly described how utility functions are used to model consumer behavior. We now look in more detail at some specific—admittedly simple—utility functions and the constraints that are imposed on them.

As described earlier, a prototype model for consumer behavior uses two independent variables: leisure time ℓ and consumable goods g. A utility function $U = f(\ell, g)$ measures consumer preferences for various combinations of leisure time and consumable goods. The following assumptions about utility functions are commonly made.

1. Utility increases if any variable increases (essentially, *more is better*).
2. Various combinations of leisure time and consumable goods have the same utility; that is, giving up some leisure time for additional consumable goods (or vice versa) results in the same utility.

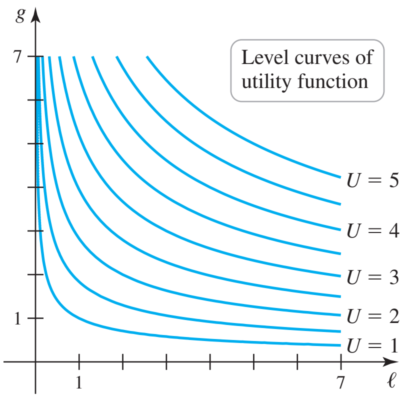


Figure 15.84

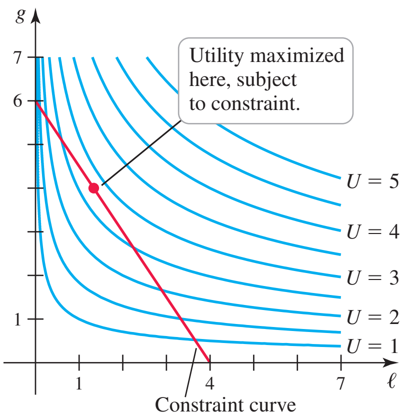


Figure 15.85

The level curves of a typical utility function are shown in **Figure 15.84**. Assumption 1 is reflected by the fact that the utility values on the level curves increase as either ℓ or g increases. Consistent with Assumption 2, a single level curve shows the combinations of ℓ and g that have the same utility; for this reason, economists call the level curves *indifference curves*. Notice that if ℓ increases, then g must decrease on a level curve to maintain the same utility, and vice versa.

Economic models assert that consumers maximize utility subject to constraints on leisure time and consumable goods. One assumption that leads to a reasonable constraint is that an increase in leisure time implies a linear decrease in consumable goods. Therefore, the constraint curve is a line with negative slope (**Figure 15.85**). When such a constraint is superimposed on the level curves of the utility function, the optimization problem becomes evident. Among all points on the constraint line, which one maximizes utility? A solution is marked in the figure; at this point, the utility has a maximum value (between 2.5 and 3.0).

EXAMPLE 3 **Constrained optimization of utility** Find the absolute maximum value of the utility function $U = f(\ell, g) = \ell^{1/3}g^{2/3}$, subject to the constraint $G(\ell, g) = 3\ell + 2g - 12 = 0$, where $\ell \geq 0$ and $g \geq 0$.

SOLUTION The constraint is closed and bounded, so we expect to find an absolute maximum value of f. The level curves of the utility function and the linear constraint are shown in Figure 15.85. The solution follows the Lagrange multiplier method with two variables. The gradient of the utility function is

$$\nabla f(\ell, g) = \left\langle \frac{\ell^{-2/3}g^{2/3}}{3}, \frac{2\ell^{1/3}g^{-1/3}}{3} \right\rangle = \frac{1}{3}\left\langle \left(\frac{g}{\ell}\right)^{2/3}, 2\left(\frac{\ell}{g}\right)^{1/3} \right\rangle.$$

The gradient of the constraint function is $\nabla G(\ell, g) = \langle 3, 2 \rangle$. Therefore, the equations that must be solved are

$$\frac{1}{3}\left(\frac{g}{\ell}\right)^{2/3} = 3\lambda, \qquad \frac{2}{3}\left(\frac{\ell}{g}\right)^{1/3} = 2\lambda, \qquad \text{and} \qquad G(\ell, g) = 3\ell + 2g - 12 = 0.$$

Eliminating λ from the first two equations leads to the condition $g = 3\ell$, which, when substituted into the constraint equation, gives the solution $\ell = \frac{4}{3}$ and $g = 4$. This point is a candidate for the location of the absolute maximum; the other candidates are the endpoints of the constraint curve, $(4, 0)$ and $(0, 6)$. The actual values of the utility function at these points are $U = f\left(\frac{4}{3}, 4\right) = 4/\sqrt[3]{3} \approx 2.8$ and $f(4, 0) = f(0, 6) = 0$. We conclude that the maximum value of f is 2.8; this solution occurs at $\ell = \frac{4}{3}$ and $g = 4$, and it is consistent with Figure 15.85.

Related Exercise 38 ◄

QUICK CHECK 4 In Figure 15.85, explain why, if you move away from the optimal point along the constraint line, the utility decreases. ◄

SECTION 15.8 EXERCISES

Getting Started

1. Explain why, at a point that maximizes or minimizes f subject to a constraint $g(x, y) = 0$, the gradient of f is parallel to the gradient of g. Use a diagram.

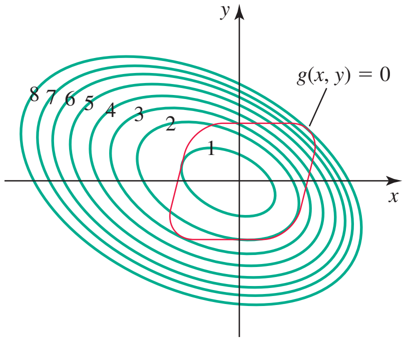
2. Describe the steps used to find the absolute maximum value and absolute minimum value of a differentiable function on a circle centered at the origin of the xy-plane.

3. For functions $f(x, y) = x + 4y$ and $g(x, y) = x^2 + y^2 - 1$, write the Lagrange multiplier conditions that must be satisfied by a point that maximizes or minimizes f subject to the constraint $g(x, y) = 0$.

4. For functions $f(x, y, z) = xyz$ and $g(x, y, z) = x^2 + 2y^2 + 3z^2 - 1$, write the Lagrange multiplier conditions that must be satisfied by a point that maximizes or minimizes f subject to the constraint $g(x, y) = 0$.

5–6. *The following figures show the level curves of f and the constraint curve $g(x, y) = 0$. Estimate the maximum and minimum values of f subject to the constraint. At each point where an extreme value occurs, indicate the direction of ∇f and a possible direction of ∇g.*

5.



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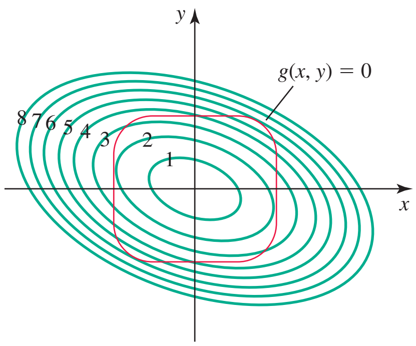


Practice Exercises

7–26. Lagrange multipliers *Each function f has an absolute maximum value and absolute minimum value subject to the given constraint. Use Lagrange multipliers to find these values.*

7. $f(x, y) = x + 2y$ subject to $x^2 + y^2 = 4$

8. $f(x, y) = xy^2$ subject to $x^2 + y^2 = 1$

9. $f(x, y) = x + y$ subject to $x^2 - xy + y^2 = 1$

10. $f(x, y) = x^2 + y^2$ subject to $2x^2 + 3xy + 2y^2 = 7$

11. $f(x, y) = xy$ subject to $x^2 + y^2 - xy = 9$

12. $f(x, y) = x - y$ subject to $x^2 + y^2 - 3xy = 20$

13. $f(x, y) = e^{xy}$ subject to $x^2 + xy + y^2 = 9$

14. $f(x, y) = x^2y$ subject to $x^2 + y^2 = 9$

15. $f(x, y) = 2x^2 + y^2$ subject to $x^2 + 2y + y^2 = 15$

16. $f(x, y) = x^2$ subject to $x^2 + xy + y^2 = 3$

17. $f(x, y, z) = x + 3y - z$ subject to $x^2 + y^2 + z^2 = 4$

18. $f(x, y, z) = xyz$ subject to $x^2 + 2y^2 + 4z^2 = 9$

19. $f(x, y, z) = x$ subject to $x^2 + y^2 + z^2 - z = 1$

20. $f(x, y, z) = x - z$ subject to $x^2 + y^2 + z^2 - y = 2$

21. $f(x, y, z) = x + y + z$ subject to $x^2 + y^2 + z^2 - xy = 5$

22. $f(x, y, z) = x + y + z$ subject to $x^2 + y^2 + z^2 - 2x - 2y = 1$

23. $f(x, y, z) = 2x + z^2$ subject to $x^2 + y^2 + 2z^2 = 25$

24. $f(x, y, z) = xy - z$ subject to $x^2 + y^2 + z^2 - xy = 1$

25. $f(x, y, z) = x^2 + y + z$ subject to $2x^2 + 2y^2 + z^2 = 2$

26. $f(x, y, z) = (xyz)^{1/2}$ subject to $x + y + z = 1$ with $x \geq 0$, $y \geq 0, z \geq 0$

27–36. Applications of Lagrange multipliers *Use Lagrange multipliers in the following problems. When the constraint curve is unbounded, explain why you have found an absolute maximum or minimum value.*

27. **Shipping regulations** A shipping company requires that the sum of length plus girth of rectangular boxes not exceed 108 in. Find the dimensions of the box with maximum volume that meets this condition. (The girth is the perimeter of the smallest side of the box.)

28. **Box with minimum surface area** Find the dimensions of the rectangular box with a volume of 16 ft^3 that has minimum surface area.

T 29. **Extreme distances to an ellipse** Find the minimum and maximum distances between the ellipse $x^2 + xy + 2y^2 = 1$ and the origin.

30. **Maximum area rectangle in an ellipse** Find the dimensions of the rectangle of maximum area with sides parallel to the coordinate axes that can be inscribed in the ellipse $4x^2 + 16y^2 = 16$.

31. **Maximum perimeter rectangle in an ellipse** Find the dimensions of the rectangle of maximum perimeter with sides parallel to the coordinate axes that can be inscribed in the ellipse $2x^2 + 4y^2 = 3$.

32. **Minimum distance to a plane** Find the point on the plane $2x + 3y + 6z - 10 = 0$ closest to the point $(-2, 5, 1)$.

33. **Minimum distance to a surface** Find the point on the surface $4x + y - 1 = 0$ closest to the point $(1, 2, -3)$.

34. **Minimum distance to a cone** Find the points on the cone $z^2 = x^2 + y^2$ closest to the point $(1, 2, 0)$.

35. **Extreme distances to a sphere** Find the minimum and maximum distances between the sphere $x^2 + y^2 + z^2 = 9$ and the point $(2, 3, 4)$.

36. **Maximum volume cylinder in a sphere** Find the dimensions of the right circular cylinder of maximum volume that can be inscribed in a sphere of radius 16.

37–40. Maximizing utility functions *Find the values of ℓ and g with $\ell \geq 0$ and $g \geq 0$ that maximize the following utility functions subject to the given constraints. Give the value of the utility function at the optimal point.*

37. $U = f(\ell, g) = 10\ell^{1/2}g^{1/2}$ subject to $3\ell + 6g = 18$

38. $U = f(\ell, g) = 32\ell^{2/3}g^{1/3}$ subject to $4\ell + 2g = 12$

39. $U = f(\ell, g) = 8\ell^{4/5}g^{1/5}$ subject to $10\ell + 8g = 40$

40. $U = f(\ell, g) = \ell^{1/6}g^{5/6}$ subject to $4\ell + 5g = 20$

41. **Explain why or why not** Determine whether the following statements are true and give an explanation or counterexample.
   a. Suppose you are standing at the center of a sphere looking at a point P on the surface of the sphere. Your line of sight to P is orthogonal to the plane tangent to the sphere at P.
   b. At a point that maximizes f on the curve $g(x, y) = 0$, the dot product $\nabla f \cdot \nabla g$ is zero.

42–47. Alternative method *Solve the following problems from Section 15.7 using Lagrange multipliers.*

42. Exercise 43 43. Exercise 44 44. Exercise 45

45. Exercise 46 46. Exercise 70 47. Exercise 63

48–51. Absolute maximum and minimum values *Find the absolute maximum and minimum values of the following functions over the given regions R. Use Lagrange multipliers to check for extreme points on the boundary.*

48. $f(x, y) = x^2 + 4y^2 + 1$; $R = \{(x, y): x^2 + 4y^2 \leq 1\}$

49. $f(x, y) = x^2 + y^2 - 2y + 1$; $R = \{(x, y): x^2 + y^2 \leq 4\}$ (This is Exercise 47, Section 15.7.)

50. $f(x, y) = 2x^2 + y^2$; $R = \{(x, y): x^2 + y^2 \leq 16\}$ (This is Exercise 48, Section 15.7.)

51. $f(x, y) = 2x^2 - 4x + 3y^2 + 2$; $R = \{(x, y): (x - 1)^2 + y^2 \leq 1\}$ (This is Exercise 51, Section 15.7.)

52. **Extreme points on flattened spheres** The equation $x^{2n} + y^{2n} + z^{2n} = 1$, where n is a positive integer, describes a flattened sphere. Define the extreme points to be the points on the flattened sphere with a maximum distance from the origin.
   a. Find all the extreme points on the flattened sphere with $n = 2$. What is the distance between the extreme points and the origin?
   b. Find all the extreme points on the flattened sphere for integers $n > 2$. What is the distance between the extreme points and the origin?
   c. Give the location of the extreme points in the limit as $n \to \infty$. What is the limiting distance between the extreme points and the origin as $n \to \infty$?

53–55. Production functions *Economists model the output of manufacturing systems using production functions that have many of the same properties as utility functions. The family of Cobb-Douglas production functions has the form $P = f(K, L) = CK^aL^{1-a}$, where K represents capital, L represents labor, and C and a are positive real numbers with $0 < a < 1$. If the cost of capital is p dollars per unit, the cost of labor is q dollars per unit, and the total available budget is B, then the constraint takes the form $pK + qL = B$. Find the values of K and L that maximize the following production functions subject to the given constraint, assuming $K \geq 0$ and $L \geq 0$.*

53. $P = f(K, L) = K^{1/2}L^{1/2}$ for $20K + 30L = 300$

54. $P = f(K, L) = 10K^{1/3}L^{2/3}$ for $30K + 60L = 360$

55. Given the production function $P = f(K, L) = K^aL^{1-a}$ and the budget constraint $pK + qL = B$, where a, p, q, and B are given, show that P is maximized when $K = \dfrac{aB}{p}$ and $L = \dfrac{(1 - a)B}{q}$.

56. Temperature of an elliptical plate The temperature of points on an elliptical plate $x^2 + y^2 + xy \leq 1$ is given by $T(x, y) = 25(x^2 + y^2)$. Find the hottest and coldest temperatures on the edge of the plate.

Explorations and Challenges

57–59. Maximizing a sum

57. Find the maximum value of $x_1 + x_2 + x_3 + x_4$ subject to the condition that $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 16$.

58. Generalize Exercise 57 and find the maximum value of $x_1 + x_2 + \cdots + x_n$ subject to the condition that $x_1^2 + x_2^2 + \cdots + x_n^2 = c^2$ for a real number c and a positive integer n.

59. Generalize Exercise 57 and find the maximum value of $a_1x_1 + a_2x_2 + \cdots + a_nx_n$ subject to the condition that $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$, for given positive real numbers $a_1, \ldots, a_n$ and a positive integer n.

60. Geometric and arithmetic means Given positive numbers $x_1, \ldots, x_n$, prove that the geometric mean $(x_1x_2 \cdots x_n)^{1/n}$ is no greater than the arithmetic mean $\dfrac{x_1 + \cdots + x_n}{n}$ in the following cases.

a. Find the maximum value of xyz, subject to $x + y + z = k$, where k is a positive real number and $x > 0$, $y > 0$, and $z > 0$. Use the result to prove that

$$(xyz)^{1/3} \leq \frac{x + y + z}{3}.$$

b. Generalize part (a) and show that

$$(x_1x_2 \cdots x_n)^{1/n} \leq \frac{x_1 + \cdots + x_n}{n}.$$

61. Problems with two constraints Given a differentiable function $w = f(x, y, z)$, the goal is to find its absolute maximum and minimum values (assuming they exist) subject to the constraints $g(x, y, z) = 0$ and $h(x, y, z) = 0$, where g and h are also differentiable.

a. Imagine a level surface of the function f and the constraint surfaces $g(x, y, z) = 0$ and $h(x, y, z) = 0$. Note that g and h intersect (in general) in a curve C on which maximum and minimum values of f must be found. Explain why ∇g and ∇h are orthogonal to their respective surfaces.

b. Explain why ∇f lies in the plane formed by ∇g and ∇h at a point of C where f has a maximum or minimum value.

c. Explain why part (b) implies that $\nabla f = \lambda \nabla g + \mu \nabla h$ at a point of C where f has a maximum or minimum value, where λ and μ (the Lagrange multipliers) are real numbers.

d. Conclude from part (c) that the equations that must be solved for maximum or minimum values of f subject to two constraints are $\nabla f = \lambda \nabla g + \mu \nabla h$, $g(x, y, z) = 0$, and $h(x, y, z) = 0$.

62–64. Two-constraint problems *Use the result of Exercise 61 to solve the following problems.*

62. The planes $x + 2z = 12$ and $x + y = 6$ intersect in a line L. Find the point on L nearest the origin.

63. Find the maximum and minimum values of $f(x, y, z) = xyz$ subject to the conditions that $x^2 + y^2 = 4$ and $x + y + z = 1$.

64. Find the maximum and minimum values of $f(x, y, z) = x^2 + y^2 + z^2$ on the curve on which the cone $z^2 = 4x^2 + 4y^2$ and the plane $2x + 4z = 5$ intersect.

65. Check assumptions Consider the function $f(x, y) = xy + x + y + 100$ subject to the constraint $xy = 4$.

a. Use the method of Lagrange multipliers to write a system of three equations with three variables x, y, and λ.

b. Solve the system in part (a) to verify that $(x, y) = (-2, -2)$ and $(x, y) = (2, 2)$ are solutions.

c. Let the curve C_1 be the branch of the constraint curve corresponding to $x > 0$. Calculate $f(2, 2)$ and determine whether this value is an absolute maximum or minimum value of f over C_1. (*Hint:* Let $h_1(x)$, for $x > 0$, equal the values of f over the curve C_1 and determine whether h_1 attains an absolute maximum or minimum value at $x = 2$.)

d. Let the curve C_2 be the branch of the constraint curve corresponding to $x < 0$. Calculate $f(-2, -2)$ and determine whether this value is an absolute maximum or minimum value of f over C_2. (*Hint:* Let $h_2(x)$, for $x < 0$, equal the values of f over the curve C_2 and determine whether h_2 attains an absolute maximum or minimum value at $x = -2$.)

e. Show that the method of Lagrange multipliers fails to find the absolute maximum and minimum values of f over the constraint curve $xy = 4$. Reconcile your explanation with the method of Lagrange multipliers.

QUICK CHECK ANSWERS

1. Note that $\nabla f(1, 1) = \langle 2x, 2y \rangle|_{(1,1)} = \langle 2, 2 \rangle$ and $\nabla g(1, 1) = \langle \frac{1}{2}(x - 3), -1 \rangle|_{(1,1)} = \langle -1, -1 \rangle$, which implies the gradients are multiples of one another, and therefore parallel. The equation of the line tangent to C at $(1, 1)$ is $y = -x + 2$; therefore, the vector $\mathbf{v} = \langle 1, -1 \rangle$ is parallel to this tangent line. Because $\nabla f(1, 1) \cdot \mathbf{v} = 0$ and $\nabla g(1, 1) \cdot \mathbf{v} = 0$, both gradients are orthogonal to the tangent line. **3.** The distance between $(3, 4, 0)$ and the cone can be arbitrarily large, so there is no maximizing solution. If the point of interest is not in the xy-plane, there is one minimizing solution. **4.** If you move along the constraint line away from the optimal solution in either direction, you cross level curves of the utility function with decreasing values. ➤

CHAPTER 15 REVIEW EXERCISES

1. **Explain why or why not** Determine whether the following statements are true and give an explanation or counterexample.
   a. The level curves of $g(x, y) = e^{x+y}$ are lines.
   b. The equation $z^2 = 2x^2 - 6y^2$ determines z as a single function of x and y.
   c. If f has continuous partial derivatives of all orders, then $f_{xxy} = f_{yyx}$.
   d. Given the surface $z = f(x, y)$, the gradient $\nabla f(a, b)$ lies in the plane tangent to the surface at $(a, b, f(a, b))$.

2–5. Domains *Find the domain of the following functions. Make a sketch of the domain in the xy-plane.*

2. $f(x, y) = \sin^{-1}\left(\dfrac{x^2 + y^2}{4}\right)$
3. $f(x, y) = \sqrt{x - y^2}$
4. $f(x, y) = \ln xy$
5. $f(x, y) = \sqrt{9x^2 + 4y^2 - 36}$

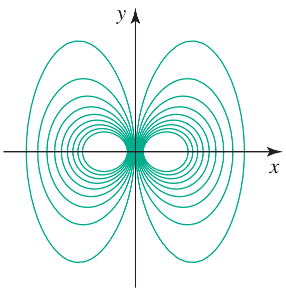
6–7. Graphs *Describe the graph of the following functions, and state the domain and range of the function.*

6. $f(x, y) = -\sqrt{x^2 + y^2}$
7. $g(x, y) = -\sqrt{x^2 + y^2 - 1}$

8–9. Level curves *Make a sketch of several level curves of the following functions. Label at least two level curves with their z-values.*

8. $f(x, y) = x^2 - y$
9. $f(x, y) = x^2 + 4y^2$

10. **Matching level curves with surfaces** Match level curve plots a–d with surfaces A–D.



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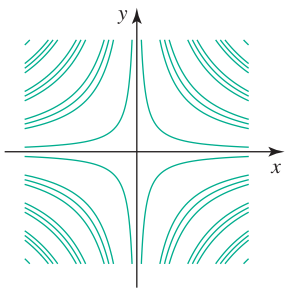

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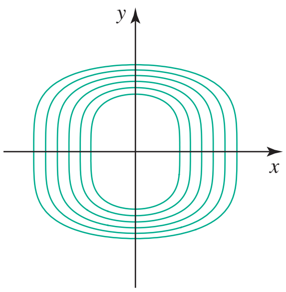

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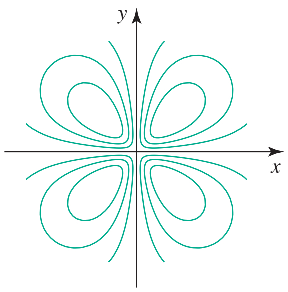

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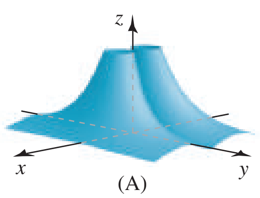

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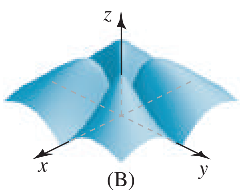

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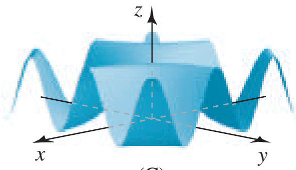

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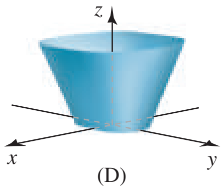


(D)

11–18. Limits *Evaluate the following limits or determine that they do not exist.*

11. $\displaystyle\lim_{(x,y)\to(4,-2)} (10x - 5y + 6xy)$
12. $\displaystyle\lim_{(x,y)\to(1,1)} \frac{xy}{x + y}$
13. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x + y}{xy}$
14. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin xy}{x^2 + y^2}$
15. $\displaystyle\lim_{(x,y)\to(-1,1)} \frac{x^2 - y^2}{x^2 - xy - 2y^2}$
16. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{25x^6y^4}{\sin^2(x^3y^2)}$
17. $\displaystyle\lim_{(x,y,z)\to(2,2,3)} \frac{x^2z - 3x^2 - y^2z + 3y^2}{xz - 3x - yz + 3y}$
18. $\displaystyle\lim_{(x,y,z)\to(3,4,7)} \frac{\sqrt{x + y} - \sqrt{z}}{x + y - z}$

19–20. Continuity *At what points of $\mathbb{R}^2$ are the following functions continuous?*

19. $f(x, y) = \ln(y - x^2 - 1)$
20. $g(x, y) = \dfrac{1}{x^2 + y^2}$

21–26. Partial derivatives *Find the first partial derivatives of the following functions.*

21. $f(x, y) = 3x^2y^5$
22. $g(x, y, z) = 4xyz^2 - \dfrac{3x}{y}$
23. $f(x, y) = \dfrac{x^2}{x^2 + y^2}$
24. $g(x, y, z) = \dfrac{xyz}{x + y}$
25. $f(x, y) = xye^{xy}$
26. $g(u, v) = u \cos v - v \sin u$

27–28. Second partial derivatives *Find the four second partial derivatives of the following functions.*

27. $f(x, y) = e^{2xy}$
28. $H(p, r) = p^2\sqrt{p + 2r}$

29–30. Laplace's equation *Verify that the following functions satisfy Laplace's equation,* $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0.$

29. $u(x, y) = y(3x^2 - y^2)$
30. $u(x, y) = \ln(x^2 + y^2)$

31. **Region between spheres** Two spheres have the same center and radii r and R, where $0 < r < R$. The volume of the region between the spheres is $V(r, R) = \dfrac{4\pi}{3}(R^3 - r^3)$.
    a. First use your intuition. If r is held fixed, how does V change as R increases? What is the sign of V_R? If R is held fixed, how does V change as r increases (up to the value of R)? What is the sign of V_r?
    b. Compute V_r and V_R. Are the results consistent with part (a)?

c. Consider spheres with $R = 3$ and $r = 1$. Does the volume change more if R is increased by $\Delta R = 0.1$ (with r fixed) or if r is decreased by $\Delta r = 0.1$ (with R fixed)?

32–35. Chain Rule *Use the Chain Rule to evaluate the following derivatives.*

32. $w'(t)$, where $w = x \cos yz$, $x = t^2 + 1$, $y = t$, and $z = t^3$

33. $w'(t)$, where $w = z \ln(x^2 + y^2)$, $x = 3e^t$, $y = 4e^t$, and $z = t$

34. w_s, w_t, w_{ss}, w_{tt}, and w_{st}, where $w = xyz$, $x = 2st$, $y = st^2$, and $z = s^2t$

35. w_r, w_s and w_t, where $w = \ln(xy^2)$, $x = rst$, and $y = r + s$

36–37. Implicit differentiation *Find dy/dx for the following implicit relations using Theorem 15.9.*

36. $2x^2 + 3xy - 3y^4 = 2$ **37.** $y \ln(x^2 + y^2) = 4$

38–39. Walking on a surface *Consider the following surfaces and parameterized curves C in the xy-plane.*

a. *In each case, find $z'(t)$ on C.*

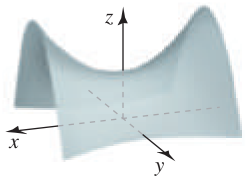
b. *Imagine that you are walking on the surface directly above C consistent with the positive orientation of C. Find the values of t for which you are walking uphill.*

38. $z = 4x^2 + y^2 - 2$; $C: x = \cos t$, $y = \sin t$, for $0 \le t \le 2\pi$

39. $z = x^2 - 2y^2 + 4$; $C: x = 2\cos t$, $y = 2\sin t$, for $0 \le t \le 2\pi$

40. Constant volume cones Suppose the radius of a right circular cone increases as $r(t) = t^a$ and the height decreases as $h(t) = t^{-b}$, for $t \ge 1$, where a and b are positive constants. What is the relationship between a and b such that the volume of the cone remains constant (that is, $V'(t) = 0$, where $V = (\pi/3)r^2h$)?

41. Directional derivatives Consider the function $f(x, y) = 2x^2 - 4y^2 + 10$, whose graph is shown in the figure.



a. Fill in the table showing the values of the directional derivative at points (a, b) in the directions given by the unit vectors **u**, **v**, and **w**.

	$(a, b) = (0, 0)$	$(a, b) = (2, 0)$	$(a, b) = (1, 1)$
$\mathbf{u} = \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$			
$\mathbf{v} = \left\langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$			
$\mathbf{w} = \left\langle -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right\rangle$			

b. Interpret each of the directional derivatives computed in part (a) at the point $(2, 0)$.

42–47. Computing directional derivatives *Compute the gradient of the following functions, evaluate it at the given point P, and evaluate the directional derivative at that point in the direction of the given vector.*

42. $f(x, y) = x^2$; $P(1, 2)$; $\mathbf{u} = \left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle$

43. $g(x, y) = x^2y^3$; $P(-1, 1)$; $\mathbf{u} = \left\langle \frac{5}{13}, \frac{12}{13} \right\rangle$

44. $f(x, y) = \frac{x}{y^2}$; $P(0, 3)$; $\mathbf{u} = \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle$

45. $h(x, y) = \sqrt{2 + x^2 + 2y^2}$; $P(2, 1)$; $\mathbf{u} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$

46. $f(x, y, z) = xe^{1+y^2+z^2}$; $P(0, 1, -2)$; $\mathbf{u} = \left\langle \frac{4}{9}, \frac{1}{9}, \frac{8}{9} \right\rangle$

47. $f(x, y, z) = \sin xy + \cos z$; $P(1, \pi, 0)$; $\mathbf{u} = \left\langle \frac{2}{7}, \frac{3}{7}, -\frac{6}{7} \right\rangle$

48–49. Direction of steepest ascent and descent

a. *Find the unit vectors that give the direction of steepest ascent and steepest descent at P.*

b. *Find a unit vector that points in a direction of no change.*

48. $f(x, y) = \ln(1 + xy)$; $P(2, 3)$

49. $f(x, y) = \sqrt{4 - x^2 - y^2}$; $P(-1, 1)$

50–51. Level curves *Consider the paraboloid $f(x, y) = 8 - 2x^2 - y^2$. For the following level curves $f(x, y) = C$ and points (a, b), compute the slope of the line tangent to the level curve at (a, b) and verify that the tangent line is orthogonal to the gradient at that point.*

50. $f(x, y) = 5$; $(a, b) = (1, 1)$

51. $f(x, y) = 0$; $(a, b) = (2, 0)$

52. Directions of zero change Find the directions in which the function $f(x, y) = 4x^2 - y^2$ has zero change at the point $(1, 1, 3)$. Express the directions in terms of unit vectors.

53. Electric potential due to a charged cylinder An infinitely long charged cylinder of radius R with its axis along the z-axis has an electric potential $V = k \ln \frac{R}{r}$, where r is the distance between a variable point $P(x, y)$ and the axis of the cylinder ($r^2 = x^2 + y^2$) and k is a physical constant. The electric field at a point (x, y) in the xy-plane is given by $\mathbf{E} = -\nabla V$, where ∇V is the two-dimensional gradient. Compute the electric field at a point (x, y) with $r > R$.

54–59. Tangent planes *Find an equation of the plane tangent to the following surfaces at the given points.*

54. $z = 2x^2 + y^2$; $(1, 1, 3)$ and $(0, 2, 4)$

55. $x^2 + \frac{y^2}{4} - \frac{z^2}{9} = 1$; $(0, 2, 0)$ and $\left(1, 1, \frac{3}{2}\right)$

56. $x^2 - 2x + y^2 + 4y + 3z^2 = 2$; $(3, -2, 1)$ and $(-1, -2, 1)$

57. $e^{xy^2z^3-1} = 1$; $(1, 1, 1)$ and $(1, -1, 1)$

58. $z - \tan^{-1} xy = 0$; $(1, 1, \pi/4)$ and $(1, \sqrt{3}, \pi/3)$

59. $\sqrt{\frac{x + y}{z}} = 1$; $(2, 2, 4)$ and $(10, -1, 9)$

T 60–61. Linear approximation

a. *Find the linear approximation to the function f at the point (a, b).*

b. *Use part (a) to estimate the given function value.*

60. $f(x, y) = 4\cos(2x - y)$; $(a, b) = \left(\frac{\pi}{4}, \frac{\pi}{4}\right)$; estimate $f(0.8, 0.8)$.

61. $f(x, y) = (x + y)e^{xy}$; $(a, b) = (2, 0)$; estimate $f(1.95, 0.05)$.

62. Changes in a function Estimate the change in the function $f(x, y) = -2y^2 + 3x^2 + xy$ when (x, y) changes from $(1, -2)$ to $(1.05, -1.9)$.

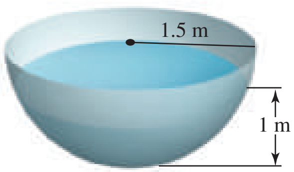
63. Volume of a cylinder The volume of a cylinder with radius r and height h is $V = \pi r^2 h$. Find the approximate percentage change in the volume when the radius decreases by 3% and the height increases by 2%.

64. Volume of an ellipsoid The volume of an ellipsoid with axes of length $2a$, $2b$, and $2c$ is $V = \pi abc$. Find the percentage change in the volume when a increases by 2%, b increases by 1.5%, and c decreases by 2.5%.

65. Water level changes A hemispherical tank with a radius of 1.50 m is filled with water to a depth of 1.00 m. Water is released from the tank, and the water level drops by 0.05 m (from 1.00 m to 0.95 m).

a. Approximate the change in the volume of water in the tank. The volume of a spherical cap is $V = \frac{1}{3}\pi h^2(3r - h)$, where r is the radius of the sphere and h is the thickness of the cap (in this case, the depth of the water).

b. Approximate the change in the surface area of the water in the tank.



66–69. Analyzing critical points *Identify the critical points of the following functions. Then determine whether each critical point corresponds to a local maximum, local minimum, or saddle point. State when your analysis is inconclusive. Confirm your results using a graphing utility.*

66. $f(x, y) = x^4 + y^4 - 16xy$

67. $f(x, y) = \dfrac{x^3}{3} - \dfrac{y^3}{3} + 2xy$

68. $f(x, y) = xy(2 + x)(y - 3)$

69. $f(x, y) = 10 - x^3 - y^3 - 3x^2 + 3y^2$

70–73. Absolute maxima and minima *Find the absolute maximum and minimum values of the following functions on the specified region R.*

70. $f(x, y) = \dfrac{x^3}{3} - \dfrac{y^3}{3} + 2xy$ on the rectangle $R = \{(x, y): 0 \le x \le 3, -1 \le y \le 1\}$

71. $f(x, y) = x^4 + y^4 - 4xy + 1$ on the square $R = \{(x, y): -2 \le x \le 2, -2 \le y \le 2\}$

72. $f(x, y) = x^2y - y^3$ on the triangle $R = \{(x, y): 0 \le x \le 2, 0 \le y \le 2 - x\}$

73. $f(x, y) = xy$ on the semicircular disk $R = \{(x, y): -1 \le x \le 1, 0 \le y \le \sqrt{1 - x^2}\}$

74. Least distance What point on the plane $x + y + 4z = 8$ is closest to the origin? Give an argument showing you have found an absolute minimum of the distance function.

75–78. Lagrange multipliers *Use Lagrange multipliers to find the absolute maximum and minimum values of f (if they exist) subject to the given constraint.*

75. $f(x, y) = 2x + y + 10$ subject to $2(x - 1)^2 + 4(y - 1)^2 = 1$

76. $f(x, y) = xy$ subject to $3x^2 - 2xy + 3y^2 = 4$

77. $f(x, y, z) = x + 2y - z$ subject to $x^2 + y^2 + z^2 = 1$

78. $f(x, y, z) = x^2y^2z$ subject to $2x^2 + y^2 + z^2 = 25$

79. Maximum perimeter rectangle Use Lagrange multipliers to find the dimensions of the rectangle with the maximum perimeter that can be inscribed with sides parallel to the coordinate axes in the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

80. Minimum surface area cylinder Use Lagrange multipliers to find the dimensions of the right circular cylinder of minimum surface area (including the circular ends) with a volume of 32π in^3. Give an argument showing you have found an absolute minimum.

81. Minimum distance to a cone Find the point(s) on the cone $z^2 - x^2 - y^2 = 0$ that are closest to the point $(1, 3, 1)$. Give an argument showing you have found an absolute minimum of the distance function.

82. Gradient of a distance function Let $P_0(a, b, c)$ be a fixed point in $\mathbb{R}^3$, and let $d(x, y, z)$ be the distance between P_0 and a variable point $P(x, y, z)$.

a. Compute $\nabla d(x, y, z)$.

b. Show that $\nabla d(x, y, z)$ points in the direction from P_0 to P and has magnitude 1 for all (x, y, z).

c. Describe the level surfaces of d and give the direction of $\nabla d(x, y, z)$ relative to the level surfaces of d.

d. Discuss $\lim_{P \to P_0} \nabla d(x, y, z)$.

83. Minimum distance to a paraboloid Use Lagrange multipliers to find the point on the paraboloid $z = x^2 + y^2$ that lies closest to the point $(5, 10, 3)$. Give an argument showing you have found an absolute minimum of the distance function.

Chapter 15 Guided Projects

Applications of the material in this chapter and related topics can be found in the following Guided Projects. For additional information, see the Preface.

- Traveling waves
- Ecological diversity
- Economic production functions

16

Multiple Integration

Chapter Preview We have now generalized limits and derivatives to functions of several variables. The next step is to carry out a similar process with respect to integration. As you know, single (one-variable) integrals are developed from Riemann sums and are used to compute areas of regions in $\mathbb{R}^2$. In an analogous way, we use Riemann sums to develop double (two-variable) and triple (three-variable) integrals, which are used to compute volumes of solid regions in $\mathbb{R}^3$. These multiple integrals have many applications in statistics, science, and engineering, including calculating the mass, the center of mass, and moments of inertia of solids with a variable density. Another significant development in this chapter is the appearance of cylindrical and spherical coordinates. These alternative coordinate systems often simplify the evaluation of integrals in three-dimensional space. The chapter closes with the two- and three-dimensional versions of the substitution (change of variables) rule. The overall lesson of the chapter is that we can integrate functions over most geometrical objects, from intervals on the x-axis to regions in the plane bounded by curves to complicated three-dimensional solids.

16.1 Double Integrals over Rectangular Regions

In Chapter 15 the concept of differentiation was extended to functions of several variables. In this chapter, we extend integration to multivariable functions. By the close of the chapter, we will have completed Table 16.1, which is a basic road map for calculus.

Table 16.1

	Derivatives	Integrals
Single variable: $f(x)$	$f'(x)$	$\int_a^b f(x)\,dx$
Several variables: $f(x, y)$ **and** $f(x, y, z)$	$\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial f}{\partial z}$	$\iint_R f(x, y)\,dA,\ \iiint_D f(x, y, z)\,dV$

Volumes of Solids

The problem of finding the net area of a region bounded by a curve led to the definite integral in Chapter 5. Recall that we began that discussion by approximating the region with a collection of rectangles and then formed a Riemann sum of the areas of the rectangles. Under appropriate conditions, as the number of rectangles increases, the sum approaches the value of the definite integral, which is the net area of the region.

We now carry out an analogous procedure with surfaces defined by functions of the form $z = f(x, y)$, where, for the moment, we assume $f(x, y) \geq 0$ on a region R in the xy-plane (**Figure 16.1a**). The goal is to determine the volume of the solid bounded by the surface and R. In general terms, the solid is first approximated by *boxes* (**Figure 16.1b**). The sum of the volumes of these boxes, which is a Riemann sum, approximates the volume of the solid. Under appropriate conditions, as the number of boxes increases, the approximations converge to the value of a *double integral*, which is the volume of the solid.

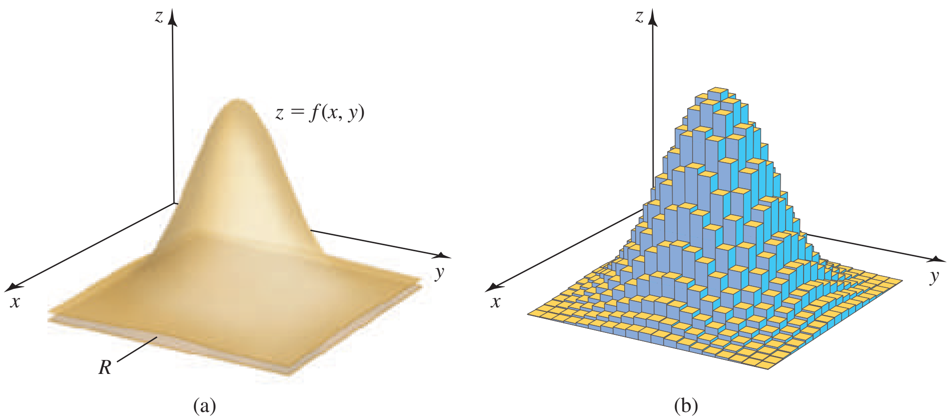


Figure 16.1

➤ We adopt the convention that Δx_k and Δy_k are the side lengths of the kth rectangle, for $k = 1, \ldots, n$, even though there are generally fewer than n different values of Δx_k and Δy_k. This convention is used throughout the chapter.

We assume $z = f(x, y)$ is a nonnegative function defined on a *rectangular* region $R = \{(x, y): a \leq x \leq b, c \leq y \leq d\}$. A **partition** of R is formed by dividing R into n rectangular subregions using lines parallel to the x- and y-axes (not necessarily uniformly spaced). The rectangles may be numbered in any systematic way; for example, left to right and then bottom to top. The side lengths of the kth rectangle are denoted Δx_k and Δy_k, so the area of the kth rectangle is $\Delta A_k = \Delta x_k \Delta y_k$. We also let (x_k^*, y_k^*) be any point in the kth rectangle, for $1 \leq k \leq n$ (**Figure 16.2**).

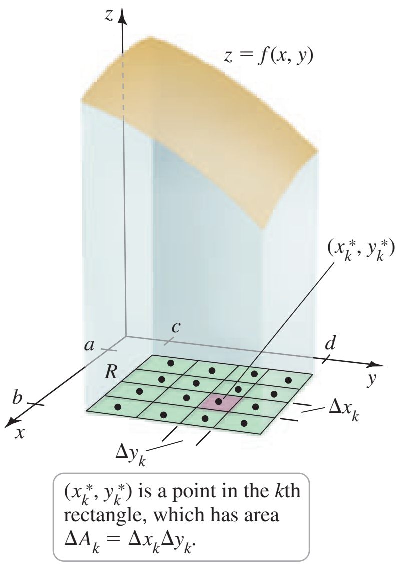


Figure 16.2

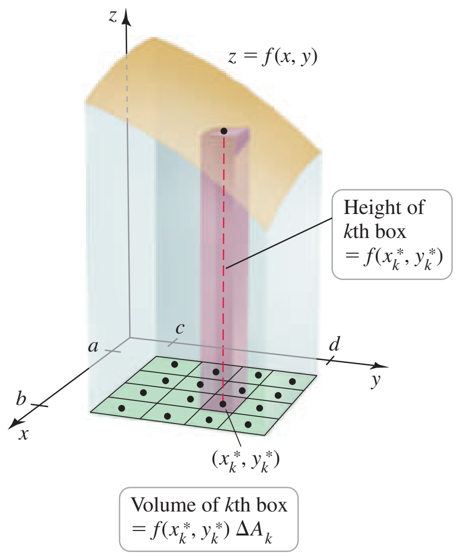


Figure 16.3

To approximate the volume of the solid bounded by the surface $z = f(x, y)$ and the region R, we construct boxes on each of the n rectangles; each box has a height of $f(x_k^*, y_k^*)$ and a base with area ΔA_k, for $1 \leq k \leq n$ (**Figure 16.3**). Therefore, the volume of the kth box is

$$f(x_k^*, y_k^*)\Delta A_k = f(x_k^*, y_k^*)\Delta x_k \Delta y_k.$$

QUICK CHECK 1 Explain why the sum for the volume is an approximation. How can the approximation be improved? ◄

The sum of the volumes of the n boxes gives an approximation to the volume of the solid:

$$V \approx \sum_{k=1}^{n} f(x_k^*, y_k^*)\,\Delta A_k.$$

We now let Δ be the maximum length of the diagonals of the rectangles in the partition. As $\Delta \rightarrow 0$, the areas of *all* the rectangles approach zero $(\Delta A_k \rightarrow 0)$ and the number of rectangles increases $(n \rightarrow \infty)$. If the approximations given by these Riemann sums have a limit as $\Delta \rightarrow 0$, then we define the volume of the solid to be that limit (**Figure 16.4**).

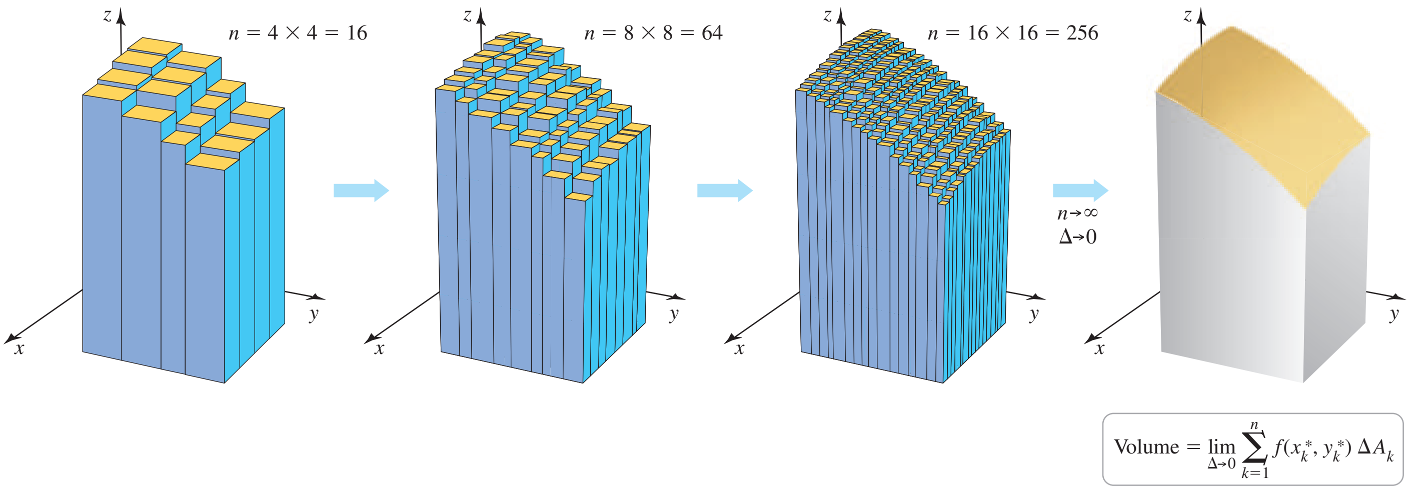


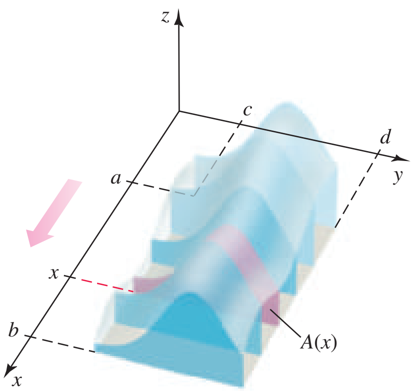
Figure 16.4

➤ The functions that we encounter in this text are integrable. Advanced methods are needed to prove that continuous functions and many functions with finite discontinuities are also integrable.

DEFINITION Double Integrals

A function f defined on a rectangular region R in the xy-plane is **integrable** on R if $\lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k$ exists for all partitions of R and for all choices of (x_k^*, y_k^*) within those partitions. The limit is the **double integral of f over R**, which we write

$$\iint_R f(x, y)\, dA = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k.$$



If a solid is sliced parallel to the y-axis and perpendicular to the xy-plane, and the cross-sectional area of the slice at the point x is $A(x)$, then the volume of the solid region is

$$V = \int_a^b A(x)\, dx.$$

If f is nonnegative on R, then the double integral equals the volume of the solid bounded by $z = f(x, y)$ and the xy-plane over R. If f is negative on parts of R, the value of the double integral may be zero or negative, and the result is interpreted as a *net volume* (by analogy with *net area* for single variable integrals).

Iterated Integrals

Evaluating double integrals using limits of Riemann sums is tedious and rarely done. Fortunately, there is a practical method for evaluating double integrals that is based on the general slicing method (Section 6.3). An example illustrates the technique.

Suppose we wish to compute the volume of the solid region bounded by the plane $z = f(x, y) = 6 - 2x - y$ over the rectangular region $R = \{(x, y): 0 \le x \le 1, 0 \le y \le 2\}$ (**Figure 16.5**). By definition, the volume is given by the double integral

$$V = \iint_R f(x, y)\, dA = \iint_R (6 - 2x - y)\, dA.$$

According to the general slicing method (see margin note and figure), we can compute this volume by taking vertical slices through the solid parallel to the yz-plane (Figure 16.5). The slice at the point x has a cross-sectional area denoted $A(x)$. In general, as x varies, the area $A(x)$ also changes, so we integrate these cross-sectional areas from $x = 0$ to $x = 1$ to obtain the volume

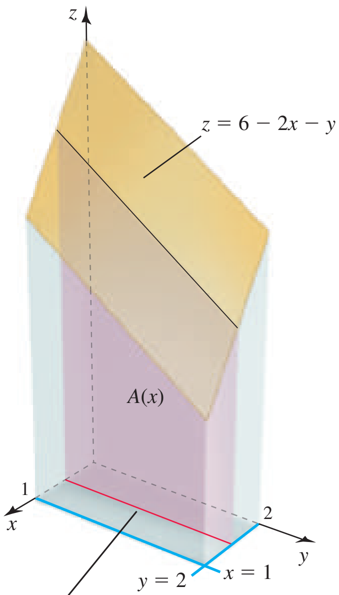
$$V = \int_0^1 A(x)\,dx.$$

The important observation is that for a fixed value of x, $A(x)$ is the area of the plane region under the curve $z = 6 - 2x - y$. This area is computed by integrating f with respect to y from $y = 0$ to $y = 2$, holding x fixed; that is,

$$A(x) = \int_0^2 (6 - 2x - y)\,dy,$$

where $0 \le x \le 1$, and x is treated as a constant in the integration. Substituting for $A(x)$, we have

$$V = \int_0^1 A(x)\,dx = \int_0^1 \underbrace{\left(\int_0^2 (6 - 2x - y)\,dy \right)}_{A(x)} dx.$$



A slice at a fixed value of x has area $A(x)$, where $0 \le x \le 1$.

Figure 16.5

The expression that appears on the right side of this equation is called an **iterated integral** (meaning repeated integral). We first evaluate the inner integral with respect to y holding x fixed; the result is a function of x. Then the outer integral is evaluated with respect to x; the result is a real number, which is the volume of the solid in Figure 16.5. Both of these integrals are ordinary one-variable integrals.

EXAMPLE 1 **Evaluating an iterated integral** Evaluate $V = \int_0^1 A(x)\,dx$, where $A(x) = \int_0^2 (6 - 2x - y)\,dy$.

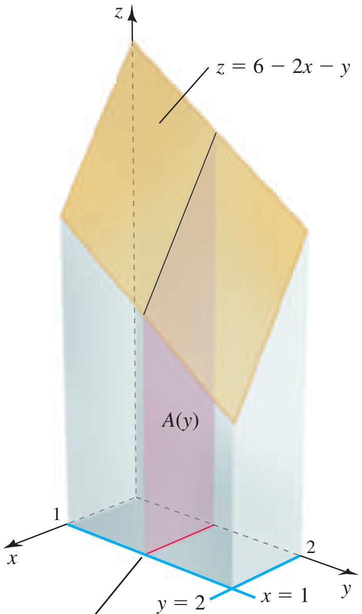
SOLUTION Using the Fundamental Theorem of Calculus, holding x constant, we have

$$\begin{aligned} A(x) &= \int_0^2 (6 - 2x - y)\,dy \\ &= \left(6y - 2xy - \frac{y^2}{2}\right)\Bigg|_0^2 && \text{Evaluate integral with respect to } y;\ x \text{ is constant.} \\ &= (12 - 4x - 2) - 0 && \text{Simplify; limits are in } y. \\ &= 10 - 4x. && \text{Simplify.} \end{aligned}$$

Substituting $A(x) = 10 - 4x$ into the volume integral, we have

$$\begin{aligned} V &= \int_0^1 A(x)\,dx \\ &= \int_0^1 (10 - 4x)\,dx && \text{Substitute for } A(x). \\ &= (10x - 2x^2)\Big|_0^1 && \text{Evaluate integral with respect to } x. \\ &= 8. && \text{Simplify.} \end{aligned}$$

Related Exercises 10, 25 ◄



A slice at a fixed value of y has area $A(y)$, where $0 \le y \le 2$.

Figure 16.6

EXAMPLE 2 **Same double integral, different order** Example 1 used slices through the solid parallel to the yz-plane. Compute the volume of the same solid using vertical slices through the solid parallel to the xz-plane, for $0 \le y \le 2$ (**Figure 16.6**).

SOLUTION In this case, $A(y)$ is the area of a slice through the solid for a fixed value of y in the interval $0 \le y \le 2$. This area is computed by integrating $z = 6 - 2x - y$ from $x = 0$ to $x = 1$, holding y fixed; that is,

$$A(y) = \int_0^1 (6 - 2x - y)\,dx,$$

where $0 \le y \le 2$.

Using the general slicing method again, the volume is

$$\begin{aligned}
V &= \int_0^2 A(y)\,dy && \text{General slicing method} \\
&= \int_0^2 \underbrace{\left(\int_0^1 (6 - 2x - y)\,dx\right)}_{A(y)} dy && \text{Substitute for } A(y). \\
&= \int_0^2 \left((6x - x^2 - yx)\Big|_0^1 \right) dy && \text{Evaluate inner integral with respect to } x;\ y \text{ is constant.} \\
&= \int_0^2 (5 - y)\,dy && \text{Simplify; limits are in } x. \\
&= \left(5y - \frac{y^2}{2}\right)\Big|_0^2 && \text{Evaluate outer integral with respect to } y. \\
&= 8. && \text{Simplify.}
\end{aligned}$$

Related Exercise 37 ◄

Several important comments are in order. First, the two iterated integrals give the same value for the double integral. Second, the notation of the iterated integral must be used carefully. When we write $\int_c^d \int_a^b f(x, y)\,dx\,dy$, it means $\int_c^d \left(\int_a^b f(x, y)\,dx\right) dy$. The *inner* integral with respect to x is evaluated first, holding y fixed, and the variable runs from $x = a$ to $x = b$. The result of that integration is a constant or a function of y, which is then integrated in the *outer* integral, with the variable running from $y = c$ to $y = d$. The order of integration is signified by the order of dx and dy.

Similarly, $\int_a^b \int_c^d f(x, y)\,dy\,dx$ means $\int_a^b \left(\int_c^d f(x, y)\,dy\right) dx$. The inner integral with respect to y is evaluated first, holding x fixed. The result is then integrated with respect to x. In both cases, the limits of integration in the iterated integrals determine the boundaries of the rectangular *region of integration*.

QUICK CHECK 2 Consider the integral $\int_3^4 \int_1^2 f(x, y)\,dx\,dy$. Give the limits of integration and the variable of integration for the first (inner) integral and the second (outer) integral. Sketch the region of integration. ◄

Examples 1 and 2 illustrate one version of *Fubini's Theorem*, a deep result that relates double integrals to iterated integrals. The first version of the theorem applies to double integrals over rectangular regions.

➤ The area of the kth rectangle in the partition is $\Delta A_k = \Delta x_k \Delta y_k$, where Δx_k and Δy_k are the lengths of the sides of that rectangle. Accordingly, the *element of area dA* in the double integral becomes $dx\,dy$ or $dy\,dx$ in the iterated integral.

THEOREM 16.1 (Fubini) Double Integrals over Rectangular Regions
Let f be continuous on the rectangular region $R = \{(x, y): a \le x \le b, c \le y \le d\}$. The double integral of f over R may be evaluated by either of two iterated integrals:

$$\iint_R f(x, y)\,dA = \int_c^d \int_a^b f(x, y)\,dx\,dy = \int_a^b \int_c^d f(x, y)\,dy\,dx.$$

The importance of Fubini's Theorem is twofold: It says that double integrals may be evaluated by iterated integrals. It also says that the order of integration in the iterated integrals does not matter (although in practice, one order of integration is often easier to use than the other).

EXAMPLE 3 A double integral Find the volume of the solid bounded by the surface $f(x, y) = 4 + 9x^2y^2$ over the region $R = \{(x, y): -1 \le x \le 1, 0 \le y \le 2\}$. Use both possible orders of integration.

SOLUTION Because $f(x, y) > 0$ on R, the volume of the region is given by the double integral $\iint_R (4 + 9x^2y^2)\,dA$. By Fubini's Theorem, the double integral is evaluated as an iterated integral. If we first integrate with respect to x, the area of a cross section of the solid for a fixed value of y is given by $A(y)$ (**Figure 16.7a**). The volume of the region is

$$\iint_R (4 + 9x^2y^2)\,dA = \int_0^2 \underbrace{\int_{-1}^1 (4 + 9x^2y^2)\,dx}_{A(y)}\,dy \quad \text{Convert to an iterated integral.}$$

$$= \int_0^2 (4x + 3x^3y^2)\Big|_{-1}^1 dy \quad \text{Evaluate inner integral with respect to } x.$$

$$= \int_0^2 (8 + 6y^2)\,dy \quad \text{Simplify.}$$

$$= (8y + 2y^3)\Big|_0^2 \quad \text{Evaluate outer integral with respect to } y.$$

$$= 32. \quad \text{Simplify.}$$

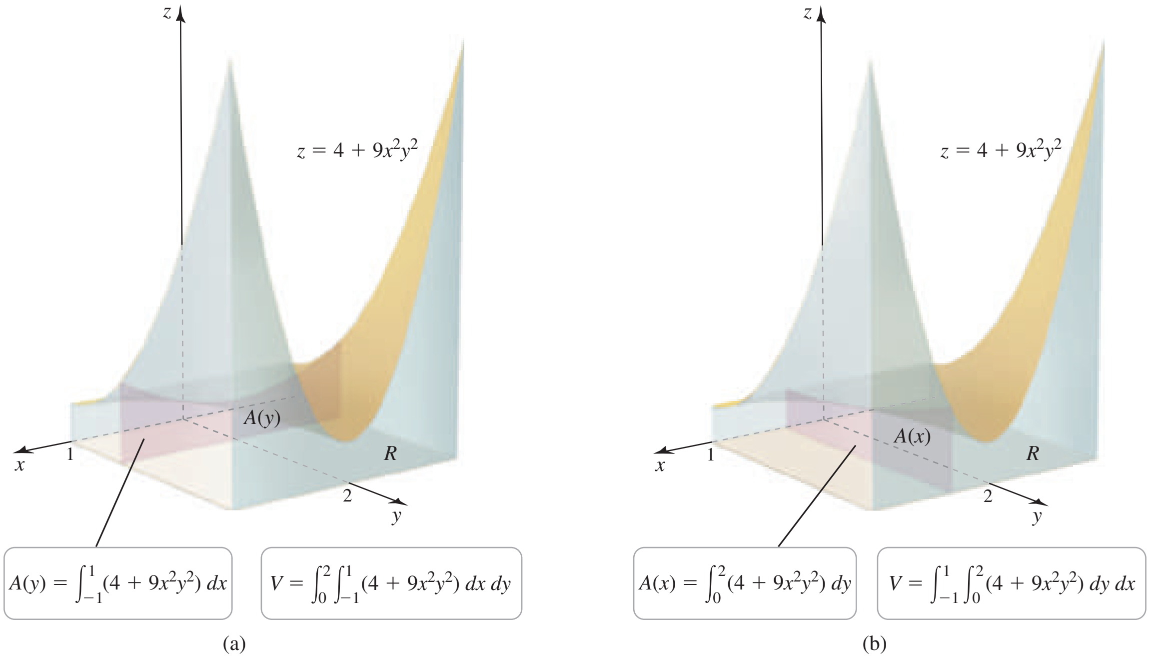


Figure 16.7

Alternatively, if we integrate first with respect to y, the area of a cross section of the solid for a fixed value of x is given by $A(x)$ (**Figure 16.7b**). The volume of the region is

$$\iint_R (4 + 9x^2y^2)\,dA = \int_{-1}^1 \underbrace{\int_0^2 (4 + 9x^2y^2)\,dy}_{A(x)}\,dx \quad \text{Convert to an iterated integral.}$$

$$= \int_{-1}^1 (4y + 3x^2y^3)\Big|_0^2 dx \quad \text{Evaluate inner integral with respect to } y.$$

$$= \int_{-1}^1 (8 + 24x^2)\,dx \quad \text{Simplify.}$$

$$= (8x + 8x^3)\Big|_{-1}^1 = 32. \quad \text{Evaluate outer integral with respect to } x.$$

As guaranteed by Fubini's Theorem, the two iterated integrals are equal, both giving the value of the double integral and the volume of the solid.

Related Exercises 26, 39 ➤

QUICK CHECK 3 Write the iterated integral $\int_{-10}^{10}\int_0^{20}(x^2y + 2xy^3)\,dy\,dx$ with the order of integration reversed. ➤

The following example shows that sometimes the order of integration must be chosen carefully either to save work or to make the integration possible.

EXAMPLE 4 **Choosing a convenient order of integration** Evaluate $\iint_R ye^{xy}\,dA$, where $R = \{(x, y): 0 \le x \le 1, 0 \le y \le \ln 2\}$.

SOLUTION The iterated integral $\int_0^1 \int_0^{\ln 2} ye^{xy}\,dy\,dx$ requires first integrating ye^{xy} with respect to y, which entails integration by parts. An easier approach is to integrate first with respect to x:

$$\int_0^{\ln 2}\int_0^1 ye^{xy}\,dx\,dy = \int_0^{\ln 2} e^{xy}\Big|_0^1\,dy \quad \text{Evaluate inner integral with respect to } x.$$

$$= \int_0^{\ln 2}(e^y - 1)\,dy \quad \text{Simplify.}$$

$$= (e^y - y)\Big|_0^{\ln 2} \quad \text{Evaluate outer integral with respect to } y.$$

$$= 1 - \ln 2. \quad \text{Simplify.}$$

Related Exercises 41, 43 ◄

Average Value

The concept of the average value of a function (Section 5.4) extends naturally to functions of two variables. Recall that the average value of the integrable function f over the interval $[a, b]$ is

$$\bar{f} = \frac{1}{b - a}\int_a^b f(x)\,dx.$$

To find the average value of an integrable function f over a region R, we integrate f over R and divide the result by the "size" of R, which is the area of R in the two-variable case.

➤ The same definition of average value applies to more general regions in the plane.

DEFINITION **Average Value of a Function over a Plane Region**

The **average value** of an integrable function f over a region R is

$$\bar{f} = \frac{1}{\text{area of } R}\iint_R f(x, y)\,dA.$$

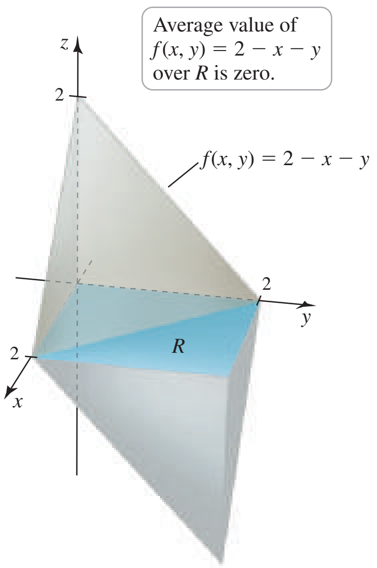


Figure 16.8

EXAMPLE 5 **Average value** Find the average value of the quantity $2 - x - y$ over the square $R = \{(x, y): 0 \le x \le 2, 0 \le y \le 2\}$ (**Figure 16.8**).

SOLUTION The area of the region R is 4. Letting $f(x, y) = 2 - x - y$, the average value of f is

$$\frac{1}{\text{area of } R}\iint_R f(x, y)\,dA = \frac{1}{4}\iint_R (2 - x - y)\,dA$$

$$= \frac{1}{4}\int_0^2\int_0^2 (2 - x - y)\,dx\,dy \quad \text{Convert to an iterated integral.}$$

$$= \frac{1}{4}\int_0^2 \left(2x - \frac{x^2}{2} - xy\right)\Big|_0^2\,dy \quad \text{Evaluate inner integral with respect to } x.$$

$$= \frac{1}{4}\int_0^2 (2 - 2y)\,dy \quad \text{Simplify.}$$

$$= 0. \quad \text{Evaluate outer integral with respect to } y.$$

Related Exercise 46 ◄

➤ An average value of 0 means that over the region R, the volume of the solid above the xy-plane and below the surface equals the volume of the solid below the xy-plane and above the surface.

SECTION 16.1 EXERCISES

Getting Started

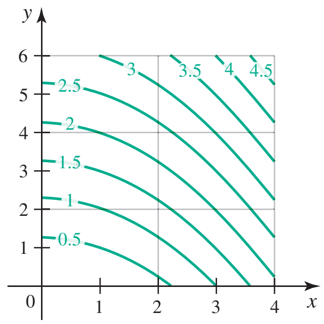
1. Write an iterated integral that gives the volume of the solid bounded by the surface $f(x, y) = xy$ over the square $R = \{(x, y): 0 \le x \le 2, 1 \le y \le 3\}$.

2. Write an iterated integral that gives the volume of a box with height 10 and base $R = \{(x, y): 0 \le x \le 5, -2 \le y \le 4\}$.

3. Write two iterated integrals that equal $\iint_R f(x, y)\, dA$, where $R = \{(x, y): -2 \le x \le 4, 1 \le y \le 5\}$.

4. Consider the integral $\int_1^3 \int_{-1}^1 (2y^2 + xy)\, dy\, dx$. State the variable of integration in the first (inner) integral and the limits of integration. State the variable of integration in the second (outer) integral and the limits of integration.

5. Region $R = \{(x, y): 0 \le x \le 4, 0 \le y \le 6\}$ is partitioned into six equal subregions (see figure, which also shows the level curves of a function f continuous on the region R). Estimate the value of $\iint_R f(x, y)\, dA$ by evaluating the Riemann sum $\sum_{k=1}^{6} f(x_k^*, y_k^*)\Delta A_k$, where (x_k^*, y_k^*) is the center of the kth subregion, for $k = 1, \ldots, 6$.



6. Draw a solid whose volume is given by the double integral $\int_0^6 \int_1^2 10\, dy\, dx$. Then evaluate the integral using geometry.

Practice Exercises

7–24. Iterated integrals *Evaluate the following iterated integrals.*

7. $\displaystyle\int_0^2 \int_0^1 4xy\, dx\, dy$

8. $\displaystyle\int_1^2 \int_0^1 (3x^2 + 4y^3)\, dy\, dx$

9. $\displaystyle\int_1^3 \int_0^2 x^2 y\, dx\, dy$

10. $\displaystyle\int_0^3 \int_{-2}^1 (2x + 3y)\, dx\, dy$

11. $\displaystyle\int_1^3 \int_0^{\pi/2} x \sin y\, dy\, dx$

12. $\displaystyle\int_1^3 \int_1^2 (y^2 + y)\, dx\, dy$

13. $\displaystyle\int_1^4 \int_0^4 \sqrt{uv}\, du\, dv$

14. $\displaystyle\int_0^{\pi/4} \int_0^3 r \sec \theta\, dr\, d\theta$

15. $\displaystyle\int_1^{\ln 5} \int_0^{\ln 3} e^{x+y}\, dx\, dy$

16. $\displaystyle\int_0^{\pi/2} \int_0^1 uv \cos(u^2 v)\, du\, dv$

17. $\displaystyle\int_0^1 \int_0^1 t^2 e^{st}\, ds\, dt$

18. $\displaystyle\int_0^2 \int_0^1 \frac{8xy}{1 + x^4}\, dx\, dy$

19. $\displaystyle\int_1^e \int_0^1 4(p + q) \ln q\, dp\, dq$

20. $\displaystyle\int_0^1 \int_0^\pi y^2 \cos xy\, dx\, dy$

21. $\displaystyle\int_1^2 \int_1^2 \frac{x}{x + y}\, dy\, dx$

22. $\displaystyle\int_0^2 \int_0^1 x^5 y^2 e^{x^3 y^3}\, dy\, dx$

23. $\displaystyle\int_0^1 \int_1^4 \frac{3y}{\sqrt{x + y^2}}\, dx\, dy$

24. $\displaystyle\int_0^1 \int_0^1 x^2 y^2 e^{x^3 y}\, dx\, dy$

25–35. Double integrals *Evaluate each double integral over the region R by converting it to an iterated integral.*

25. $\displaystyle\iint_R (x + 2y)\, dA;\ R = \{(x, y): 0 \le x \le 3, 1 \le y \le 4\}$

26. $\displaystyle\iint_R (x^2 + xy)\, dA;\ R = \{(x, y): 1 \le x \le 2, -1 \le y \le 1\}$

27. $\displaystyle\iint_R s^2 t \sin(st^2)\, dA;\ R = \{(s, t): 0 \le s \le \pi, 0 \le t \le 1\}$

28. $\displaystyle\iint_R \frac{x}{1 + xy}\, dA;\ R = \{(x, y): 0 \le x \le 1, 0 \le y \le 1\}$

29. $\displaystyle\iint_R \sqrt{\frac{x}{y}}\, dA;\ R = \{(x, y): 0 \le x \le 1, 1 \le y \le 4\}$

30. $\displaystyle\iint_R xy \sin x^2\, dA;\ R = \{(x, y): 0 \le x \le \sqrt{\pi/2}, 0 \le y \le 1\}$

31. $\displaystyle\iint_R e^{x+2y}\, dA;\ R = \{(x, y): 0 \le x \le \ln 2, 1 \le y \le \ln 3\}$

32. $\displaystyle\iint_R (x^2 - y^2)^2\, dA;\ R = \{(x, y): -1 \le x \le 2, 0 \le y \le 1\}$

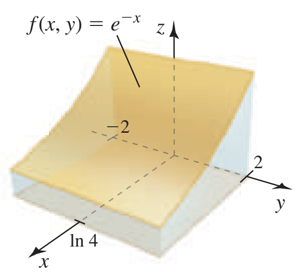
33. $\displaystyle\iint_R (x^5 - y^5)^2\, dA;\ R = \{(x, y): 0 \le x \le 1, -1 \le y \le 1\}$

34. $\displaystyle\iint_R \cos(x\sqrt{y})\, dA;\ R = \{(x, y): 0 \le x \le 1, 0 \le y \le \pi^2/4\}$

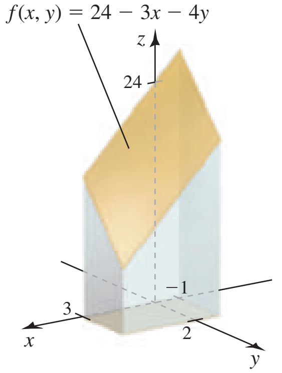
35. $\displaystyle\iint_R x^3 y \cos(x^2 y^2)\, dA;\ R = \{(x, y): 0 \le x \le \sqrt{\pi/2}, 0 \le y \le 1\}$

36–39. Volumes of solids *Find the volume of the following solids.*

36. The solid beneath the cylinder $f(x, y) = e^{-x}$ and above the region $R = \{(x, y): 0 \le x \le \ln 4, -2 \le y \le 2\}$



37. The solid beneath the plane $f(x, y) = 24 - 3x - 4y$ and above the region $R = \{(x, y): -1 \le x \le 3, 0 \le y \le 2\}$



38. The solid in the first octant bounded above by the surface $z = 9xy\sqrt{1 - x^2}\sqrt{4 - y^2}$ and below by the xy-plane

39. The solid in the first octant bounded by the surface $z = xy^2\sqrt{1 - x^2}$ and the planes $z = 0$ and $y = 3$

40–45. Choose a convenient order *When converted to an iterated integral, the following double integrals are easier to evaluate in one order than the other. Find the best order and evaluate the integral.*

40. $\displaystyle\iint_R y \cos xy\, dA$; $R = \{(x, y): 0 \le x \le 1, 0 \le y \le \pi/3\}$

41. $\displaystyle\iint_R (y + 1)e^{x(y+1)}\, dA$; $R = \{(x, y): 0 \le x \le 1, -1 \le y \le 1\}$

42. $\displaystyle\iint_R x \sec^2 xy\, dA$; $R = \{(x, y): 0 \le x \le \pi/3, 0 \le y \le 1\}$

43. $\displaystyle\iint_R 6x^5 e^{x^3 y}\, dA$; $R = \{(x, y): 0 \le x \le 2, 0 \le y \le 2\}$

44. $\displaystyle\iint_R y^3 \sin(xy^2)\, dA$; $R = \{(x, y): 0 \le x \le 2, 0 \le y \le \sqrt{\pi/2}\}$

45. $\displaystyle\iint_R \frac{x}{(1 + xy)^2}\, dA$; $R = \{(x, y): 0 \le x \le 4, 1 \le y \le 2\}$

46–48. Average value *Compute the average value of the following functions over the region R.*

46. $f(x, y) = 4 - x - y$; $R = \{(x, y): 0 \le x \le 2, 0 \le y \le 2\}$

47. $f(x, y) = e^{-y}$; $R = \{(x, y): 0 \le x \le 6, 0 \le y \le \ln 2\}$

48. $f(x, y) = \sin x \sin y$; $R = \{(x, y): 0 \le x \le \pi, 0 \le y \le \pi\}$

49. Average value Find the average squared distance between the points of $R = \{(x, y): -2 \le x \le 2, 0 \le y \le 2\}$ and the origin.

50. Average value Find the average squared distance between the points of $R = \{(x, y): 0 \le x \le 3, 0 \le y \le 3\}$ and the point $(3, 3)$.

51. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. The region of integration for $\int_4^6 \int_1^3 4\, dx\, dy$ is a square.

b. If f is continuous on $\mathbb{R}^2$, then $\int_4^6 \int_1^3 f(x, y)\, dx\, dy = \int_4^6 \int_1^3 f(x, y)\, dy\, dx$.

c. If f is continuous on $\mathbb{R}^2$, then $\int_4^6 \int_1^3 f(x, y)\, dx\, dy = \int_1^3 \int_4^6 f(x, y)\, dy\, dx$.

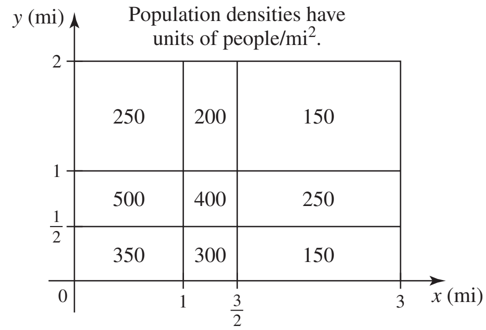
52. Symmetry Evaluate the following integrals using symmetry arguments. Let $R = \{(x, y): -a \le x \le a, -b \le y \le b\}$, where a and b are positive real numbers.

a. $\displaystyle\iint_R xye^{-(x^2+y^2)}\, dA$ **b.** $\displaystyle\iint_R \frac{\sin(x - y)}{x^2 + y^2 + 1}\, dA$

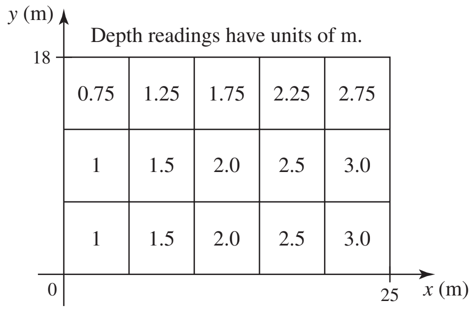
53. Computing populations The population densities in nine districts of a rectangular county are shown in the figure.

a. Use the fact that population $=$ (population density) $\times$ (area) to estimate the population of the county.

b. Explain how the calculation of part (a) is related to Riemann sums and double integrals.



T 54. Approximating water volume The varying depth of an 18 m $\times$ 25 m swimming pool is measured in 15 different rectangles of equal area (see figure). Approximate the volume of water in the pool.



Explorations and Challenges

55. Cylinders Let S be the solid in $\mathbb{R}^3$ between the cylinder $z = f(x)$ and the region $R = \{(x, y): a \le x \le b, c \le y \le d\}$, where $f(x) \ge 0$ on R. Explain why $\int_c^d \int_a^b f(x)\, dx\, dy$ equals the area of the constant cross section of S multiplied by $(d - c)$, which is the volume of S.

56. Product of integrals Suppose $f(x, y) = g(x)h(y)$, where g and h are continuous functions for all real values of x and y.

a. Show that $\int_c^d \int_a^b f(x, y)\, dx\, dy = \left(\int_a^b g(x)\, dx\right)\left(\int_c^d h(y)\, dy\right)$. Interpret this result geometrically.

b. Write $\left(\int_a^b g(x)\, dx\right)^2$ as an iterated integral.

c. Use the result of part (a) to evaluate $\int_0^{2\pi} \int_{10}^{30} e^{-4y^2} \cos x\, dy\, dx$.

57. Solving for a parameter Let $R = \{(x, y): 0 \le x \le \pi, 0 \le y \le a\}$. For what values of a, with $0 \le a \le \pi$, is $\iint_R \sin(x + y)\, dA$ equal to 1?

58–59. Zero average value *Find the value of $a > 0$ such that the average value of the following functions over $R = \{(x, y): 0 \le x \le a, 0 \le y \le a\}$ is zero.*

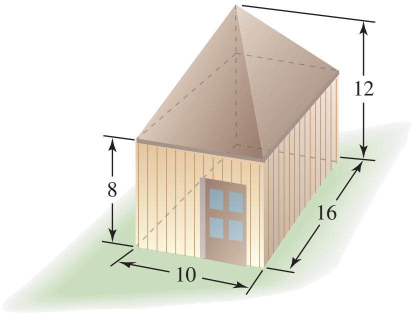
58. $f(x, y) = x + y - 8$ **59.** $f(x, y) = 4 - x^2 - y^2$

60. **Maximum integral** Consider the plane $x + 3y + z = 6$ over the rectangle R with vertices at $(0, 0)$, $(a, 0)$, $(0, b)$, and (a, b), where the vertex (a, b) lies on the line where the plane intersects the xy-plane (so $a + 3b = 6$). Find the point (a, b) for which the volume of the solid between the plane and R is a maximum.

61. **Density and mass** Suppose a thin rectangular plate, represented by a region R in the xy-plane, has a density given by the function $\rho(x, y)$; this function gives the *area density* in units such as grams per square centimeter $(\mathrm{g/cm^2})$. The mass of the plate is $\iint_R \rho(x, y)\, dA$. Assume $R = \{(x, y): 0 \le x \le \pi/2, 0 \le y \le \pi\}$ and find the mass of the plates with the following density functions.

a. $\rho(x, y) = 1 + \sin x$ **b.** $\rho(x, y) = 1 + \sin y$
c. $\rho(x, y) = 1 + \sin x \sin y$

62. **Approximating volume** Propose a method based on Riemann sums to approximate the volume of the shed shown in the figure (the peak of the roof is directly above the rear corner of the shed). Carry out the method and provide an estimate of the volume.



63. **An identity** Suppose the second partial derivatives of f are continuous on $R = \{(x, y): 0 \le x \le a, 0 \le y \le b\}$. Simplify
$$\iint_R \frac{\partial^2 f}{\partial x\, \partial y}\, dA.$$

QUICK CHECK ANSWERS

1. The sum gives the volume of a collection of rectangular boxes, and these boxes do not exactly fill the solid region under the surface. The approximation is improved by using more boxes. **2.** Inner integral: x runs from $x = 1$ to $x = 2$; outer integral: y runs from $y = 3$ to $y = 4$. The region is the rectangle $\{(x, y): 1 \le x \le 2, 3 \le y \le 4\}$.
3. $\int_0^{20} \int_{-10}^{10} (x^2 y + 2xy^3)\, dx\, dy$ ◄

16.2 Double Integrals over General Regions

Evaluating double integrals over rectangular regions is a useful place to begin our study of multiple integrals. Problems of practical interest, however, usually involve nonrectangular regions of integration. The goal of this section is to extend the methods presented in Section 16.1 so that they apply to more general regions of integration.

General Regions of Integration

Consider a function f defined over a closed, bounded *nonrectangular* region R in the xy-plane. As with rectangular regions, we use a partition consisting of rectangles, but now, such a partition does not cover R exactly. In this case, only the n rectangles that lie entirely within R are considered to be in the partition (Figure 16.9). When f is nonnegative on R, the volume of the solid bounded by the surface $z = f(x, y)$ and the xy-plane over R is approximated by the Riemann sum

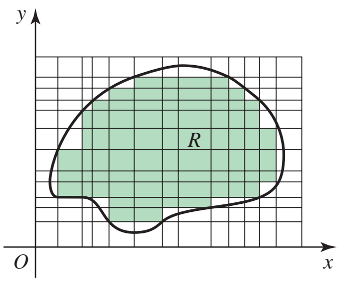


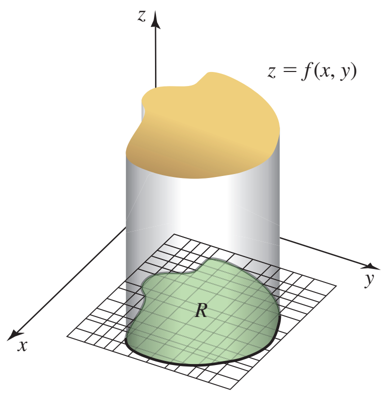
Figure 16.9

$$V \approx \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k,$$

where $\Delta A_k = \Delta x_k \Delta y_k$ is the area of the kth rectangle and (x_k^*, y_k^*) is any point in the kth rectangle, for $1 \le k \le n$. As before, we define Δ to be the maximum length of the diagonals of the rectangles in the partition.

Under the assumptions that f is continuous on R and that the boundary of R consists of a finite number of smooth curves, two things occur as $\Delta \to 0$ and the number of rectangles increases $(n \to \infty)$.

- The rectangles in the partition fill R more and more completely; that is, the union of the rectangles approaches R.
- Over all partitions and all choices of (x_k^*, y_k^*) within a partition, the Riemann sums approach a (unique) limit.



Volume of solid $= \iint_R f(x, y)\, dA$

$= \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*)\, \Delta A_k$

Figure 16.10

The limit approached by the Riemann sums is the **double integral of f over R**; that is,

$$\iint_R f(x, y)\, dA = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k.$$

When this limit exists, f is **integrable** over R. If f is nonnegative on R, then the double integral equals the volume of the solid bounded by the surface $z = f(x, y)$ and the xy-plane over R (Figure 16.10).

The double integral $\iint_R f(x, y)\, dA$ has another common interpretation. Suppose R represents a thin plate whose density at the point (x, y) is $f(x, y)$. The units of density are mass per unit area, so the product $f(x_k^*, y_k^*)\Delta A_k$ approximates the mass of the kth rectangle in R. Summing the masses of the rectangles gives an approximation to the total mass of R. In the limit as $n \to \infty$ and $\Delta \to 0$, the double integral equals the mass of the plate.

Iterated Integrals

Double integrals over nonrectangular regions are also evaluated using iterated integrals. However, in this more general setting, the order of integration is critical. Most of the double integrals we encounter fall into one of two categories determined by the shape of the region R.

The first type of region has the property that its lower and upper boundaries are the graphs of continuous functions $y = g(x)$ and $y = h(x)$, respectively, for $a \le x \le b$. Such regions have any of the forms shown in Figure 16.11.

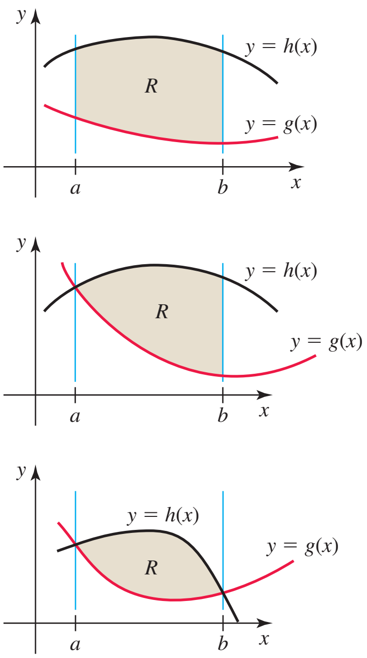


Figure 16.11

Once again, we appeal to the general slicing method. Assume for the moment that f is nonnegative on R and consider the solid bounded by the surface $z = f(x, y)$ and R (Figure 16.12). Imagine taking vertical slices through the solid parallel to the yz-plane. The cross section through the solid at a fixed value of x extends from the lower curve $y = g(x)$ to the upper curve $y = h(x)$. The area of that cross section is

$$A(x) = \int_{g(x)}^{h(x)} f(x, y)\, dy, \qquad \text{for } a \le x \le b.$$

The volume of the solid is given by a double integral; it is evaluated by integrating the cross-sectional areas $A(x)$ from $x = a$ to $x = b$:

$$\iint_R f(x, y)\, dA = \int_a^b \underbrace{\int_{g(x)}^{h(x)} f(x, y)\, dy}_{A(x)}\, dx.$$

The limits of integration in the iterated integral describe the boundaries of the region of integration R.

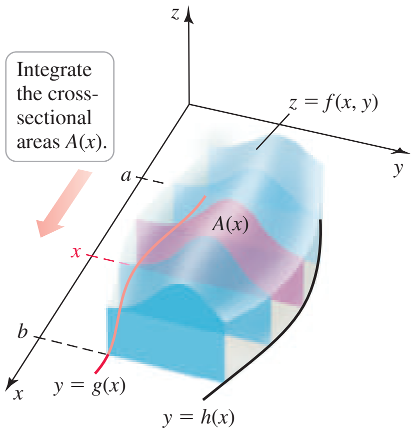


Figure 16.12

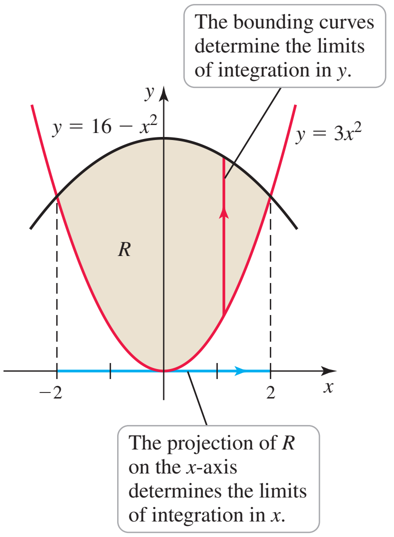


Figure 16.13

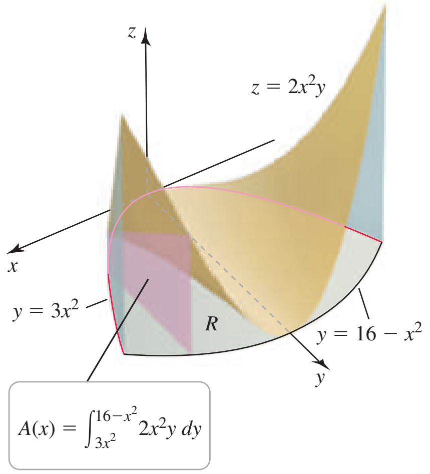


Figure 16.14

QUICK CHECK 1 A region R is bounded by the x- and y-axes and the line $x + y = 2$. Suppose you integrate first with respect to y. Give the limits of the iterated integral over R. ◀

EXAMPLE 1 **Evaluating a double integral** Express the integral $\iint_R 2x^2y\,dA$ as an iterated integral, where R is the region bounded by the parabolas $y = 3x^2$ and $y = 16 - x^2$. Then evaluate the integral.

SOLUTION The region R is bounded below and above by the graphs of $g(x) = 3x^2$ and $h(x) = 16 - x^2$, respectively. Solving $3x^2 = 16 - x^2$, we find that these curves intersect at $x = -2$ and $x = 2$, which are the limits of integration in the x-direction (Figure 16.13).

Figure 16.14 shows the solid bounded by the surface $z = 2x^2y$ and the region R. A typical vertical cross section through the solid parallel to the yz-plane at a fixed value of x has area

$$A(x) = \int_{3x^2}^{16-x^2} 2x^2y\,dy.$$

Integrating these cross-sectional areas between $x = -2$ and $x = 2$, the iterated integral becomes

$$\iint_R 2x^2y\,dA = \int_{-2}^{2} \underbrace{\int_{3x^2}^{16-x^2} 2x^2y\,dy}_{A(x)}\,dx \qquad \text{Convert to an iterated integral.}$$

$$= \int_{-2}^{2} x^2y^2\Big|_{3x^2}^{16-x^2} dx \qquad \text{Evaluate inner integral with respect to } y.$$

$$= \int_{-2}^{2} x^2\left((16 - x^2)^2 - (3x^2)^2\right) dx \qquad \text{Simplify.}$$

$$= \int_{-2}^{2} (-8x^6 - 32x^4 + 256x^2)\,dx \qquad \text{Simplify.}$$

$$\approx 663.2. \qquad \text{Evaluate outer integral with respect to } x.$$

Because $z = 2x^2y \geq 0$ on R, the value of the integral is the volume of the solid shown in Figure 16.14.

Related Exercises 12, 46 ◀

Change of Perspective Suppose the region of integration R is bounded on the left and right by the graphs of continuous functions $x = g(y)$ and $x = h(y)$, respectively, on the interval $c \leq y \leq d$. Such regions may take any of the forms shown in Figure 16.15.

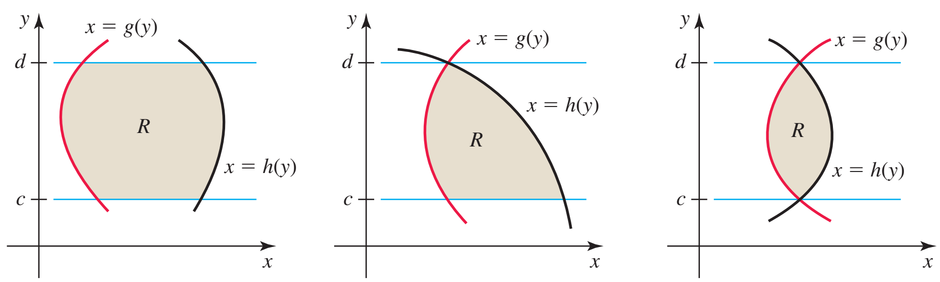


Figure 16.15

To find the volume of the solid bounded by the surface $z = f(x, y)$ and R, we now take vertical slices parallel to the xz-plane. The double integral $\iint_R f(x, y)\,dA$ is then converted to an iterated integral in which the inner integration is with respect to x over the interval $g(y) \leq x \leq h(y)$ and the outer integration is with respect to y over the interval $c \leq y \leq d$. The evaluation of double integrals in these two cases is summarized in the following theorem.

➤ Theorem 16.2 is another version of Fubini's Theorem. With integrals over nonrectangular regions, the order of integration cannot be simply switched; that is,

$$\int_a^b \int_{g(x)}^{h(x)} f(x, y)\, dy\, dx \neq \int_{g(x)}^{h(x)} \int_a^b f(x, y)\, dx\, dy.$$

The *element of area dA* corresponds to the area of a small rectangle in the partition. Comparing the double integral to the iterated integral, we see that the element of area is $dA = dy\,dx$ or $dA = dx\,dy$, which is consistent with the area formula for rectangles.

THEOREM 16.2 Double Integrals over Nonrectangular Regions

Let R be a region bounded below and above by the graphs of the continuous functions $y = g(x)$ and $y = h(x)$, respectively, and by the lines $x = a$ and $x = b$ (Figure 16.11). If f is continuous on R, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_{g(x)}^{h(x)} f(x, y)\, dy\, dx.$$

Let R be a region bounded on the left and right by the graphs of the continuous functions $x = g(y)$ and $x = h(y)$, respectively, and the lines $y = c$ and $y = d$ (Figure 16.15). If f is continuous on R, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_{g(y)}^{h(y)} f(x, y)\, dx\, dy.$$

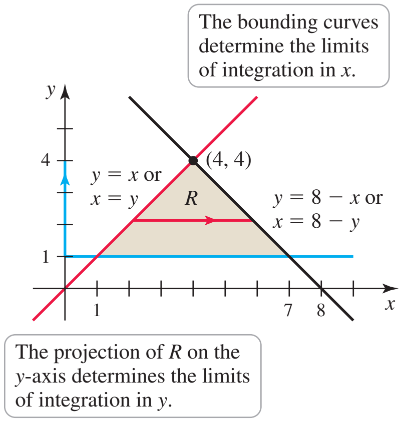


Figure 16.16

EXAMPLE 2 Computing a volume Find the volume of the solid below the surface $f(x, y) = 2 + \dfrac{1}{y}$ and above the region R in the xy-plane bounded by the lines $y = x$, $y = 8 - x$, and $y = 1$. Notice that $f(x, y) > 0$ on R.

SOLUTION The region R is bounded on the left by $x = y$ and bounded on the right by $y = 8 - x$, or $x = 8 - y$ (Figure 16.16). These lines intersect at the point $(4, 4)$. We take vertical slices through the solid parallel to the xz-plane from $y = 1$ to $y = 4$. To visualize these slices, it helps to draw lines through R parallel to the x-axis.

Integrating the cross-sectional areas of slices from $y = 1$ to $y = 4$, the volume of the solid beneath the graph of f and above R (Figure 16.17) is given by

$$\begin{aligned}
\iint_R \left(2 + \frac{1}{y}\right) dA &= \int_1^4 \int_y^{8-y} \left(2 + \frac{1}{y}\right) dx\, dy && \text{Convert to an iterated integral.}\\
&= \int_1^4 \left(2 + \frac{1}{y}\right) x \Big|_y^{8-y}\, dy && \text{Evaluate inner integral with respect to } x.\\
&= \int_1^4 \left(2 + \frac{1}{y}\right)(8 - 2y)\, dy && \text{Simplify.}\\
&= \int_1^4 \left(14 - 4y + \frac{8}{y}\right) dy && \text{Simplify.}\\
&= \left(14y - 2y^2 + 8 \ln |y|\right)\Big|_1^4 && \text{Evaluate outer integral with respect to } y.\\
&= 12 + 8 \ln 4 \approx 23.09. && \text{Simplify.}
\end{aligned}$$

Related Exercise 74 ◄

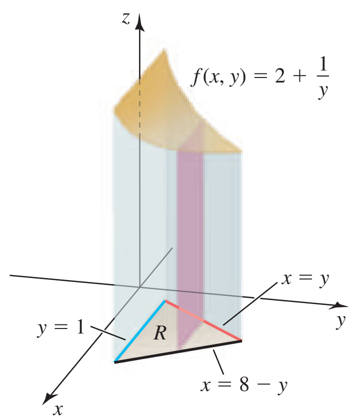


Figure 16.17

QUICK CHECK 2 Could the integral in Example 2 be evaluated by integrating first (inner integral) with respect to y? ◄

Choosing and Changing the Order of Integration

Occasionally, a region of integration is bounded above and below by a pair of curves *and* the region is bounded on the right and left by a pair of curves. For example, the region R in Figure 16.18 is bounded above by $y = x^{1/3}$ and below by $y = x^2$, and it is bounded on the right by $x = \sqrt{y}$ and on the left by $x = y^3$. In these cases, we can choose either of two orders of integration; however, one order of integration may be preferable. The following examples illustrate the valuable techniques of choosing and changing the order of integration.

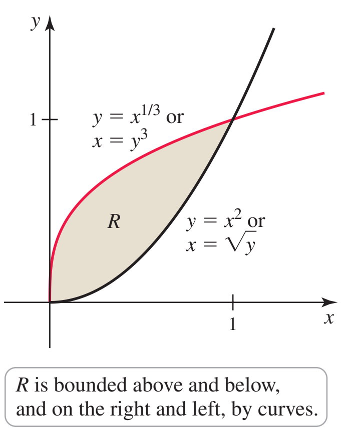


Figure 16.18

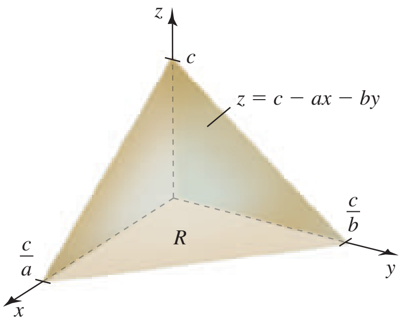


Figure 16.19

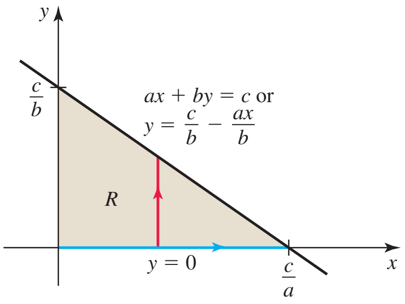


Figure 16.20

➤ In Example 3, it is just as easy to view R as being bounded on the left and the right by the lines $x = 0$ and $x = c/a - by/a$, respectively, and integrating first with respect to x.

EXAMPLE 3 **Volume of a tetrahedron** Find the volume of the tetrahedron (pyramid with four triangular faces) in the first octant bounded by the plane $z = c - ax - by$ and the coordinate planes ($x = 0$, $y = 0$, and $z = 0$). Assume a, b, and c are positive real numbers (**Figure 16.19**).

SOLUTION Let R be the triangular base of the tetrahedron in the xy-plane; it is bounded by the x- and y-axes and the line $ax + by = c$ (found by setting $z = 0$ in the equation of the plane; **Figure 16.20**). We can view R as being bounded below and above by the lines $y = 0$ and $y = c/b - ax/b$, respectively. The boundaries on the left and right are then $x = 0$ and $x = c/a$, respectively. Therefore, the volume of the solid region between the plane and R is

$$\iint_R (c - ax - by)\,dA = \int_0^{c/a}\int_0^{c/b-ax/b}(c - ax - by)\,dy\,dx \qquad \text{Convert to an iterated integral.}$$

$$= \int_0^{c/a}\left(cy - axy - \frac{by^2}{2}\right)\Big|_0^{c/b-ax/b}dx \qquad \text{Evaluate inner integral with respect to } y.$$

$$= \int_0^{c/a}\frac{(ax - c)^2}{2b}\,dx \qquad \text{Simplify and factor.}$$

$$= \frac{c^3}{6ab}. \qquad \text{Evaluate outer integral with respect to } x.$$

This result illustrates the volume formula for a tetrahedron. The lengths of the legs of the triangular base are c/a and c/b, which means the area of the base is $c^2/(2ab)$. The height of the tetrahedron is c. The general volume formula is

$$V = \frac{c^3}{6ab} = \frac{1}{3}\underbrace{\frac{c^2}{2ab}}_{\text{area of base}} \cdot \underbrace{c}_{\text{height}} = \frac{1}{3}(\text{area of base})(\text{height}).$$

Related Exercise 73 ◀

EXAMPLE 4 **Changing the order of integration** Consider the iterated integral $\int_0^{\sqrt{\pi}}\int_y^{\sqrt{\pi}}\sin x^2\,dx\,dy$. Sketch the region of integration determined by the limits of integration and then evaluate the iterated integral.

SOLUTION The region of integration is $R = \{(x, y): y \le x \le \sqrt{\pi}, 0 \le y \le \sqrt{\pi}\}$, which is a triangle (**Figure 16.21a**). Evaluating the iterated integral as given (integrating first with respect to x) requires integrating $\sin x^2$, a function whose antiderivative is not expressible in terms of elementary functions. Therefore, this order of integration is not feasible.

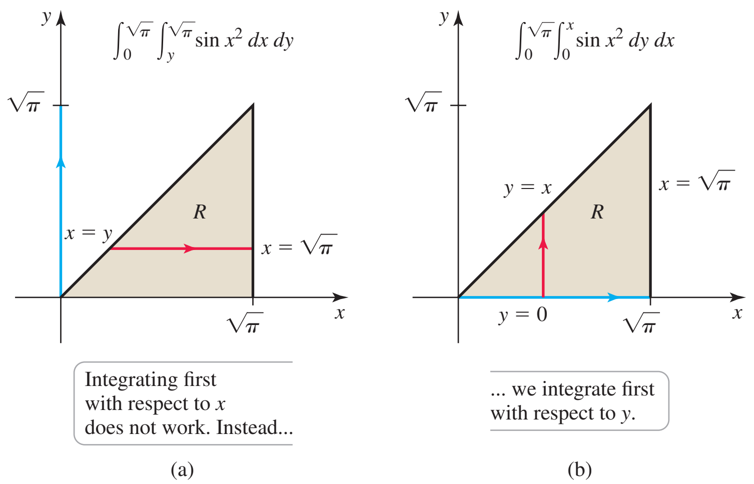


Figure 16.21

Instead, we change our perspective (**Figure 16.21b**) and integrate first with respect to y. With this order of integration, y runs from $y = 0$ to $y = x$ in the inner integral, and x runs from $x = 0$ to $x = \sqrt{\pi}$ in the outer integral:

$$\begin{aligned}
\iint_R \sin x^2 \, dA &= \int_0^{\sqrt{\pi}} \int_0^x \sin x^2 \, dy \, dx \\
&= \int_0^{\sqrt{\pi}} y \sin x^2 \Big|_0^x \, dx && \text{Evaluate inner integral with respect to } y\text{; } \sin x^2 \text{ is constant.} \\
&= \int_0^{\sqrt{\pi}} x \sin x^2 \, dx && \text{Simplify.} \\
&= -\frac{1}{2} \cos x^2 \Big|_0^{\sqrt{\pi}} && \text{Evaluate outer integral with respect to } x. \\
&= 1. && \text{Simplify.}
\end{aligned}$$

This example shows that the order of integration can make a practical difference.

Related Exercises 58, 64 ◀

QUICK CHECK 3 Change the order of integration of the integral $\int_0^1 \int_0^y f(x, y) \, dx \, dy$. ◀

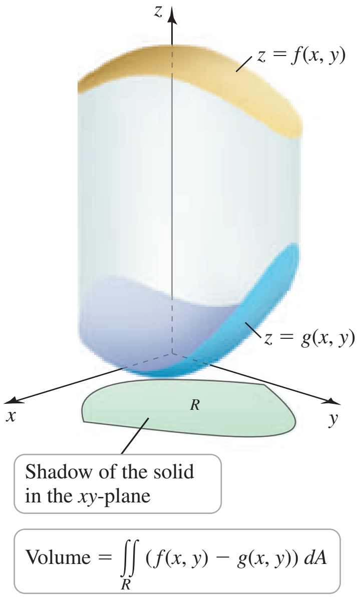


Figure 16.22

Regions Between Two Surfaces

An extension of the preceding ideas allows us to solve more general volume problems. Let $z = f(x, y)$ and $z = g(x, y)$ be continuous functions with $f(x, y) \geq g(x, y)$ on a region R in the xy-plane. Suppose we wish to compute the volume of the solid between the two surfaces over the region R (**Figure 16.22**). Forming a Riemann sum for the volume, the height of a typical box within the solid is the vertical distance $f(x, y) - g(x, y)$ between the upper and lower surfaces. Therefore, the volume of the solid between the surfaces is

$$V = \iint_R (f(x, y) - g(x, y)) \, dA.$$

EXAMPLE 5 **Region bounded by two surfaces** Find the volume of the solid bounded by the parabolic cylinder $z = 1 + x^2$ and the planes $z = 5 - y$ and $y = 0$ (**Figure 16.23**).

SOLUTION The upper surface bounding the solid is $z = 5 - y$ and the lower surface is $z = 1 + x^2$; these two surfaces intersect along a curve C. Solving $5 - y = 1 + x^2$, we find that $y = 4 - x^2$, which is the projection of C onto the xy-plane. The back wall of the solid is the plane $y = 0$, and its projection onto the xy-plane is the x-axis. This line $(y = 0)$ intersects the parabola $y = 4 - x^2$ at $x = \pm 2$. Therefore, the region of integration (Figure 16.23) is

$$R = \{(x, y): 0 \leq y \leq 4 - x^2, -2 \leq x \leq 2\}.$$

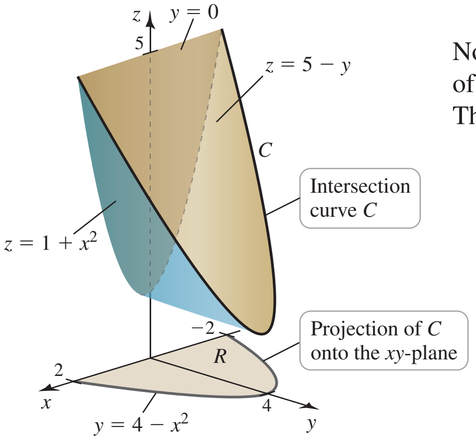


Figure 16.23

➤ To use symmetry to simplify a double integral, you must check that both the region of integration and the integrand have the same symmetry.

Notice that both R and the solid are symmetric about the yz-plane. Therefore, the volume of the entire solid is twice the volume of that part of the solid that lies in the first octant. The volume of the solid is

$$\begin{aligned}
&2 \int_0^2 \int_0^{4-x^2} (\underbrace{(5 - y)}_{f(x, y)} - \underbrace{(1 + x^2)}_{g(x, y)}) \, dy \, dx \\
&= 2 \int_0^2 \int_0^{4-x^2} (4 - x^2 - y) \, dy \, dx && \text{Simplify the integrand.} \\
&= 2 \int_0^2 \left((4 - x^2)y - \frac{y^2}{2} \right) \Big|_0^{4-x^2} dx && \text{Evaluate inner integral with respect to } y. \\
&= \int_0^2 (x^4 - 8x^2 + 16) \, dx && \text{Simplify.} \\
&= \left(\frac{x^5}{5} - \frac{8x^3}{3} + 16x \right) \Big|_0^2 && \text{Evaluate outer integral with respect to } x. \\
&= \frac{256}{15}. && \text{Simplify.}
\end{aligned}$$

Related Exercises 78–79 ◀

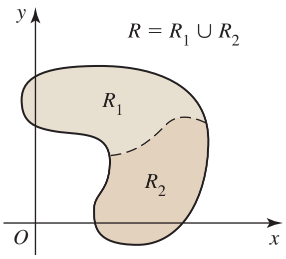
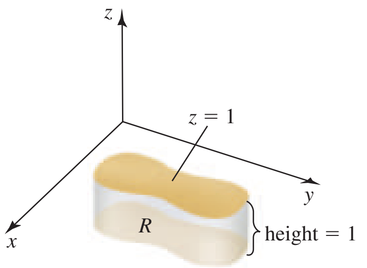


Figure 16.24



Volume of solid = (area of R) × (height)
= area of $R = \iint_R 1\, dA$

Figure 16.25

➤ We are solving a familiar area problem first encountered in Section 6.2. Suppose R is bounded above by $y = h(x)$ and below by $y = g(x)$, for $a \le x \le b$. Using a double integral, the area of R is

$$\iint_R dA = \int_a^b \int_{g(x)}^{h(x)} dy\, dx = \int_a^b (h(x) - g(x))\, dx,$$

which is a result obtained in Section 6.2.

Decomposition of Regions

We occasionally encounter regions that are more complicated than those considered so far. A technique called *decomposition* allows us to subdivide a region of integration into two (or more) subregions. If the integrals over the subregions can be evaluated separately, the results are added to obtain the value of the original integral. For example, the region R in Figure 16.24 is divided into two nonoverlapping subregions R_1 and R_2. By partitioning these regions and using Riemann sums, it can be shown that

$$\iint_R f(x, y)\, dA = \iint_{R_1} f(x, y)\, dA + \iint_{R_2} f(x, y)\, dA.$$

This method is illustrated in Example 6. The analog of decomposition with single variable integrals is the property $\int_a^b f(x)\, dx = \int_a^p f(x)\, dx + \int_p^b f(x)\, dx$.

Finding Area by Double Integrals

An interesting application of double integrals arises when the integrand is $f(x, y) = 1$. The integral $\iint_R 1\, dA$ gives the volume of the solid between the horizontal plane $z = 1$ and the region R. Because the height of this solid is 1, its volume equals (numerically) the area of R (Figure 16.25). Therefore, we have a way to compute areas of regions in the xy-plane using double integrals.

> **Areas of Regions by Double Integrals**
>
> Let R be a region in the xy-plane. Then
>
> $$\text{area of } R = \iint_R dA.$$

EXAMPLE 6 **Area of a plane region** Find the area of the region R bounded by $y = x^2$, $y = -x + 12$, and $y = 4x + 12$ (Figure 16.26).

SOLUTION The region R in its entirety is bounded neither above and below by two curves, nor on the left and right by two curves. However, when decomposed along the y-axis, R may be viewed as two regions R_1 and R_2, each of which is bounded above and below by a pair of curves. Notice that the parabola $y = x^2$ and the line $y = -x + 12$ intersect in the first quadrant at the point $(3, 9)$, while the parabola and the line $y = 4x + 12$ intersect in the second quadrant at the point $(-2, 4)$.

To find the area of R, we integrate the function $f(x, y) = 1$ over R_1 and R_2; the area is

$$\iint_{R_1} 1\, dA + \iint_{R_2} 1\, dA \qquad \text{Decompose region.}$$

$$= \int_{-2}^{0} \int_{x^2}^{4x+12} 1\, dy\, dx + \int_0^3 \int_{x^2}^{-x+12} 1\, dy\, dx \qquad \text{Convert to iterated integrals.}$$

$$= \int_{-2}^{0} (4x + 12 - x^2)\, dx + \int_0^3 (-x + 12 - x^2)\, dx \qquad \text{Evaluate inner integrals with respect to } y.$$

$$= \left(2x^2 + 12x - \frac{x^3}{3}\right)\Bigg|_{-2}^{0} + \left(-\frac{x^2}{2} + 12x - \frac{x^3}{3}\right)\Bigg|_0^3 \qquad \text{Evaluate outer integrals with respect to } x.$$

$$= \frac{40}{3} + \frac{45}{2} = \frac{215}{6}. \qquad \text{Simplify.}$$

Related Exercise 86 ◀

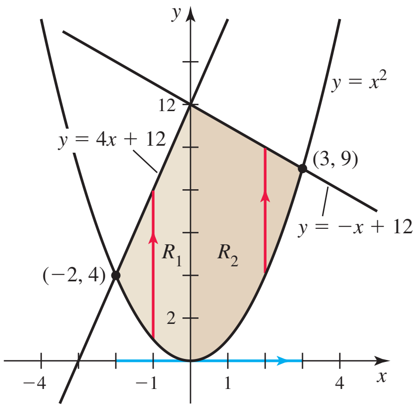


Figure 16.26

QUICK CHECK 4 Consider the triangle R with vertices $(-1, 0)$, $(1, 0)$, and $(0, 1)$ as a region of integration. If we integrate first with respect to x, does R need to be decomposed? If we integrate first with respect to y, does R need to be decomposed? ◀

SECTION 16.2 EXERCISES

Getting Started

1. Describe and sketch a region that is bounded above and below by two curves.

2. Describe and sketch a region that is bounded on the left and on the right by two curves.

3. Which order of integration is preferable to integrate $f(x, y) = xy$ over $R = \{(x, y): y - 1 \le x \le 1 - y, 0 \le y \le 1\}$?

4. Which order of integration would you use to find the area of the region bounded by the x-axis and the lines $y = 2x + 3$ and $y = 3x - 4$ using a double integral?

5. Change the order of integration in the integral $\int_0^1 \int_{y^2}^{\sqrt{y}} f(x, y)\, dx\, dy$.

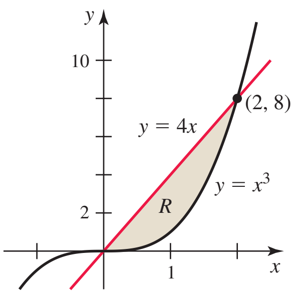
6. Sketch the region of integration for $\int_{-2}^{2} \int_{x^2}^{4} e^{xy}\, dy\, dx$.

7. Sketch the region of integration for $\int_0^2 \int_0^{2x} dy\, dx$ and use geometry to evaluate the iterated integral.

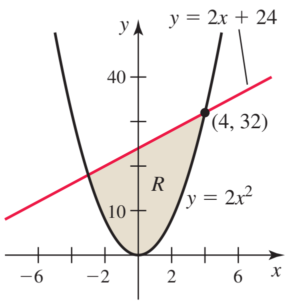
8. Describe a solid whose volume equals $\int_{-4}^{4} \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} 10\, dy\, dx$ and evaluate this iterated integral using geometry.

9–10. *Consider the region R shown in the figure and write an iterated integral of a continuous function f over R.*

9.



10.



Practice Exercises

11–27. Evaluating integrals *Evaluate the following integrals.*

11. $\int_0^1 \int_x^1 6y\, dy\, dx$

12. $\int_0^1 \int_0^{2x} 15xy^2\, dy\, dx$

13. $\int_0^2 \int_{x^2}^{2x} xy\, dy\, dx$

14. $\int_{-\pi/4}^{\pi/4} \int_{\sin x}^{\cos x} dy\, dx$

15. $\int_{-2}^{2} \int_{x^2}^{8-x^2} x\, dy\, dx$

16. $\int_0^{\ln 2} \int_{e^x}^{2} dy\, dx$

17. $\int_0^1 \int_0^x 2e^{x^2} dy\, dx$

18. $\int_0^{\sqrt[3]{\pi/2}} \int_0^x y \cos x^3\, dy\, dx$

19. $\int_0^{\ln 2} \int_{e^y}^{2} \frac{y}{x}\, dx\, dy$

20. $\int_0^4 \int_y^{2y} xy\, dx\, dy$

21. $\int_0^{\pi/2} \int_y^{\pi/2} 6 \sin(2x - 3y)\, dx\, dy$

22. $\int_0^{\pi/2} \int_0^{\cos y} e^{\sin y}\, dx\, dy$

23. $\int_0^{\pi/2} \int_0^{y \cos y} dx\, dy$

24. $\int_0^1 \int_{\tan^{-1} x}^{\pi/4} 2x\, dy\, dx$

25. $\int_0^4 \int_{-\sqrt{16-y^2}}^{\sqrt{16-y^2}} 2xy\, dx\, dy$

26. $\int_0^1 \int_0^x 2e^x\, dy\, dx$

27. $\int_{\pi/2}^{\pi} \int_0^{y^2} \cos \frac{x}{y}\, dx\, dy$

28–34. Regions of integration *Sketch each region R and write an iterated integral of a continuous function f over R. Use the order dy dx.*

28. $R = \{(x, y): 0 \le x \le 2, 3x^2 \le y \le -6x + 24\}$

29. $R = \{(x, y): 1 \le x \le 2, x + 1 \le y \le 2x + 4\}$

30. $R = \{(x, y): 0 \le x \le 4, x^2 \le y \le 8\sqrt{x}\}$

31. R is the triangular region with vertices $(0, 0)$, $(0, 2)$, and $(1, 0)$.

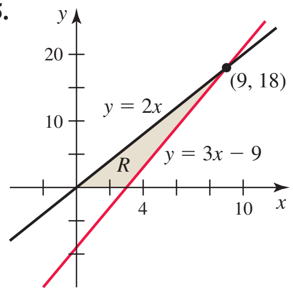
32. R is the triangular region with vertices $(0, 0)$, $(0, 2)$, and $(1, 1)$.

33. R is the region in the first quadrant bounded by a circle of radius 1 centered at the origin.

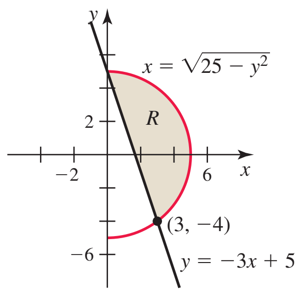
34. R is the region in the first quadrant bounded by the y-axis and the parabolas $y = x^2$ and $y = 1 - x^2$.

35–42. Regions of integration *Write an iterated integral of a continuous function f over the region R. Use the order dy dx. Start by sketching the region of integration if it is not supplied.*

35.



36.



37. R is the region bounded by $y = 4 - x$, $y = 1$, and $x = 0$.

38. $R = \{(x, y): 0 \le x \le y(1 - y)\}$.

39. R is the region bounded by $y = 2x + 3$, $y = 3x - 7$, and $y = 0$.

40. R is the region in quadrants 2 and 3 bounded by the semicircle with radius 3 centered at $(0, 0)$.

41. R is the region bounded by the triangle with vertices $(0, 0)$, $(2, 0)$, and $(1, 1)$.

42. R is the region in the first quadrant bounded by the x-axis, the line $x = 6 - y$, and the curve $y = \sqrt{x}$.

43–56. Evaluating integrals *Evaluate the following integrals. A sketch is helpful.*

43. $\iint_R xy\, dA$; R is bounded by $x = 0$, $y = 2x + 1$, and $y = -2x + 5$.

44. $\iint_R (x + y)\, dA$; R is the region in the first quadrant bounded by $x = 0$, $y = x^2$, and $y = 8 - x^2$.

45. $\iint_R y^2\, dA$; R is bounded by $x = 1$, $y = 2x + 2$, and $y = -x - 1$.

46. $\iint_R x^2 y\, dA$; R is the region in quadrants 1 and 4 bounded by the semicircle of radius 4 centered at $(0, 0)$.

47. $\iint_R 12y\, dA$; R is bounded by $y = 2 - x$, $y = \sqrt{x}$, and $y = 0$.

48. $\iint_R y^2\, dA$; R is bounded by $y = 1$, $y = 1 - x$, and $y = x - 1$.

49. $\iint_R 3xy\, dA$; R is the region in the first quadrant bounded by $y = 2 - x$, $y = 0$, and $x = 4 - y^2$.

50. $\iint_R (x + y)\, dA$; R is bounded by $y = |x|$ and $y = 4$.

51. $\iint_R 3x^2\, dA$; R is bounded by $y = 0$, $y = 2x + 4$, and $y = x^3$.

52. $\iint_R 8xy\, dA$; $R = \{(x, y): 0 \le y \le \sec x, 0 \le x \le \pi/4\}$

53. $\iint_R (x + y)\, dA$; R is the region bounded by $y = 1/x$ and $y = 5/2 - x$.

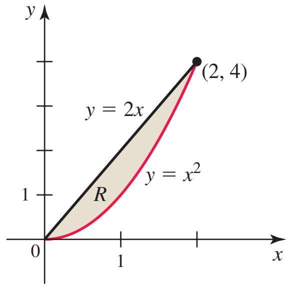
54. $\iint_R \dfrac{y}{1 + x + y^2}\, dA$; $R = \{(x, y): 0 \le \sqrt{x} \le y, 0 \le y \le 1\}$

55. $\iint_R x \sec^2 y\, dA$; $R = \{(x, y): 0 \le y \le x^2, 0 \le x \le \sqrt{\pi}/2\}$

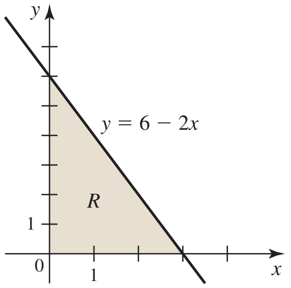
56. $\iint_R \dfrac{8xy}{1 + x^2 + y^2}\, dA$; $R = \{(x, y): 0 \le y \le x, 0 \le x \le 2\}$

57–62. Changing order of integration *Reverse the order of integration in the following integrals.*

57. $\displaystyle\int_0^2 \int_{x^2}^{2x} f(x, y)\, dy\, dx$



58. $\displaystyle\int_0^3 \int_0^{6-2x} f(x, y)\, dy\, dx$



59. $\displaystyle\int_{1/2}^1 \int_0^{-\ln y} f(x, y)\, dx\, dy$

60. $\displaystyle\int_0^1 \int_1^{e^y} f(x, y)\, dx\, dy$

61. $\displaystyle\int_0^1 \int_0^{\cos^{-1} y} f(x, y)\, dx\, dy$

62. $\displaystyle\int_1^e \int_0^{\ln x} f(x, y)\, dy\, dx$

63–68. Changing order of integration *Reverse the order of integration and evaluate the integral.*

63. $\displaystyle\int_0^1 \int_y^1 e^{x^2}\, dx\, dy$

64. $\displaystyle\int_0^\pi \int_x^\pi \sin y^2\, dy\, dx$

65. $\displaystyle\int_0^{1/2} \int_{y^2}^{1/4} y \cos(16\pi x^2)\, dx\, dy$

66. $\displaystyle\int_0^4 \int_{\sqrt{x}}^2 \frac{x}{y^5 + 1}\, dy\, dx$

67. $\displaystyle\int_0^{\sqrt[3]{\pi}} \int_y^{\sqrt[3]{\pi}} x^4 \cos(x^2 y)\, dx\, dy$

68. $\displaystyle\int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4 - y}\, dy\, dx$

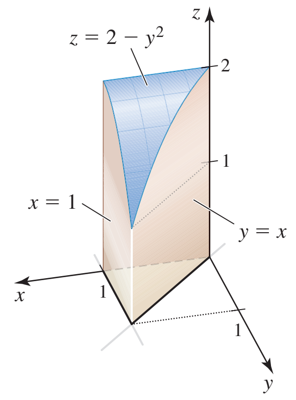
69–70. Two integrals to one *Draw the regions of integration and write the following integrals as a single iterated integral.*

69. $\displaystyle\int_0^1 \int_{e^y}^e f(x, y)\, dx\, dy + \int_{-1}^0 \int_{e^{-y}}^e f(x, y)\, dx\, dy$

70. $\displaystyle\int_{-4}^0 \int_0^{\sqrt{16-x^2}} f(x, y)\, dy\, dx + \int_0^4 \int_0^{4-x} f(x, y)\, dy\, dx$

71–80. Volumes *Find the volume of the following solids.*

71. The solid bounded by the cylinder $z = 2 - y^2$, the xy-plane, the xz-plane, and the planes $y = x$ and $x = 1$



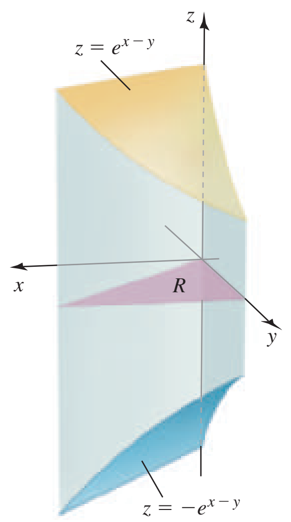
72. The solid bounded between the cylinder $z = 2\sin^2 x$ and the xy-plane over the region $R = \{(x, y): 0 \le x \le y \le \pi\}$

73. The tetrahedron bounded by the coordinate planes ($x = 0$, $y = 0$, and $z = 0$) and the plane $z = 8 - 2x - 4y$

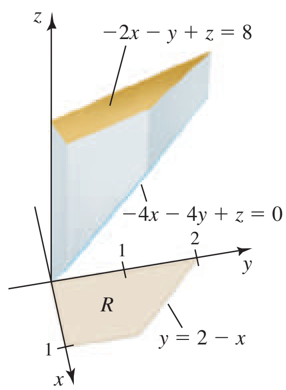
74. The solid in the first octant bounded by the coordinate planes and the surface $z = 1 - y - x^2$

75. The segment of the cylinder $x^2 + y^2 = 1$ bounded above by the plane $z = 12 + x + y$ and below by $z = 0$

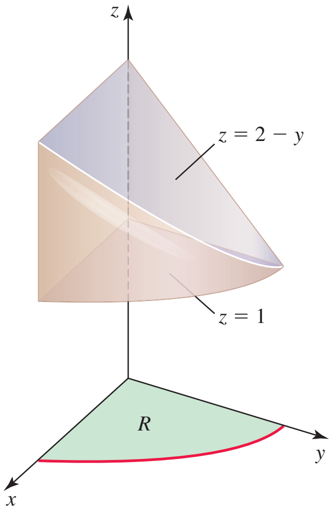
76. The solid S between the surfaces $z = e^{x-y}$ and $z = -e^{x-y}$, where S intersects the xy-plane in the region $R = \{(x, y): 0 \le x \le y, 0 \le y \le 1\}$



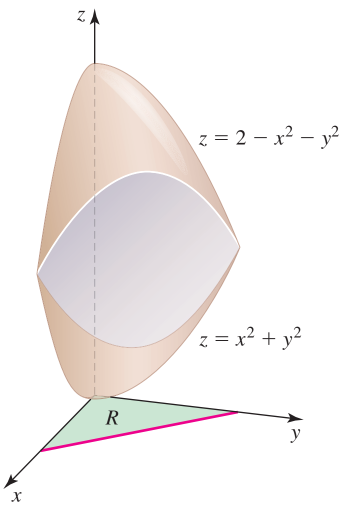
77. The solid above the region $R = \{(x, y): 0 \le x \le 1, 0 \le y \le 2 - x\}$ and between the planes $-4x - 4y + z = 0$ and $-2x - y + z = 8$



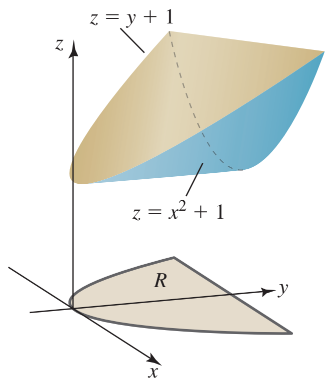
78. The solid in the first octant bounded by the planes $x = 0$, $y = 0$, $z = 1$, and $z = 2 - y$, and the cylinder $y = 1 - x^2$



79. The solid above the region $R = \{(x, y): 0 \le x \le 1,\ 0 \le y \le 1 - x\}$ bounded by the paraboloids $z = x^2 + y^2$ and $z = 2 - x^2 - y^2$ and the coordinate planes in the first octant



80. The solid bounded by the parabolic cylinder $z = x^2 + 1$, and the planes $z = y + 1$ and $y = 1$



T 81–84. Volume using technology *Find the volume of the following solids. Use a computer algebra system to evaluate an appropriate iterated integral.*

81. The column with a square base $R = \{(x, y): |x| \le 1, |y| \le 1\}$ cut by the plane $z = 4 - x - y$

82. The solid between the paraboloid $z = x^2 + y^2$ and the plane $z = 1 - 2y$

83. The wedge sliced from the cylinder $x^2 + y^2 = 1$ by the planes $z = a(2 - x)$ and $z = a(x - 2)$, where $a > 0$

84. The solid bounded by the elliptical cylinder $x^2 + 3y^2 = 12$, the plane $z = 0$, and the paraboloid $z = 3x^2 + y^2 + 1$

85–90. Area of plane regions *Use double integrals to compute the area of the following regions.*

85. The region bounded by the parabola $y = x^2$ and the line $y = 4$

86. The region bounded by the parabola $y = x^2$ and the line $y = x + 2$

87. The region in the first quadrant bounded by $y = e^x$ and $x = \ln 2$

88. The region bounded by $y = 1 + \sin x$ and $y = 1 - \sin x$ on the interval $[0, \pi]$

89. The region in the first quadrant bounded by $y = x^2$, $y = 5x + 6$, and $y = 6 - x$

90. The region bounded by the lines $x = 0$, $x = 4$, $y = x$, and $y = 2x + 1$

91. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. In the iterated integral $\int_c^d \int_a^b f(x, y)\,dx\,dy$, the limits a and b must be constants or functions of x.

b. In the iterated integral $\int_c^d \int_a^b f(x, y)\,dx\,dy$, the limits c and d must be functions of y.

c. Changing the order of integration gives $\int_0^2 \int_1^y f(x, y)\,dx\,dy = \int_1^y \int_0^2 f(x, y)\,dy\,dx$.

Explorations and Challenges

92. Related integrals Evaluate each integral.

a. $\displaystyle\int_0^4 \int_0^4 (4 - x - y)\,dx\,dy$ **b.** $\displaystyle\int_0^4 \int_0^4 |4 - x - y|\,dx\,dy$

93. Sliced block Find the volume of the solid bounded by the planes $x = 0$, $x = 5$, $z = y - 1$, $z = -2y - 1$, $z = 0$, and $z = 2$.

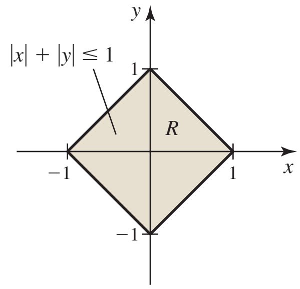
94. Square region Consider the region $R = \{(x, y): |x| + |y| \le 1\}$ shown in the figure.

a. Use a double integral to show that the area of R is 2.

b. Find the volume of the square column whose base is R and whose upper surface is $z = 12 - 3x - 4y$.

c. Find the volume of the solid above R and beneath the cylinder $x^2 + z^2 = 1$.

d. Find the volume of the pyramid whose base is R and whose vertex is on the z-axis at $(0, 0, 6)$.



95–96. Average value *Use the definition for the average value of a function over a region R (Section 16.1),* $\bar{f} = \dfrac{1}{\text{area of } R} \iint_R f(x, y)\,dA$.

95. Find the average value of $a - x - y$ over the region $R = \{(x, y): x + y \le a, x \ge 0, y \ge 0\}$, where $a > 0$.

96. Find the average value of $z = a^2 - x^2 - y^2$ over the region $R = \{(x, y): x^2 + y^2 \le a^2\}$, where $a > 0$.

T 97–98. Area integrals *Consider the following regions R. Use a computer algebra system to evaluate the integrals.*

a. *Sketch the region R.*

b. *Evaluate* $\iint_R dA$ *to determine the area of the region.*

c. *Evaluate* $\iint_R xy\,dA$.

97. R is the region between both branches of $y = 1/x$ and the lines $y = x + 3/2$ and $y = x - 3/2$.

98. R is the region bounded by the ellipse $x^2/18 + y^2/36 = 1$ with $y \le 4x/3$.

99–102. Improper integrals *Many improper double integrals may be handled using the techniques for improper integrals in one variable (Section 8.9). For example, under suitable conditions on f,*

$$\int_a^\infty \int_{g(x)}^{h(x)} f(x, y)\,dy\,dx = \lim_{b\to\infty} \int_a^b \int_{g(x)}^{h(x)} f(x, y)\,dy\,dx.$$

Use or extend the one-variable methods for improper integrals to evaluate the following integrals.

99. $\displaystyle\int_1^\infty \int_0^{e^{-x}} xy\,dy\,dx$

100. $\displaystyle\int_1^\infty \int_0^{1/x^2} \frac{2y}{x}\,dy\,dx$

101. $\displaystyle\int_0^\infty \int_0^\infty e^{-x-y}\,dy\,dx$

102. $\displaystyle\int_{-\infty}^\infty \int_{-\infty}^\infty \frac{1}{(x^2+1)(y^2+1)}\,dy\,dx$

QUICK CHECK ANSWERS

1. Inner integral: $0 \le y \le 2 - x$; outer integral: $0 \le x \le 2$ **2.** Yes; however, two separate iterated integrals would be required. **3.** $\int_0^1 \int_x^1 f(x, y)\,dy\,dx$ **4.** No; yes ◄

16.3 Double Integrals in Polar Coordinates

➤ Recall the conversions between Cartesian and polar coordinates (Section 12.2):

$x = r\cos\theta$, $y = r\sin\theta$, or

$r^2 = x^2 + y^2$, $\tan\theta = y/x$.

In Chapter 12, we explored polar coordinates and saw that in certain situations, they simplify problems considerably. The same is true when it comes to integration over plane regions. In this section, we learn how to formulate double integrals in polar coordinates and how to change double integrals from Cartesian coordinates to polar coordinates.

Moving from Rectangular to Polar Coordinates

Suppose we want to find the volume of the solid bounded by the paraboloid $z = 9 - x^2 - y^2$ and the xy-plane (Figure 16.27). The intersection of the paraboloid and the xy-plane ($z = 0$) is the curve $9 - x^2 - y^2 = 0$, or $x^2 + y^2 = 9$. Therefore, the region of integration R is the disk of radius 3 in the xy-plane, centered at the origin, which, when expressed in Cartesian coordinates, is $R = \{(x, y): -\sqrt{9 - x^2} \le y \le \sqrt{9 - x^2}, -3 \le x \le 3\}$. Using the relationship $r^2 = x^2 + y^2$ for converting Cartesian to polar coordinates, the region of integration expressed in polar coordinates is simply $R = \{(r, \theta): 0 \le r \le 3, 0 \le \theta \le 2\pi\}$. Furthermore, the paraboloid expressed in polar coordinates is $z = 9 - r^2$. This problem (which is solved in Example 1) illustrates how both the integrand and the region of integration in a double integral can be simplified by working in polar coordinates.

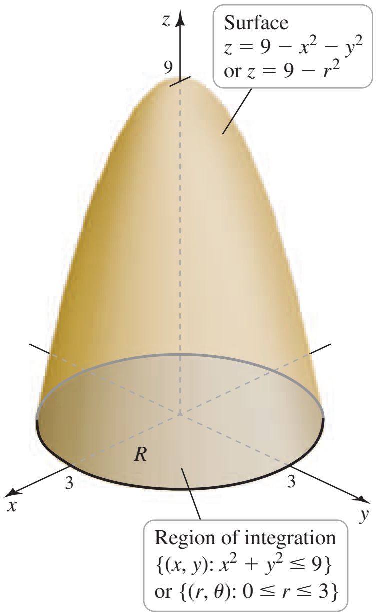


Figure 16.27

The region of integration in this problem is an example of a **polar rectangle**. In polar coordinates, it has the form $R = \{(r, \theta): 0 \le a \le r \le b, \alpha \le \theta \le \beta\}$, where $\beta - \alpha \le 2\pi$ and a, b, α, and β are constants (Figure 16.28). Polar rectangles are the analogs of rectangles in Cartesian coordinates. For this reason, the methods used in Section 16.1 for evaluating double integrals over rectangles can be extended to polar rectangles. The goal is to evaluate integrals of the form $\iint_R f(x, y)\,dA$, where f is a continuous function on the polar rectangle R. If f is nonnegative on R, this integral equals the volume of the solid bounded by the surface $z = f(x, y)$ and the region R in the xy-plane.

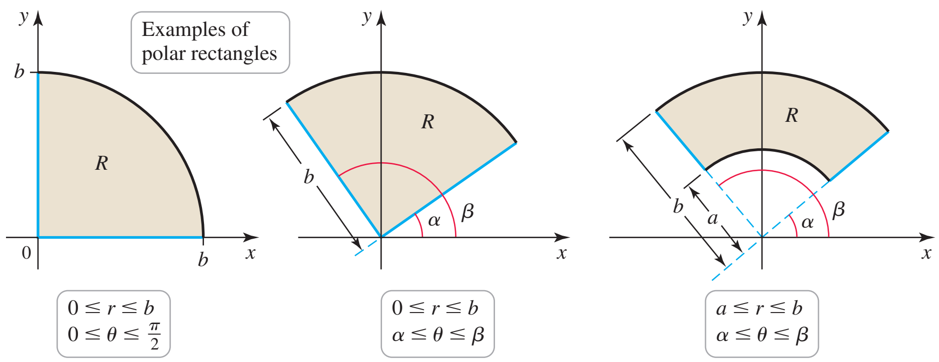


Figure 16.28

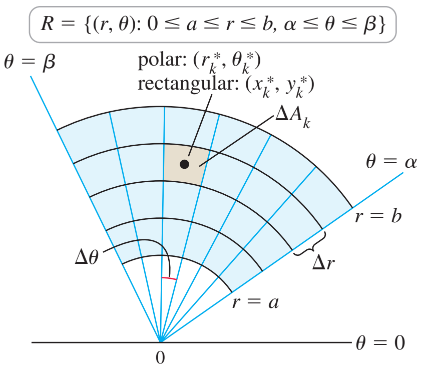


Figure 16.29

Our approach is to divide $[a, b]$ into M subintervals of equal length $\Delta r = (b - a)/M$. We similarly divide $[\alpha, \beta]$ into m subintervals of equal length $\Delta\theta = (\beta - \alpha)/m$. Now look at the arcs of the circles centered at the origin with radii

$$r = a, r = a + \Delta r, r = a + 2\Delta r, \ldots, r = b$$

and the rays

$$\theta = \alpha, \theta = \alpha + \Delta\theta, \theta = a + 2\Delta\theta, \ldots, \theta = \beta$$

emanating from the origin (**Figure 16.29**). The arcs and rays divide the region R into $n = Mm$ polar rectangles that we number in a convenient way from $k = 1$ to $k = n$. The area of the kth rectangle is denoted ΔA_k, and we let (r_k^*, θ_k^*) be the polar coordinates of an arbitrary point in that rectangle. Note that this point also has the Cartesian coordinates $(x_k^*, y_k^*) = (r_k^* \cos\theta_k^*, r_k^* \sin\theta_k^*)$. If f is continuous on R, the volume of the solid region beneath the surface $z = f(x, y)$ and above R may be computed with Riemann sums using either ordinary rectangles (as in Sections 16.1 and 16.2) or polar rectangles. Here, we now use polar rectangles.

Consider the "box" whose base is the kth polar rectangle and whose height is $f(x_k^*, y_k^*)$; its volume is $f(x_k^*, y_k^*)\Delta A_k$, for $k = 1, \ldots, n$. Therefore, the volume of the solid region beneath the surface $z = f(x, y)$ with a base R is approximately

$$V = \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k.$$

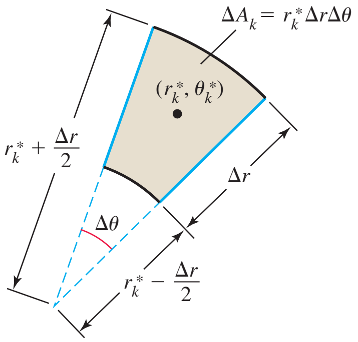
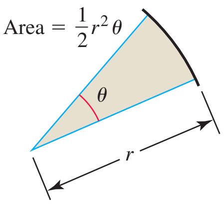


Figure 16.30

This approximation to the volume is a Riemann sum. We let Δ be the maximum value of Δr and $\Delta\theta$. If f is continuous on R, then as $n \to \infty$ and $\Delta \to 0$, the sum approaches a double integral; that is,

$$\iint_R f(x, y)\, dA = \lim_{\Delta\to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k = \lim_{\Delta\to 0} \sum_{k=1}^{n} f(r_k^* \cos\theta_k^*, r_k^* \sin\theta_k^*)\Delta A_k. \quad (1)$$

➤ Recall that the area of a sector of a circle of radius r subtended by an angle θ is $\frac{1}{2}r^2\theta$.



The next step is to express ΔA_k in terms of Δr and $\Delta\theta$. **Figure 16.30** shows the kth polar rectangle, with an area of ΔA_k. The point (r_k^*, θ_k^*) (in polar coordinates) is chosen so that the outer arc of the polar rectangle has radius $r_k^* + \Delta r/2$ and the inner arc has radius $r_k^* - \Delta r/2$. The area of the polar rectangle is

$$\begin{aligned}
\Delta A_k &= (\text{area of outer sector}) - (\text{area of inner sector}) \\
&= \frac{1}{2}\left(r_k^* + \frac{\Delta r}{2}\right)^2 \Delta\theta - \frac{1}{2}\left(r_k^* - \frac{\Delta r}{2}\right)^2 \Delta\theta \qquad \text{Area of sector} = \frac{1}{2}r^2\Delta\theta \\
&= r_k^* \Delta r \Delta\theta. \qquad \text{Expand and simplify.}
\end{aligned}$$

Substituting this expression for ΔA_k into equation (1), we have

$$\iint_R f(x, y)\, dA = \lim_{\Delta\to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k = \lim_{\Delta\to 0} \sum_{k=1}^{n} f(r_k^* \cos\theta_k^*, r_k^* \sin\theta_k^*) r_k^* \Delta r \Delta\theta.$$

This observation leads to a theorem that allows us to write a double integral in x and y as an iterated integral of $f(r\cos\theta, r\sin\theta)r$ in polar coordinates. It is an example of a change of variables, explained more generally in Section 16.7.

➤ The most common error in evaluating integrals in polar coordinates is to omit the factor r that appears in the integrand. In Cartesian coordinates, the element of area is $dx\,dy$; in polar coordinates, the element of area is $r\,dr\,d\theta$, and without the factor of r, area is not measured correctly.

THEOREM 16.3 Change of Variables for Double Integrals over Polar Rectangle Regions

Let f be continuous on the region R in the xy-plane expressed in polar coordinates as $R = \{(r, \theta): 0 \le a \le r \le b, \alpha \le \theta \le \beta\}$, where $\beta - \alpha \le 2\pi$. Then f is integrable over R, and the double integral of f over R is

$$\iint_R f(x, y)\, dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta.$$

QUICK CHECK 1 Describe in polar coordinates the region in the first quadrant between the circles of radius 1 and 2. ◄

EXAMPLE 1 **Volume of a paraboloid cap** Find the volume of the solid bounded by the paraboloid $z = 9 - x^2 - y^2$ and the xy-plane.

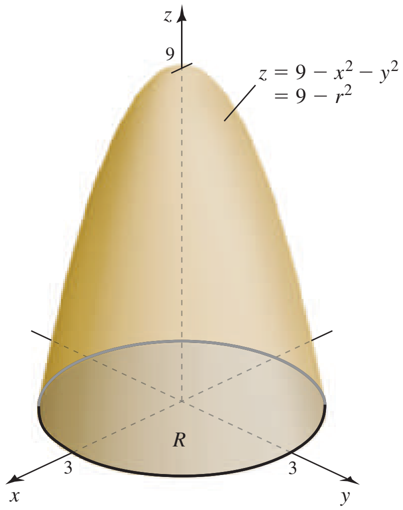
SOLUTION Using $x^2 + y^2 = r^2$, the surface is described in polar coordinates by $z = 9 - r^2$. The paraboloid intersects the xy-plane ($z = 0$) when $z = 9 - r^2 = 0$, or $r = 3$. Therefore, the intersection curve is the circle of radius 3 centered at the origin. The resulting region of integration is the disk $R = \{(r, \theta): 0 \le r \le 3, 0 \le \theta \le 2\pi\}$ (**Figure 16.31**). Integrating over R in polar coordinates, the volume is

$$V = \int_0^{2\pi}\int_0^3 \underbrace{(9 - r^2)}_{z}\, r\, dr\, d\theta \qquad \text{Iterated integral for volume}$$

$$= \int_0^{2\pi}\left(\frac{9r^2}{2} - \frac{r^4}{4}\right)\bigg|_0^3 d\theta \qquad \text{Evaluate inner integral with respect to } r.$$

$$= \int_0^{2\pi}\frac{81}{4}\, d\theta = \frac{81\pi}{2}. \qquad \text{Evaluate outer integral with respect to } \theta.$$

Related Exercises 12, 16 ◂



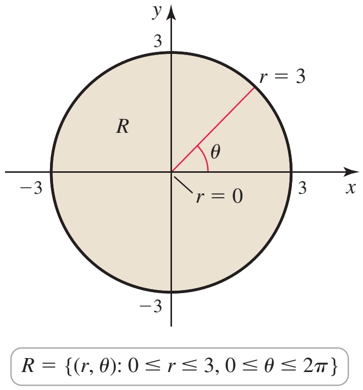


Figure 16.31

QUICK CHECK 2 Express the functions $f(x, y) = (x^2 + y^2)^{5/2}$ and $h(x, y) = x^2 - y^2$ in polar coordinates. ◂

EXAMPLE 2 **Region bounded by two surfaces** Find the volume of the region bounded by the paraboloid $z = x^2 + y^2$ and the cone $z = 2 - \sqrt{x^2 + y^2}$.

SOLUTION As discussed in Section 16.2, the volume of a solid bounded by two surfaces $z = f(x, y)$ and $z = g(x, y)$ over a region R in the xy-plane is given by $\iint_R (f(x, y) - g(x, y))\, dA$, where $f(x, y) \ge g(x, y)$ over R. Because the paraboloid $z = x^2 + y^2$ lies below the cone $z = 2 - \sqrt{x^2 + y^2}$ (**Figure 16.32**), the volume of the solid bounded by the surfaces is

$$V = \iint_R \left(\left(2 - \sqrt{x^2 + y^2}\right) - (x^2 + y^2)\right) dA,$$

where the boundary of R is the curve of intersection C of the surfaces projected onto the xy-plane. To find C, we set the equations of the surfaces equal to one another. Writing $x^2 + y^2 = 2 - \sqrt{x^2 + y^2}$ seems like a good start, but it leads to algebraic difficulties. Instead, we write the equation of the cone as $\sqrt{x^2 + y^2} = 2 - z$ and then substitute this equation into the equation for the paraboloid:

$$z = (2 - z)^2 \qquad z = x^2 + y^2 \text{ (paraboloid) and } \sqrt{x^2 + y^2} = 2 - z \text{ (cone)}$$

$$z^2 - 5z + 4 = 0 \qquad \text{Simplify.}$$

$$(z - 1)(z - 4) = 0 \qquad \text{Factor.}$$

$$z = 1 \text{ or } z = 4. \qquad \text{Solve for } z.$$

The solution $z = 4$ is an extraneous root (see Quick Check 3). Setting $z = 1$ in the equation of either the paraboloid or the cone leads to $x^2 + y^2 = 1$, which is an equation of the curve C in the plane $z = 1$. Projecting C onto the xy-plane, we conclude that the region of integration (written in polar coordinates) is $R = \{(r, \theta): 0 \le r \le 1, 0 \le \theta \le 2\pi\}$.

Converting the original volume integral into polar coordinates and evaluating it over R, we have

$$V = \iint_R \left(\left(2 - \sqrt{x^2 + y^2}\right) - (x^2 + y^2)\right) dA \qquad \text{Double integral for volume}$$

$$= \int_0^{2\pi}\int_0^1 (2 - r - r^2)\, r\, dr\, d\theta \qquad \text{Convert to polar coordinates; } x^2 + y^2 = r^2.$$

$$= \int_0^{2\pi}\left(r^2 - \frac{1}{3}r^3 - \frac{1}{4}r^4\right)\bigg|_0^1 d\theta \qquad \text{Evaluate the inner integral.}$$

$$= \int_0^{2\pi}\frac{5}{12}\, d\theta = \frac{5\pi}{6}. \qquad \text{Evaluate the outer integral.}$$

Related Exercises 33, 40 ◂

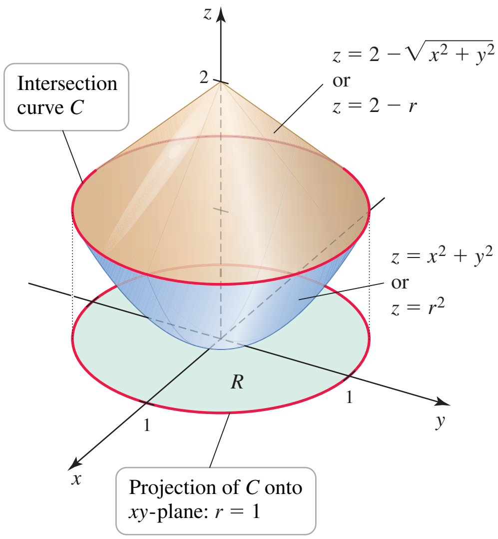


Figure 16.32

QUICK CHECK 3 Give a geometric explanation for the extraneous root $z = 4$ found in Example 2. ◂

EXAMPLE 3 **Annular region** Find the volume of the region beneath the surface $z = xy + 10$ and above the annular region $R = \{(r, \theta): 2 \le r \le 4, 0 \le \theta \le 2\pi\}$. (An *annulus* is the region between two concentric circles.)

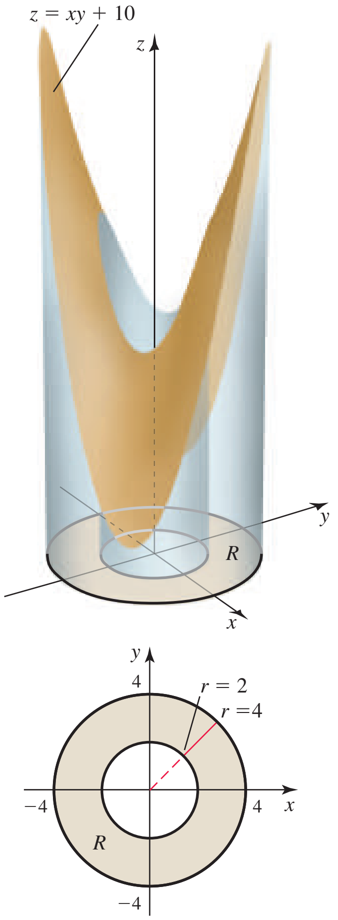
SOLUTION The region of integration suggests working in polar coordinates (Figure 16.33). Substituting $x = r\cos\theta$ and $y = r\sin\theta$, the integrand becomes

$$\begin{aligned} xy + 10 &= (r\cos\theta)(r\sin\theta) + 10 && \text{Substitute for } x \text{ and } y.\\ &= r^2 \sin\theta\cos\theta + 10 && \text{Simplify.}\\ &= \tfrac{1}{2}r^2\sin 2\theta + 10. && \sin 2\theta = 2\sin\theta\cos\theta \end{aligned}$$

Substituting the integrand into the volume integral, we have

$$\begin{aligned} V &= \int_0^{2\pi}\int_2^4 \left(\tfrac{1}{2}r^2\sin 2\theta + 10\right) r\,dr\,d\theta && \text{Iterated integral for volume}\\ &= \int_0^{2\pi}\int_2^4 \left(\tfrac{1}{2}r^3\sin 2\theta + 10r\right) dr\,d\theta && \text{Simplify.}\\ &= \int_0^{2\pi}\left(\frac{r^4}{8}\sin 2\theta + 5r^2\right)\Big|_2^4 d\theta && \text{Evaluate inner integral with respect to } r.\\ &= \int_0^{2\pi}(30\sin 2\theta + 60)\,d\theta && \text{Simplify.}\\ &= (15(-\cos 2\theta) + 60\theta)\Big|_0^{2\pi} = 120\pi. && \text{Evaluate outer integral with respect to } \theta. \end{aligned}$$

Related Exercises 22, 38 ◄



$R = \{(r, \theta): 2 \le r \le 4, 0 \le \theta \le 2\pi\}$

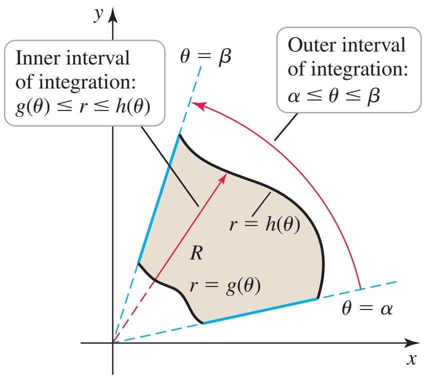
Figure 16.33

More General Polar Regions

In Section 16.2 we generalized double integrals over rectangular regions to double integrals over nonrectangular regions. In an analogous way, the method for integrating over a polar rectangle may be extended to more general regions. Consider a region (described in polar coordinates) bounded by two rays $\theta = \alpha$ and $\theta = \beta$, where $\beta - \alpha \le 2\pi$, and two curves $r = g(\theta)$ and $r = h(\theta)$ (Figure 16.34):

$$R = \{(r, \theta): 0 \le g(\theta) \le r \le h(\theta), \alpha \le \theta \le \beta\}.$$

The double integral $\iint_R f(x, y)\,dA$ is expressed as an iterated integral in which the inner integral has limits $r = g(\theta)$ and $r = h(\theta)$, and the outer integral runs from $\theta = \alpha$ to $\theta = \beta$; the integrand is transformed into polar coordinates as before. If f is nonnegative on R, the double integral gives the volume of the solid bounded by the surface $z = f(x, y)$ and R.



$R = \{(r, \theta): 0 \le g(\theta) \le r \le h(\theta), \alpha \le \theta \le \beta\}$

Figure 16.34

➤ For the type of region described in Theorem 16.4, with the boundaries in the radial direction expressed as functions of θ, the inner integral is always with respect to r.

THEOREM 16.4 **Change of Variables for Double Integrals over More General Polar Regions**
Let f be continuous on the region R in the xy-plane expressed in polar coordinates as

$$R = \{(r, \theta): 0 \le g(\theta) \le r \le h(\theta), \alpha \le \theta \le \beta\},$$

where $0 < \beta - \alpha \le 2\pi$. Then

$$\iint_R f(x, y)\,dA = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta.$$

EXAMPLE 4 **Specifying regions** Write an iterated integral in polar coordinates for $\iint_R g(r, \theta)\,dA$ for the following regions R in the xy-plane.

a. The region outside the circle $r = 2$ (with radius 2 centered at $(0, 0)$) and inside the circle $r = 4\cos\theta$ (with radius 2 centered at $(2, 0)$)

b. The region inside both circles of part (a)

➤ Recall from Section 12.2 that the polar equation $r = 2a\sin\theta$ describes a circle of radius $|a|$ with center $(0, a)$. The polar equation $r = 2a\cos\theta$ describes a circle of radius $|a|$ with center $(a, 0)$.

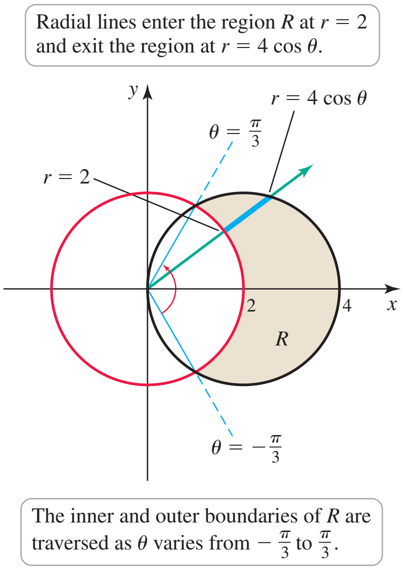


Figure 16.35

SOLUTION

a. Equating the two expressions for r, we have $4\cos\theta = 2$ or $\cos\theta = \frac{1}{2}$, so the circles intersect when $\theta = \pm\pi/3$ (**Figure 16.35**). The inner boundary of R is the circle $r = 2$, and the outer boundary is the circle $r = 4\cos\theta$. Therefore, the region of integration is $R = \{(r, \theta): 2 \le r \le 4\cos\theta, -\pi/3 \le \theta \le \pi/3\}$ and the iterated integral is

$$\iint_R g(r, \theta)\, dA = \int_{-\pi/3}^{\pi/3} \int_2^{4\cos\theta} g(r, \theta)\, r\, dr\, d\theta.$$

b. From part (a), we know that the circles intersect when $\theta = \pm\pi/3$. The region R consists of three subregions R_1, R_2, and R_3 (**Figure 16.36a**).

- For $-\pi/2 \le \theta \le -\pi/3$, R_1 is bounded by $r = 0$ (inner curve) and $r = 4\cos\theta$ (outer curve) (**Figure 16.36b**).
- For $-\pi/3 \le \theta \le \pi/3$, R_2 is bounded by $r = 0$ (inner curve) and $r = 2$ (outer curve) (**Figure 16.36c**).
- For $\pi/3 \le \theta \le \pi/2$, R_3 is bounded by $r = 0$ (inner curve) and $r = 4\cos\theta$ (outer curve) (**Figure 16.36d**).

Therefore, the double integral is expressed in three parts:

$$\iint_R g(r, \theta)\, dA = \int_{-\pi/2}^{-\pi/3} \int_0^{4\cos\theta} g(r, \theta)\, r\, dr\, d\theta + \int_{-\pi/3}^{\pi/3} \int_0^{2} g(r, \theta)\, r\, dr\, d\theta$$
$$+ \int_{\pi/3}^{\pi/2} \int_0^{4\cos\theta} g(r, \theta)\, r\, dr\, d\theta.$$

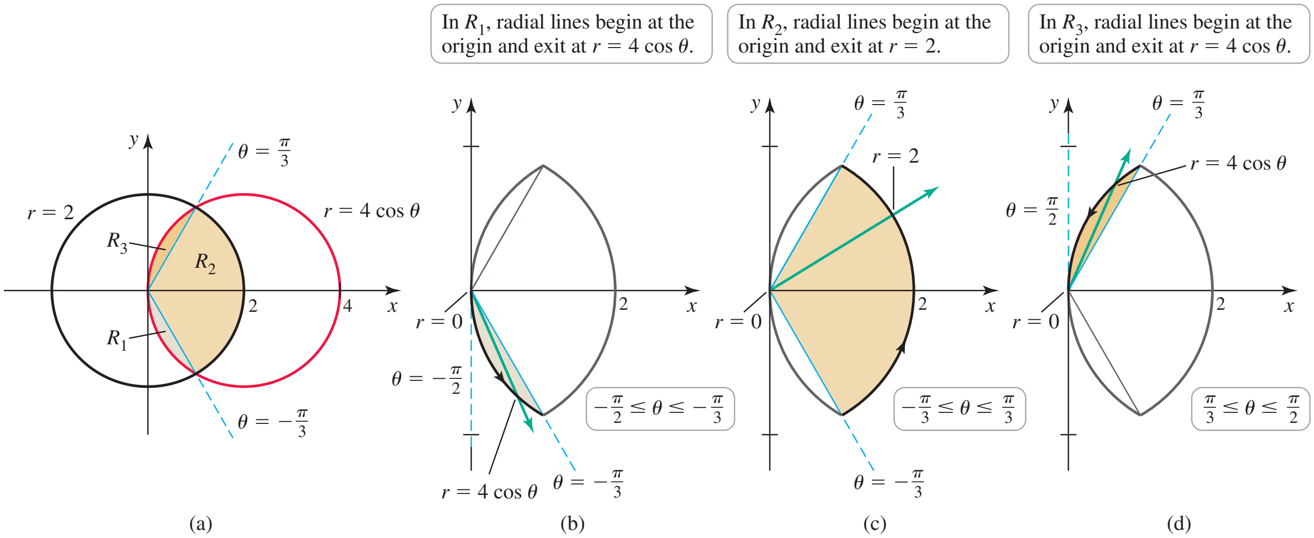


Figure 16.36

Related Exercise 44

Areas of Regions

In Cartesian coordinates, the area of a region R in the xy-plane is computed by integrating the function $f(x, y) = 1$ over R; that is, $A = \iint_R dA$. This fact extends to polar coordinates.

> Do not forget the factor of r in the area integral!

Area of Polar Regions

The area of the polar region $R = \{(r, \theta): 0 \le g(\theta) \le r \le h(\theta), \alpha \le \theta \le \beta\}$, where $0 < \beta - \alpha \le 2\pi$, is

$$A = \iint_R dA = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} r\, dr\, d\theta.$$

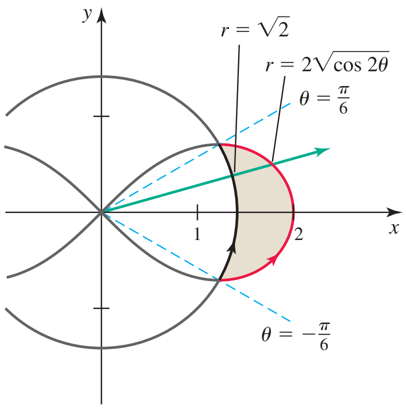


Figure 16.37

QUICK CHECK 4 Express the area of the disk $R = \{(r, \theta): 0 \leq r \leq a, 0 \leq \theta \leq 2\pi\}$ in terms of a double integral in polar coordinates. ◄

EXAMPLE 5 Area within a lemniscate Compute the area of the region in the first and fourth quadrants outside the circle $r = \sqrt{2}$ and inside the lemniscate $r^2 = 4\cos 2\theta$ (**Figure 16.37**).

SOLUTION The equation of the circle can be written as $r^2 = 2$. Equating the two expressions for r^2, the circle and the lemniscate intersect when $2 = 4\cos 2\theta$, or $\cos 2\theta = \frac{1}{2}$. The angles in the first and fourth quadrants that satisfy this equation are $\theta = \pm\pi/6$ (Figure 16.37). The region between the two curves is bounded by the inner curve $r = g(\theta) = \sqrt{2}$ and the outer curve $r = h(\theta) = 2\sqrt{\cos 2\theta}$. Therefore, the area of the region is

$$\begin{aligned}
A &= \int_{-\pi/6}^{\pi/6}\int_{\sqrt{2}}^{2\sqrt{\cos 2\theta}} r\,dr\,d\theta \\
&= \int_{-\pi/6}^{\pi/6}\left(\frac{r^2}{2}\right)\Bigg|_{\sqrt{2}}^{2\sqrt{\cos 2\theta}} d\theta && \text{Evaluate inner integral with respect to } r. \\
&= \int_{-\pi/6}^{\pi/6}(2\cos 2\theta - 1)\,d\theta && \text{Simplify.} \\
&= (\sin 2\theta - \theta)\Big|_{-\pi/6}^{\pi/6} && \text{Evaluate outer integral with respect to } \theta. \\
&= \sqrt{3} - \frac{\pi}{3}. && \text{Simplify.}
\end{aligned}$$

Related Exercises 50–51 ◄

Average Value over a Planar Polar Region

We have encountered the average value of a function in several different settings. To find the average value of a function over a region in polar coordinates, we again integrate the function over the region and divide by the area of the region.

EXAMPLE 6 Average *y*-coordinate Find the average value of the *y*-coordinates of the points in the semicircular disk of radius a given by $R = \{(r, \theta): 0 \leq r \leq a, 0 \leq \theta \leq \pi\}$.

SOLUTION The double integral that gives the average value we seek is $\bar{y} = \frac{1}{\text{area of } R}\iint_R y\,dA$. We use the facts that the area of R is $\pi a^2/2$ and the y-coordinates of points in the semicircular disk are given by $y = r\sin\theta$. Evaluating the average value integral, we find that

$$\begin{aligned}
\bar{y} &= \frac{1}{\pi a^2/2}\int_0^{\pi}\int_0^{a} r\sin\theta\, r\,dr\,d\theta \\
&= \frac{2}{\pi a^2}\int_0^{\pi}\sin\theta\left(\frac{r^3}{3}\right)\Bigg|_0^{a} d\theta && \text{Evaluate inner integral with respect to } r. \\
&= \frac{2}{\pi a^2}\,\frac{a^3}{3}\int_0^{\pi}\sin\theta\,d\theta && \text{Simplify.} \\
&= \frac{2a}{3\pi}(-\cos\theta)\Big|_0^{\pi} && \text{Evaluate outer integral with respect to } \theta. \\
&= \frac{4a}{3\pi}. && \text{Simplify.}
\end{aligned}$$

Note that $4/(3\pi) \approx 0.42$, so the average value of the y-coordinates is less than half the radius of the disk.

Related Exercise 53 ◄

SECTION 16.3 EXERCISES

Getting Started

1. Draw the polar region $\{(r, \theta): 1 \le r \le 2, 0 \le \theta \le \pi/2\}$. Why is it called a polar rectangle?

2. Write the double integral $\iint_R f(x, y)\, dA$ as an iterated integral in polar coordinates when $R = \{(r, \theta): a \le r \le b, \alpha \le \theta \le \beta\}$.

3. Sketch in the xy-plane the region of integration for the integral $\int_{-\pi/6}^{\pi/6}\int_{1/2}^{\cos 2\theta} g(r, \theta)\, r\, dr\, d\theta$.

4. Explain why the element of area in Cartesian coordinates $dx\, dy$ becomes $r\, dr\, d\theta$ in polar coordinates.

5. How do you find the area of a polar region $R = \{(r, \theta): g(\theta) \le r \le h(\theta), \alpha \le \theta \le \beta\}$?

6. How do you find the average value of a function over a region that is expressed in polar coordinates?

7–10. Polar rectangles *Sketch the following polar rectangles.*

7. $R = \{(r, \theta): 0 \le r \le 5, 0 \le \theta \le \pi/2\}$

8. $R = \{(r, \theta): 2 \le r \le 3, \pi/4 \le \theta \le 5\pi/4\}$

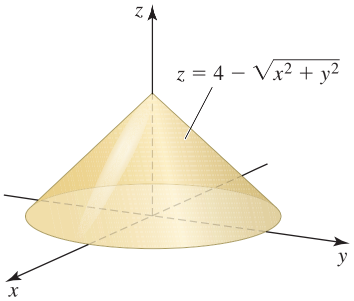
9. $R = \{(r, \theta): 1 \le r \le 4, -\pi/4 \le \theta \le 2\pi/3\}$

10. $R = \{(r, \theta): 4 \le r \le 5, -\pi/3 \le \theta \le \pi/2\}$

Practice Exercises

11–14. Volume of solids *Find the volume of the solid bounded by the surface $z = f(x, y)$ and the xy-plane.*

11. $f(x, y) = 4 - \sqrt{x^2 + y^2}$

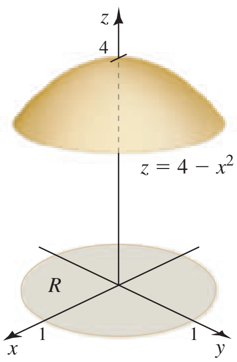


12. $f(x, y) = 16 - 4(x^2 + y^2)$ **13.** $f(x, y) = e^{-(x^2+y^2)/8} - e^{-2}$

14. $f(x, y) = \dfrac{20}{1 + x^2 + y^2} - 2$

15–18. Solids bounded by paraboloids *Find the volume of the solid below the paraboloid $z = 4 - x^2 - y^2$ and above the following polar rectangles.*

15. $R = \{(r, \theta): 0 \le r \le 1, 0 \le \theta \le 2\pi\}$



16. $R = \{(r, \theta): 0 \le r \le 2, 0 \le \theta \le 2\pi\}$

17. $R = \{(r, \theta): 1 \le r \le 2, 0 \le \theta \le 2\pi\}$

18. $R = \{(r, \theta): 1 \le r \le 2, -\pi/2 \le \theta \le \pi/2\}$

19–20. Solids bounded by hyperboloids *Find the volume of the solid below the hyperboloid $z = 5 - \sqrt{1 + x^2 + y^2}$ and above the following polar rectangles.*

19. $R = \{(r, \theta): \sqrt{3} \le r \le 2\sqrt{2}, 0 \le \theta \le 2\pi\}$

20. $R = \{(r, \theta): \sqrt{3} \le r \le \sqrt{15}, -\pi/2 \le \theta \le \pi\}$

21–30. Cartesian to polar coordinates *Evaluate the following integrals using polar coordinates. Assume (r, θ) are polar coordinates. A sketch is helpful.*

21. $\displaystyle\iint_R (x^2 + y^2)\, dA;\ R = \{(r, \theta): 0 \le r \le 4, 0 \le \theta \le 2\pi\}$

22. $\displaystyle\iint_R 2xy\, dA;\ R = \{(r, \theta): 1 \le r \le 3, 0 \le \theta \le \pi/2\}$

23. $\displaystyle\iint_R 2xy\, dA;\ R = \{(x, y): x^2 + y^2 \le 9, y \ge 0\}$

24. $\displaystyle\iint_R \frac{dA}{1 + x^2 + y^2};\ R = \{(r, \theta): 1 \le r \le 2, 0 \le \theta \le \pi\}$

25. $\displaystyle\iint_R \frac{dA}{\sqrt{16 - x^2 - y^2}};\ R = \{(x, y): x^2 + y^2 \le 4, x \ge 0, y \ge 0\}$

26. $\displaystyle\iint_R e^{-x^2-y^2}\, dA;\ R = \{(x, y): x^2 + y^2 \le 9\}$

27. $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (x^2 + y^2)^{3/2}\, dy\, dx$

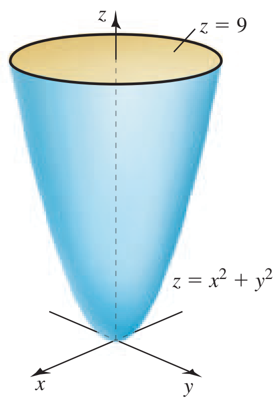
28. $\displaystyle\int_0^3\int_0^{\sqrt{9-x^2}} \sqrt{x^2 + y^2}\, dy\, dx$

29. $\displaystyle\iint_R \sqrt{x^2 + y^2}\, dA;\ R = \{(x, y): 1 \le x^2 + y^2 \le 4\}$

30. $\displaystyle\int_{-4}^{4}\int_0^{\sqrt{16-y^2}} (16 - x^2 - y^2)\, dx\, dy$

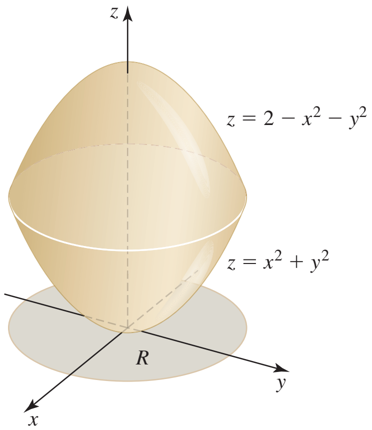
31–40. Volume between surfaces *Find the volume of the following solids.*

31. The solid bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 9$

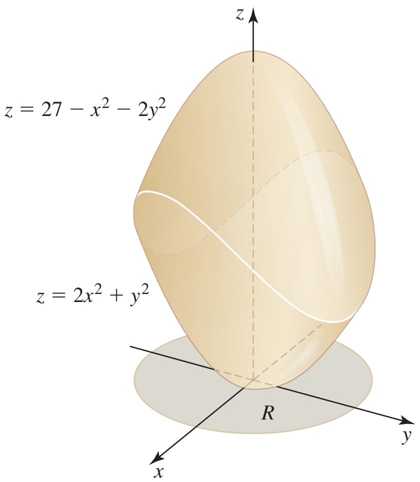


32. The solid bounded by the paraboloid $z = 2 - x^2 - y^2$ and the plane $z = 1$

33. The solid bounded by the paraboloids $z = x^2 + y^2$ and $z = 2 - x^2 - y^2$

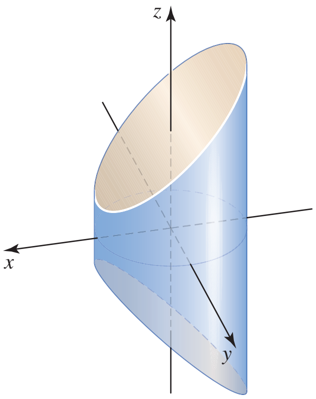


34. The solid bounded by the paraboloids $z = 2x^2 + y^2$ and $z = 27 - x^2 - 2y^2$



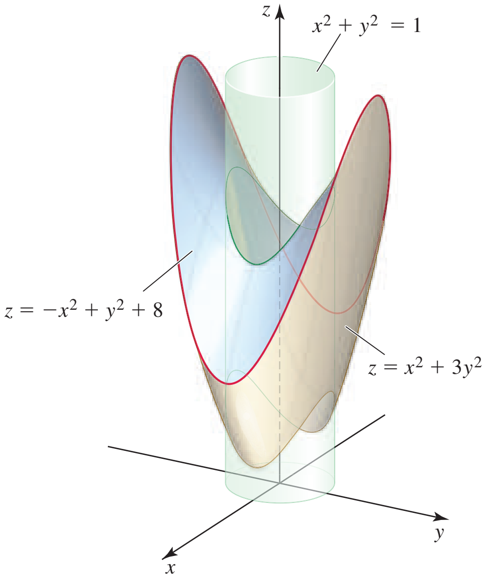
35. The solid bounded below by the paraboloid $z = x^2 + y^2 - x - y$ and above by the plane $x + y + z = 4$

36. The solid bounded by the cylinder $x^2 + y^2 = 4$ and the planes $z = 3 - x$ and $z = x - 3$

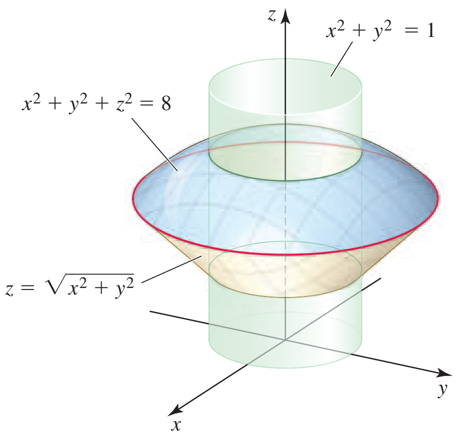


37. The solid bounded by the paraboloid $z = 18 - x^2 - 3y^2$ and the hyperbolic paraboloid $z = x^2 - y^2$

38. The solid outside the cylinder $x^2 + y^2 = 1$ that is bounded above by the hyperbolic paraboloid $z = -x^2 + y^2 + 8$ and below by the paraboloid $z = x^2 + 3y^2$



39. The solid outside the cylinder $x^2 + y^2 = 1$ that is bounded above by the sphere $x^2 + y^2 + z^2 = 8$ and below by the cone $z = \sqrt{x^2 + y^2}$



40. The solid bounded by the cone $z = 2 - \sqrt{x^2 + y^2}$ and the upper half of a hyperboloid of two sheets $z = \sqrt{1 + x^2 + y^2}$

41–46. Describing general regions *Sketch the following regions R. Then express* $\iint_R g(r, \theta)\, dA$ *as an iterated integral over R in polar coordinates.*

41. The region inside the limaçon $r = 1 + \frac{1}{2}\cos\theta$

42. The region inside the leaf of the rose $r = 2\sin 2\theta$ in the first quadrant

43. The region inside the lobe of the lemniscate $r^2 = 2\sin 2\theta$ in the first quadrant

44. The region outside the circle $r = 2$ and inside the circle $r = 4\sin\theta$

45. The region outside the circle $r = 1$ and inside the rose $r = 2\sin 3\theta$ in the first quadrant

46. The region outside the circle $r = 1/2$ and inside the cardioid $r = 1 + \cos\theta$

47–52. Computing areas *Use a double integral to find the area of the following regions.*

47. The annular region $\{(r, \theta): 1 \le r \le 2, 0 \le \theta \le \pi\}$

48. The region bounded by the cardioid $r = 2(1 - \sin\theta)$

49. The region bounded by all leaves of the rose $r = 2\cos 3\theta$

50. The region inside both the cardioid $r = 1 - \cos\theta$ and the circle $r = 1$

51. The region inside both the cardioid $r = 1 + \sin\theta$ and the cardioid $r = 1 + \cos\theta$

52. The region bounded by the spiral $r = 2\theta$, for $0 \le \theta \le \pi$, and the x-axis

53–54. Average values *Find the following average values.*

53. The average distance between points of the disk $\{(r, \theta): 0 \le r \le a\}$ and the origin

54. The average value of $1/r^2$ over the annulus $\{(r, \theta): 2 \le r \le 4\}$

55. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. Let R be the unit disk centered at $(0, 0)$. Then $\iint_R (x^2 + y^2)\, dA = \int_0^{2\pi}\int_0^1 r^2\, dr\, d\theta$.

b. The average distance between the points of the hemisphere $z = \sqrt{4 - x^2 - y^2}$ and the origin is 2 (calculus not required).

c. The integral $\int_0^1 \int_0^{\sqrt{1-y^2}} e^{x^2+y^2}\, dx\, dy$ is easier to evaluate in polar coordinates than in Cartesian coordinates.

56. Areas of circles Use integration to show that the circles $r = 2a\cos\theta$ and $r = 2a\sin\theta$ have the same area, which is πa^2.

57. Filling bowls with water Which bowl holds the most water when all the bowls are filled to a depth of 4 units?

- The paraboloid $z = x^2 + y^2$, for $0 \le z \le 4$
- The cone $z = \sqrt{x^2 + y^2}$, for $0 \le z \le 4$
- The hyperboloid $z = \sqrt{1 + x^2 + y^2}$, for $1 \le z \le 5$

58. Equal volumes To what height (above the bottom of the bowl) must the cone and paraboloid bowls of Exercise 57 be filled to hold the same volume of water as the hyperboloid bowl filled to a depth of 4 units ($1 \le z \le 5$)?

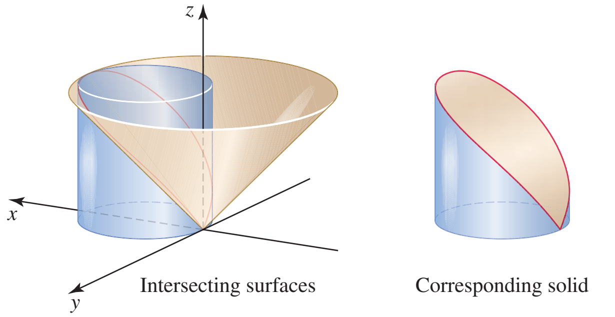
59. Volume of a hyperbolic paraboloid Consider the surface $z = x^2 - y^2$.

a. Find the region in the xy-plane in polar coordinates for which $z \ge 0$.

b. Let $R = \{(r, \theta): 0 \le r \le a, -\pi/4 \le \theta \le \pi/4\}$, which is a sector of a circle of radius a. Find the volume of the region below the hyperbolic paraboloid and above the region R.

60. Volume of a sphere Use double integrals in polar coordinates to verify that the volume of a sphere of radius a is $\frac{4}{3}\pi a^3$.

61. Volume Find the volume of the solid bounded by the cylinder $(x - 1)^2 + y^2 = 1$, the plane $z = 0$, and the cone $z = \sqrt{x^2 + y^2}$ (see figure). (*Hint:* Use symmetry.)



62. Volume Find the volume of the solid bounded by the paraboloid $z = 2x^2 + 2y^2$, the plane $z = 0$, and the cylinder $x^2 + (y - 1)^2 = 1$. (*Hint:* Use symmetry.)

Explorations and Challenges

63–64. Miscellaneous integrals *Evaluate the following integrals using the method of your choice. A sketch is helpful.*

63. $\displaystyle\iint_R \frac{dA}{4 + \sqrt{x^2 + y^2}}$; $R = \left\{(r, \theta): 0 \le r \le 2, \frac{\pi}{2} \le \theta \le \frac{3\pi}{2}\right\}$

64. $\displaystyle\iint_R \frac{x - y}{x^2 + y^2 + 1}\, dA$; R is the region bounded by the unit circle centered at the origin.

65–68. Improper integrals *Improper integrals arise in polar coordinates when the radial coordinate r becomes arbitrarily large. Under certain conditions, these integrals are treated in the usual way:*

$$\int_\alpha^\beta \int_a^\infty g(r, \theta)\, r\, dr\, d\theta = \lim_{b\to\infty} \int_\alpha^\beta \int_a^b g(r, \theta)\, r\, dr\, d\theta.$$

Use this technique to evaluate the following integrals.

65. $\displaystyle\int_0^{\pi/2} \int_1^\infty \frac{\cos\theta}{r^3}\, r\, dr\, d\theta$

66. $\displaystyle\iint_R \frac{dA}{(x^2 + y^2)^{5/2}}$; $R = \{(r, \theta): 1 \le r < \infty, 0 \le \theta \le 2\pi\}$

67. $\displaystyle\iint_R e^{-x^2-y^2}\, dA$; $R = \left\{(r, \theta): 0 \le r < \infty, 0 \le \theta \le \frac{\pi}{2}\right\}$

68. $\displaystyle\iint_R \frac{dA}{(1 + x^2 + y^2)^2}$; R is the first quadrant.

T 69. Slicing a hemispherical cake A cake is shaped like a hemisphere of radius 4 with its base on the xy-plane. A wedge of the cake is removed by making two slices from the center of the cake outward, perpendicular to the xy-plane and separated by an angle of φ.

a. Use a double integral to find the volume of the slice for $\varphi = \pi/4$. Use geometry to check your answer.

b. Now suppose the cake is sliced horizontally at $z = a > 0$ and let D be the piece of cake above the plane $z = a$. For what approximate value of a is the volume of D equal to the volume in part (a)?

T 70. Mass from density data The following table gives the density (in units of g/cm^2) at selected points (in polar coordinates) of a thin semicircular plate of radius 3. Estimate the mass of the plate and explain your method.

	$\theta = 0$	$\theta = \pi/4$	$\theta = \pi/2$	$\theta = 3\pi/4$	$\theta = \pi$
$r = 1$	2.0	2.1	2.2	2.3	2.4
$r = 2$	2.5	2.7	2.9	3.1	3.3
$r = 3$	3.2	3.4	3.5	3.6	3.7

71. A mass calculation Suppose the density of a thin plate represented by the polar region R is $\rho(r, \theta)$ (in units of mass per area). The mass of the plate is $\iint_R \rho(r, \theta)\, dA$. Find the mass of the thin half annulus $R = \{(r, \theta): 1 \le r \le 4, 0 \le \theta \le \pi\}$ with a density $\rho(r, \theta) = 4 + r\sin\theta$.

72. Area formula In Section 12.3 it was shown that the area of a region enclosed by the polar curve $r = g(\theta)$ and the rays $\theta = \alpha$ and $\theta = \beta$, where $\beta - \alpha \le 2\pi$, is $A = \frac{1}{2}\int_\alpha^\beta r^2\, d\theta$. Prove this result using the area formula with double integrals.

73. Normal distribution An important integral in statistics associated with the normal distribution is $I = \int_{-\infty}^{\infty} e^{-x^2}\, dx$. It is evaluated in the following steps.

a. In Section 8.9, it is shown that $\int_0^\infty e^{-x^2}\, dx$ converges (in the narrative following Example 7). Use this result to explain why $\int_{-\infty}^{\infty} e^{-x^2}\, dx$ converges.

b. Assume

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\, dy\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\, dx\, dy,$$

where we have chosen the variables of integration to be x and y and then written the product as an iterated integral. Evaluate this integral in polar coordinates and show that $I = \sqrt{\pi}$. Why is the solution $I = -\sqrt{\pi}$ rejected?

c. Evaluate $\int_0^\infty e^{-x^2}\, dx$, $\int_0^\infty xe^{-x^2}\, dx$, and $\int_0^\infty x^2 e^{-x^2}\, dx$ (using part (a) if needed).

74. Existence of integrals For what values of p does the integral $\iint_R \frac{dA}{(x^2 + y^2)^p}$ exist in the following cases? Assume (r, θ) are polar coordinates.

a. $R = \{(r, \theta): 1 \le r < \infty, 0 \le \theta \le 2\pi\}$

b. $R = \{(r, \theta): 0 \le r \le 1, 0 \le \theta \le 2\pi\}$

75. Integrals in strips Consider the integral

$$I = \iint_R \frac{dA}{(1 + x^2 + y^2)^2}$$

where $R = \{(x, y): 0 \le x \le 1, 0 \le y \le a\}$.

a. Evaluate I for $a = 1$. (*Hint:* Use polar coordinates.)

b. Evaluate I for arbitrary $a > 0$.

c. Let $a \to \infty$ in part (b) to find I over the infinite strip $R = \{(x, y): 0 \le x \le 1, 0 \le y < \infty\}$.

QUICK CHECK ANSWERS

1. $R = \{(r, \theta): 1 \le r \le 2, 0 \le \theta \le \pi/2\}$
2. $r^5, r^2(\cos^2\theta - \sin^2\theta) = r^2\cos 2\theta$
3. $z = 2 - \sqrt{x^2 + y^2}$ is the lower half of the double-napped cone $(2 - z)^2 = x^2 + y^2$. Imagine both halves of this cone in Figure 16.32: It is apparent that the paraboloid $z = x^2 + y^2$ intersects the cone twice, once when $z = 1$ and once when $z = 4$. **4.** $\int_0^{2\pi}\int_0^a r\, dr\, d\theta = \pi a^2$ ◄

16.4 Triple Integrals

At this point, you may see a pattern that is developing with respect to integration. In Chapter 5, we introduced integrals of single-variable functions. In the first three sections of this chapter, we moved up one dimension to double integrals of two-variable functions. In this section, we take another step and investigate triple integrals of three-variable functions. There is no end to the progression of multiple integrals. It is possible to define integrals with respect to any number of variables. For example, problems in statistics and statistical mechanics involve integration over regions of many dimensions.

Triple Integrals in Rectangular Coordinates

Consider a function $w = f(x, y, z)$ that is defined on a closed and bounded region D of $\mathbb{R}^3$. The graph of f lies in four-dimensional space and is the set of points $(x, y, z, f(x, y, z))$, where (x, y, z) is in D. Despite the difficulty in representing f in $\mathbb{R}^3$, we may still define the integral of f over D. We first create a partition of D by slicing the region with three sets of planes that run parallel to the xz-, yz-, and xy-planes (Figure 16.38). This partition subdivides D into small boxes that are ordered in a convenient way from $k = 1$ to $k = n$. The partition includes all boxes that are wholly contained in D. The kth box has side lengths Δx_k, Δy_k, and Δz_k, and volume $\Delta V_k = \Delta x_k \Delta y_k \Delta z_k$. We let (x_k^*, y_k^*, z_k^*) be an arbitrary point in the kth box, for $k = 1, \ldots, n$.

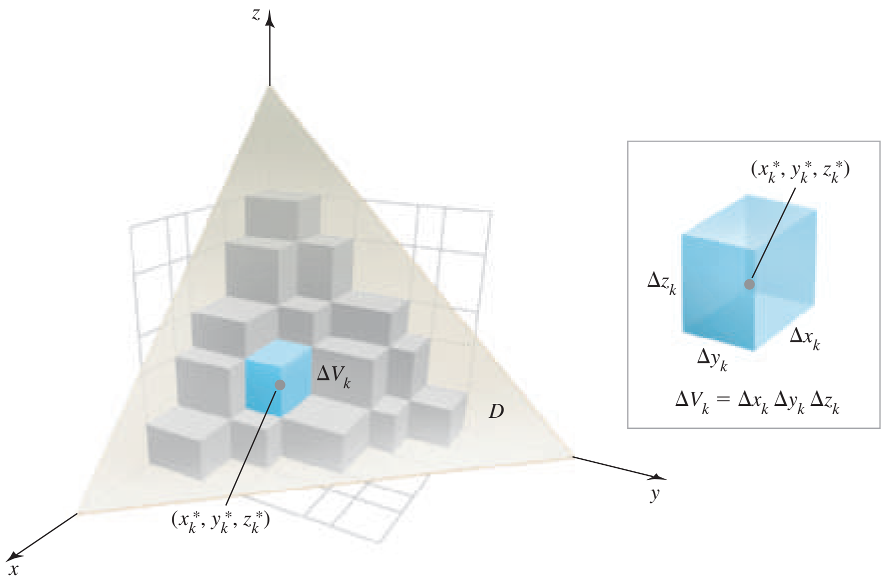


Figure 16.38

A Riemann sum is now formed, in which the kth term is the function value $f(x_k^*, y_k^*, z_k^*)$ multiplied by the volume of the kth box:

$$\sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\Delta V_k.$$

We let Δ denote the maximum length of the diagonals of the boxes. As the number of boxes n increases, while Δ approaches zero, two things happen.

- For commonly encountered regions, the region formed by the collection of boxes approaches the region D.
- If f is continuous, the Riemann sum approaches a limit.

The limit of the Riemann sum is the **triple integral of f over D**, and we write

$$\iiint_D f(x, y, z)\, dV = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\Delta V_k.$$

The kth box in the partition has volume $\Delta V_k = \Delta x_k \Delta y_k \Delta z_k$, where Δx_k, Δy_k, and Δz_k are the side lengths of the box. Accordingly, the *element of volume* in the triple integral, which we denote dV, becomes $dx\,dy\,dz$ (or some rearrangement of dx, dy, and dz) in an iterated integral.

➤ Notice the analogy between double and triple integrals:

$$\text{area of } R = \iint_R dA \quad \text{and}$$

$$\text{volume of } D = \iiint_D dV.$$

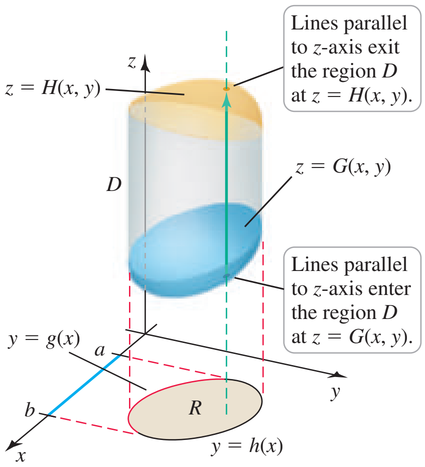
The use of triple integrals to compute the mass of an object is discussed in detail in Section 16.6.

QUICK CHECK 1 List the six orders in which the three differentials dx, dy, and dz may be written. ◄

We give two immediate interpretations of a triple integral. First, if $f(x, y, z) = 1$, then the Riemann sum simply adds up the volumes of the boxes in the partition. In the limit as $\Delta \to 0$, the triple integral $\iiint_D dV$ gives the volume of the region D. Second, suppose D is a solid three-dimensional object and its density varies from point to point according to the function $f(x, y, z)$. The units of density are mass per unit volume, so the product $f(x_k^*, y_k^*, z_k^*)\Delta V_k$ approximates the mass of the kth box in D. Summing the masses of the boxes gives an approximation to the total mass of D. In the limit as $\Delta \to 0$, the triple integral gives the mass of the object.

As with double integrals, a version of Fubini's Theorem expresses a triple integral in terms of an iterated integral in x, y, and z. The situation becomes interesting because with three variables, there are *six* possible orders of integration.

Finding Limits of Integration We discuss one of the six orders of integration in detail; the others are examined in the examples. Suppose a region D in $\mathbb{R}^3$ is bounded above by



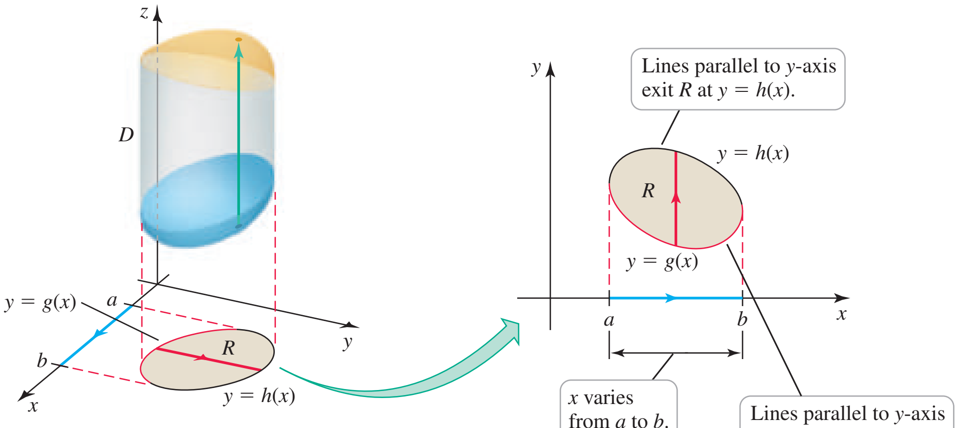
$$\iiint_D f(x, y, z)\, dV = \iint_R \left(\int_{G(x,y)}^{H(x,y)} f(x, y, z)\, dz \right) dA$$

Figure 16.39

a surface $z = H(x, y)$ and below by a surface $z = G(x, y)$ (**Figure 16.39**). These two surfaces determine the limits of integration in the z-direction. The next step is to project the region D onto the xy-plane to form a region that we call R (**Figure 16.40**). You can think of R as the shadow of D in the xy-plane. At this point, we can begin to write the triple integral as an iterated integral. So far, we have

$$\iiint_D f(x, y, z)\, dV = \iint_R \left(\int_{G(x,y)}^{H(x,y)} f(x, y, z)\, dz \right) dA.$$

Now assume R is bounded above and below by the curves $y = h(x)$ and $y = g(x)$, respectively, and bounded on the right and left by the lines $x = a$ and $x = b$, respectively (Figure 16.40). The remaining integration over R is carried out as a double integral (Section 16.2).



$$\iiint_D f(x, y, z)\, dV = \int_a^b \int_{g(x)}^{h(x)} \int_{G(x,y)}^{H(x,y)} f(x, y, z)\, dz\, dy\, dx$$

Figure 16.40

Table 16.2

Integral	Variable	Interval
Inner	z	$G(x, y) \le z \le H(x, y)$
Middle	y	$g(x) \le y \le h(x)$
Outer	x	$a \le x \le b$

The intervals that describe D are summarized in Table 16.2, which can then be used to formulate the limits of integration. To integrate over all points of D, we carry out the following steps.

1. Integrate with respect to z from $z = G(x, y)$ to $z = H(x, y)$; the result (in general) is a function of x and y.
2. Integrate with respect to y from $y = g(x)$ to $y = h(x)$; the result (in general) is a function of x.
3. Integrate with respect to x from $x = a$ to $x = b$; the result is (always) a real number.

➤ Theorem 16.5 is a version of Fubini's Theorem. Five other versions could be written for the other orders of integration.

THEOREM 16.5 Triple Integrals

Let f be continuous over the region

$$D = \{(x, y, z): a \le x \le b,\ g(x) \le y \le h(x),\ G(x, y) \le z \le H(x, y)\},$$

where g, h, G, and H are continuous functions. Then f is integrable over D and the triple integral is evaluated as the iterated integral

$$\iiint_D f(x, y, z)\, dV = \int_a^b \int_{g(x)}^{h(x)} \int_{G(x,y)}^{H(x,y)} f(x, y, z)\, dz\, dy\, dx.$$

We now illustrate this procedure with several examples.

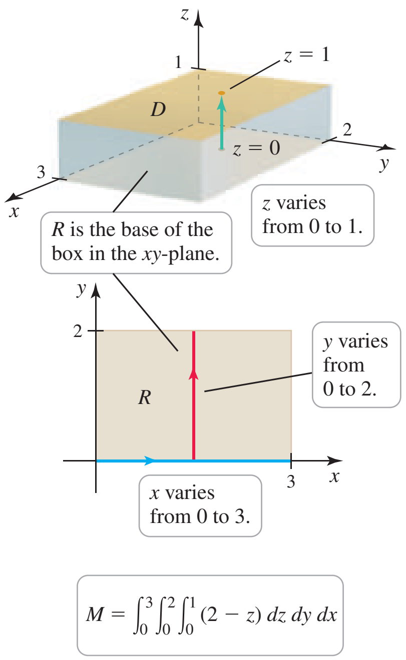


Figure 16.41

Table 16.3

Integral	Variable	Interval
Inner	z	$0 \le z \le 1$
Middle	y	$0 \le y \le 2$
Outer	x	$0 \le x \le 3$

QUICK CHECK 2 Write the integral in Example 1 in the orders $dx\, dy\, dz$ and $dx\, dz\, dy$. ◂

EXAMPLE 1 **Mass of a box** A solid box D is bounded by the planes $x = 0$, $x = 3$, $y = 0$, $y = 2$, $z = 0$, and $z = 1$. The density of the box decreases linearly in the positive z-direction and is given by $f(x, y, z) = 2 - z$. Find the mass of the box.

SOLUTION The mass of the box is found by integrating the density $f(x, y, z) = 2 - z$ over the box. Because the limits of integration for all three variables are constant, the iterated integral may be written in any order. Using the order of integration $dz\, dy\, dx$ (Figure 16.41), the limits of integration are shown in Table 16.3.

The mass of the box is

$$
\begin{aligned}
M &= \iiint_D (2 - z)\, dV \\
&= \int_0^3 \int_0^2 \int_0^1 (2 - z)\, dz\, dy\, dx && \text{Convert to an iterated integral.} \\
&= \int_0^3 \int_0^2 \left(2z - \frac{z^2}{2}\right)\Big|_0^1 dy\, dx && \text{Evaluate inner integral with respect to } z. \\
&= \int_0^3 \int_0^2 \frac{3}{2}\, dy\, dx && \text{Simplify.} \\
&= \int_0^3 \left(\frac{3y}{2}\right)\Big|_0^2 dx && \text{Evaluate middle integral with respect to } y. \\
&= \int_0^3 3\, dx = 9. && \text{Evaluate outer integral with respect to } x \text{ and simplify.}
\end{aligned}
$$

The result makes sense: The density of the box varies linearly from 1 (at the top of the box) to 2 (at the bottom); if the box had a constant density of 1, its mass would be (volume) $\times$ (density) $= 6$; if the box had a constant density of 2, its mass would be 12. The actual mass is the average of 6 and 12, as you might expect.

Any other order of integration produces the same result. For example, with the order $dy\, dx\, dz$, the iterated integral is

$$M = \iiint_D (2 - z)\, dV = \int_0^1 \int_0^3 \int_0^2 (2 - z)\, dy\, dx\, dz = 9.$$

Related Exercises 8–9 ◂

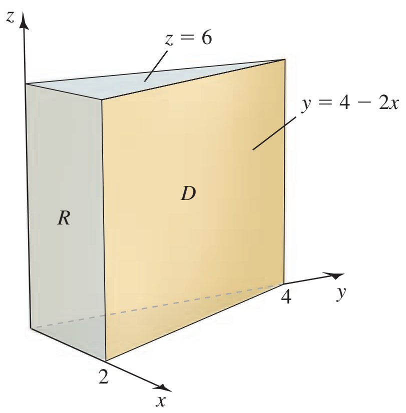


Figure 16.42

EXAMPLE 2 **Volume of a prism** Find the volume of the prism D in the first octant bounded by the planes $y = 4 - 2x$ and $z = 6$ (Figure 16.42).

SOLUTION The prism may be viewed in several different ways. Letting the base of the prism be in the xz-plane, the upper surface of the prism is the plane $y = 4 - 2x$, and the lower surface is $y = 0$. The projection of the prism onto the xz-plane is the rectangle $R = \{(x, z): 0 \le x \le 2, 0 \le z \le 6\}$. One possible order of integration in this case is $dy\, dx\, dz$.

Inner integral with respect to y: A line through the prism parallel to the y-axis enters the prism through the rectangle R at $y = 0$ and exits the prism at the plane $y = 4 - 2x$. Therefore, we first integrate with respect to y over the interval $0 \le y \le 4 - 2x$ (Figure 16.43a).

Middle integral with respect to x: The limits of integration for the middle and outer integrals must cover the region R in the xz-plane. A line parallel to the x-axis enters R at $x = 0$ and exits R at $x = 2$. So we integrate with respect to x over the interval $0 \le x \le 2$ (Figure 16.43b).

Outer integral with respect to z: To cover all of R, the line segments from $x = 0$ to $x = 2$ must run from $z = 0$ to $z = 6$. So we integrate with respect to z over the interval $0 \le z \le 6$ (Figure 16.43b).

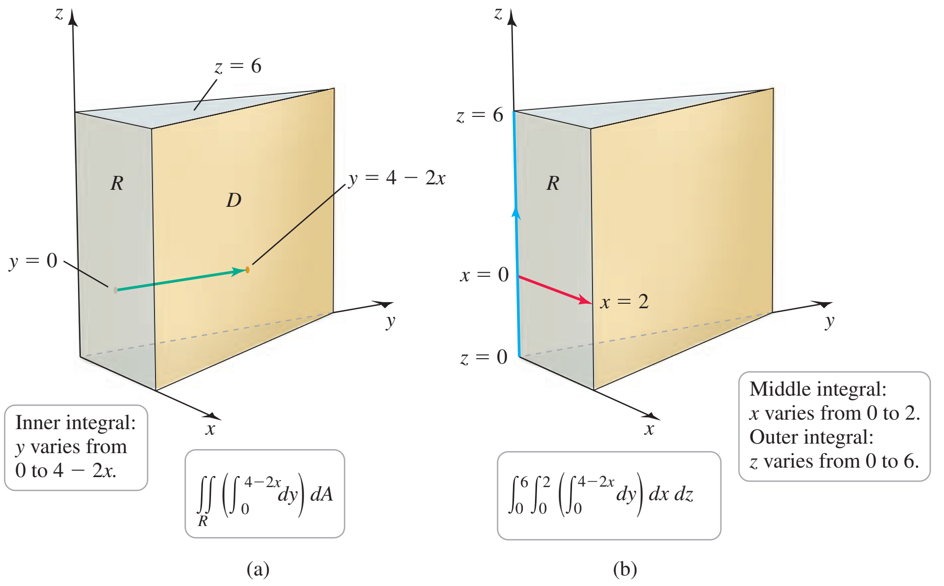


Figure 16.43

Integrating $f(x, y, z) = 1$, the volume of the prism is

$$
\begin{aligned}
V &= \iiint_D dV = \int_0^6 \int_0^2 \int_0^{4-2x} dy\, dx\, dz \\
&= \int_0^6 \int_0^2 (4 - 2x)\, dx\, dz && \text{Evaluate inner integral with respect to } y. \\
&= \int_0^6 (4x - x^2)\Big|_0^2 dz && \text{Evaluate middle integral with respect to } x. \\
&= \int_0^6 4\, dz && \text{Simplify.} \\
&= 24. && \text{Evaluate outer integral with respect to } z.
\end{aligned}
$$

Related Exercises 15, 18 ◀

➤ The volume of the prism could also be found using geometry: The area of the triangular base in the xy-plane is 4 and the height of the prism is 6. Therefore, the volume is $6 \cdot 4 = 24$.

QUICK CHECK 3 Write the integral in Example 2 in the orders $dz\, dy\, dx$ and $dx\, dy\, dz$. ◀

EXAMPLE 3 **A volume integral** Find the volume of the solid D bounded by the paraboloids $y = x^2 + 3z^2 + 1$ and $y = 5 - 3x^2 - z^2$ (Figure 16.44a).

SOLUTION The right boundary of D is the surface $y = 5 - 3x^2 - z^2$ and the left boundary is $y = x^2 + 3z^2 + 1$. These surfaces are functions of x and z, so they determine the limits of integration for the inner integral in the y-direction.

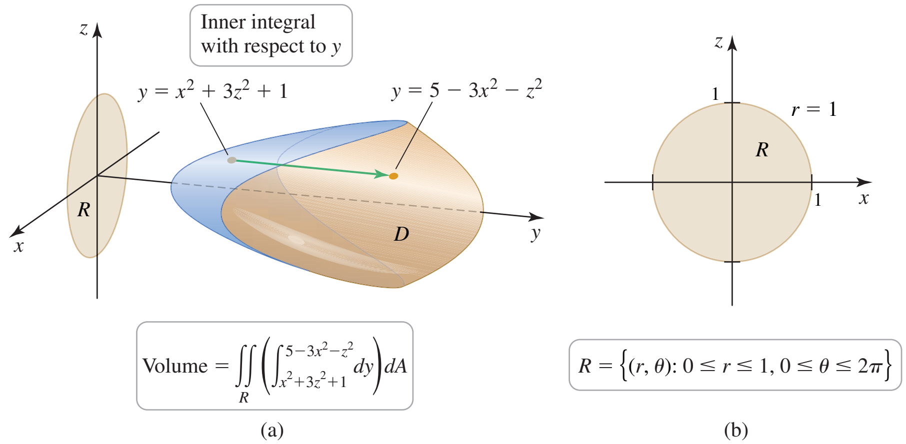


Figure 16.44

A key step in the calculation is finding the curve of intersection between the two surfaces and projecting it onto the xz-plane to form the boundary of the region R, where R is the projection of D onto the xz-plane. Equating the y-coordinates of the surfaces, we have $x^2 + 3z^2 + 1 = 5 - 3x^2 - z^2$, which, when simplified, is the equation of a unit circle centered at the origin in the xz-plane:

$$x^2 + z^2 = 1.$$

Observe that a line through the solid parallel to the y-axis enters the solid at $y = x^2 + 3z^2 + 1$ and exits at $y = 5 - 3x^2 - z^2$. Therefore, for fixed values of x and z, we integrate in y over the interval $x^2 + 3z^2 + 1 \le y \le 5 - 3x^2 - z^2$ (Figure 16.44a). After evaluating the inner integral with respect to y, we have

$$V = \iint_R \left(\int_{x^2+3z^2+1}^{5-3x^2-z^2} dy \right) dA = \iint_R \left(y \Big|_{x^2+3z^2+1}^{5-3x^2-z^2} \right) dA = \iint_R 4(1 - x^2 - z^2)\, dA.$$

The region R is bounded by a circle, so it is advantageous to evaluate the remaining double integral in polar coordinates, where θ and r have the same meaning in the xz-plane as they do in the xy-plane. Note that R is expressed in polar coordinates as $R = \{(r, \theta): 0 \le r \le 1, 0 \le \theta \le 2\pi\}$ (**Figure 16.44b**). Using the relationship $x^2 + z^2 = r^2$, we change variables and evaluate the double integral:

$$\begin{aligned}
V &= \iint_R 4(1 - x^2 - z^2)\, dA \\
&= \int_0^{2\pi} \int_0^1 4(1 - r^2)\, r\, dr\, d\theta && \text{Convert to polar coordinates: } x^2 + z^2 = r^2 \text{ and } dA = r\, dr\, d\theta. \\
&= \int_0^{2\pi} \int_0^1 2u\, du\, d\theta && \text{Let } u = 1 - r^2 \Rightarrow du = -2r\, dr. \\
&= \int_0^{2\pi} \left(u^2 \Big|_0^1 \right) d\theta && \text{Evaluate inner integral.} \\
&= \int_0^{2\pi} d\theta = 2\pi. && \text{Evaluate outer integral.}
\end{aligned}$$

Related Exercises 23, 25 ◄

Changing the Order of Integration

As with double integrals, choosing an appropriate order of integration may simplify the evaluation of a triple integral. Therefore, it is important to become proficient at changing the order of integration.

EXAMPLE 4 **Changing the order of integration** Consider the integral

$$\int_0^{\sqrt[4]{\pi}} \int_0^z \int_y^z 12y^2z^3 \sin x^4\, dx\, dy\, dz.$$

a. Sketch the region of integration D.

b. Evaluate the integral by changing the order of integration.

SOLUTION

a. We begin by finding the projection of the region of integration D on the appropriate coordinate plane; call the projection R. Because the inner integration is with respect to x, R lies in the yz-plane, and it is determined by the limits on the middle and outer integrals. We see that

$$R = \{(y, z): 0 \le y \le z, 0 \le z \le \sqrt[4]{\pi}\},$$

which is a triangular region in the yz-plane bounded by the z-axis and the lines $y = z$ and $z = \sqrt[4]{\pi}$. Using the limits on the inner integral, for each point in R we let x vary from the plane $x = y$ to the plane $x = z$. In so doing, the points fill an inverted tetrahedron in the first octant with its vertex at the origin, which is D (**Figure 16.45**).

b. It is difficult to evaluate the integral in the given order $(dx\, dy\, dz)$ because the antiderivative of $\sin x^4$ is not expressible in terms of elementary functions. If we integrate first with respect to y, we introduce a factor in the integrand that enables us to use a substitution

➤ How do we know to switch the order of integration so the inner integral is with respect to y? Often we do not know in advance whether a new order of integration will work, and some trial and error is needed. In this case, either y^2 or z^3 is easier to integrate than $\sin x^4$, so either y or z is a likely variable for the inner integral.

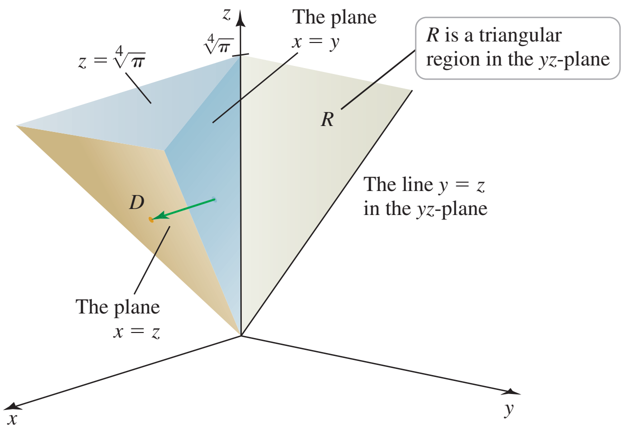


Figure 16.45

to integrate $\sin x^4$. With the order of integration $dy\,dx\,dz$, the bounds of integration for the inner integral extend from the plane $y = 0$ to the plane $y = x$ (**Figure 16.46a**). Furthermore, the projection of D onto the xz-plane is the region R, which must be covered by the middle and outer integrals (**Figure 16.46b**). In this case, we draw a line segment parallel to the x-axis to see that the limits of the middle integral run from $x = 0$ to $x = z$. Then we include all these segments from $z = 0$ to $z = \sqrt[4]{\pi}$ to obtain the outer limits of integration in z. The integration proceeds as follows:

$$\int_0^{\sqrt[4]{\pi}}\int_0^z\int_0^x 12y^2z^3\sin x^4\,dy\,dx\,dz = \int_0^{\sqrt[4]{\pi}}\int_0^z (4y^3z^3\sin x^4)\Big|_0^x\,dx\,dz \qquad \text{Evaluate inner integral with respect to } y.$$

$$= \int_0^{\sqrt[4]{\pi}}\int_0^z 4x^3z^3\sin x^4\,dx\,dz \qquad \text{Simplify.}$$

$$= \int_0^{\sqrt[4]{\pi}} z^3(-\cos x^4)\Big|_0^z\,dz \qquad \text{Evaluate middle integral with respect to } x;\ u = x^4.$$

$$= \int_0^{\sqrt[4]{\pi}} z^3(1 - \cos z^4)\,dz \qquad \text{Simplify.}$$

$$= \left(\frac{z^4}{4} - \frac{\sin z^4}{4}\right)\Big|_0^{\sqrt[4]{\pi}} \qquad \text{Evaluate outer integral with respect to } z;\ u = z^4.$$

$$= \frac{\pi}{4}. \qquad \text{Simplify.}$$

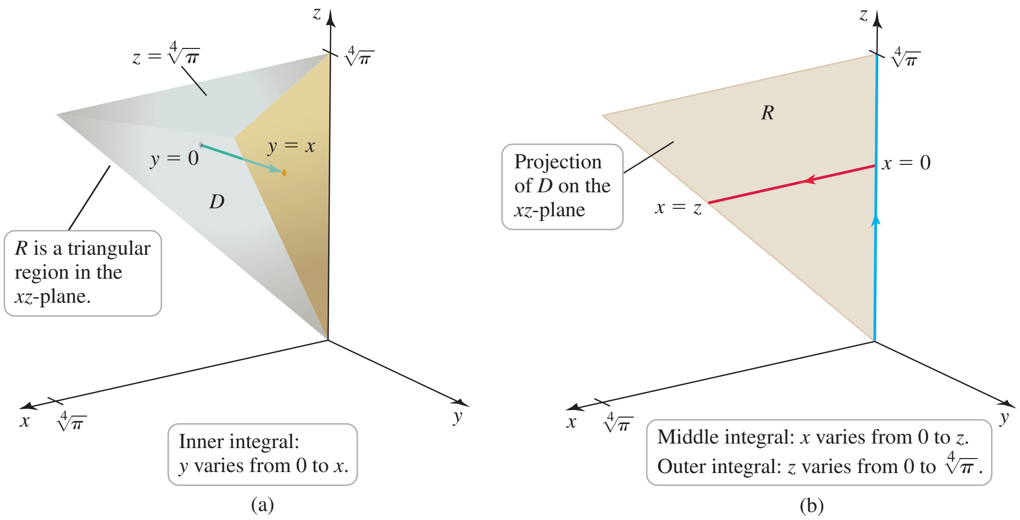


Figure 16.46

Related Exercises 47, 56 ◀

Average Value of a Function of Three Variables

The idea of the average value of a function extends naturally from the one- and two-variable cases. The average value of a function of three variables is found by integrating the function over the region of interest and dividing by the volume of the region.

DEFINITION Average Value of a Function of Three Variables

If f is continuous on a region D of $\mathbb{R}^3$, then the **average value** of f over D is

$$\bar{f} = \frac{1}{\text{volume of } D} \iiint_D f(x, y, z)\, dV.$$

EXAMPLE 5 Average temperature Consider a block of a conducting material occupying the region

$$D = \{(x, y, z): 0 \le x \le 2, 0 \le y \le 2, 0 \le z \le 1\}.$$

Due to heat sources on its boundaries, the temperature in the block is given by $T(x, y, z) = 250xy \sin \pi z$. Find the average temperature of the block.

SOLUTION We must integrate the temperature function over the block and divide by the volume of the block, which is 4. One way to evaluate the temperature integral is as follows:

$$\iiint_D 250xy \sin \pi z\, dV = 250 \int_0^2 \int_0^2 \int_0^1 xy \sin \pi z\, dz\, dy\, dx \qquad \text{Convert to an iterated integral.}$$

$$= 250 \int_0^2 \int_0^2 xy \frac{1}{\pi}(-\cos \pi z)\Big|_0^1 dy\, dx \qquad \text{Evaluate inner integral with respect to } z.$$

$$= \frac{500}{\pi} \int_0^2 \int_0^2 xy\, dy\, dx \qquad \text{Simplify.}$$

$$= \frac{500}{\pi} \int_0^2 x \left(\frac{y^2}{2}\right)\Big|_0^2 dx \qquad \text{Evaluate middle integral with respect to } y.$$

$$= \frac{1000}{\pi} \int_0^2 x\, dx \qquad \text{Simplify.}$$

$$= \frac{1000}{\pi}\left(\frac{x^2}{2}\right)\Big|_0^2 = \frac{2000}{\pi}. \qquad \text{Evaluate outer integral with respect to } x.$$

Dividing by the volume of the region, we find that the average temperature is $(2000/\pi)/4 = 500/\pi \approx 159.2$.

Related Exercises 51–52 ◄

QUICK CHECK 4 Without integrating, what is the average value of $f(x, y, z) = \sin x \sin y \sin z$ on the cube $\{(x, y, z): -1 \le x \le 1, -1 \le y \le 1, -1 \le z \le 1\}$? Use symmetry arguments. ◄

SECTION 16.4 EXERCISES

Getting Started

1. Sketch the region $D = \{(x, y, z): x^2 + y^2 \le 4, 0 \le z \le 4\}$.
2. Write an iterated integral for $\iiint_D f(x, y, z)\, dV$, where D is the box $\{(x, y, z): 0 \le x \le 3, 0 \le y \le 6, 0 \le z \le 4\}$.
3. Write an iterated integral for $\iiint_D f(x, y, z)\, dV$, where D is a sphere of radius 9 centered at $(0, 0, 0)$. Use the order $dz\, dy\, dx$.
4. Sketch the region of integration for the integral
$$\int_0^1 \int_0^{\sqrt{1-z^2}} \int_0^{\sqrt{1-y^2-z^2}} f(x, y, z)\, dx\, dy\, dz.$$
5. Write the integral in Exercise 4 in the order $dy\, dx\, dz$.
6. Write an integral for the average value of $f(x, y, z) = xyz$ over the region bounded by the paraboloid $z = 9 - x^2 - y^2$ and the xy-plane (assuming the volume of the region is known).

Practice Exercises

7–14. Integrals over boxes *Evaluate the following integrals. A sketch of the region of integration may be useful.*

7. $\int_{-2}^2 \int_3^6 \int_0^2 dx\, dy\, dz$
8. $\int_{-1}^1 \int_{-1}^2 \int_0^1 6xyz\, dy\, dx\, dz$
9. $\int_{-2}^2 \int_1^2 \int_1^e \frac{xy^2}{z}\, dz\, dx\, dy$
10. $\int_0^{\ln 4} \int_0^{\ln 3} \int_0^{\ln 2} e^{-x+y+z}\, dx\, dy\, dz$

11. $\displaystyle\int_0^{\pi/2}\int_0^1\int_0^{\pi/2} \sin \pi x \cos y \sin 2z \, dy\, dx\, dz$

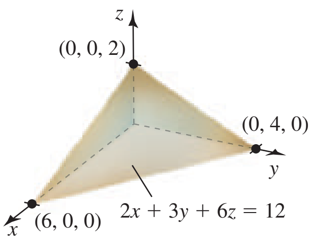
12. $\displaystyle\int_0^2\int_1^2\int_0^1 yze^x \, dx\, dz\, dy$

13. $\displaystyle\iiint_D (xy + xz + yz)\, dV$; $D = \{(x, y, z): -1 \le x \le 1, -2 \le y \le 2, -3 \le z \le 3\}$

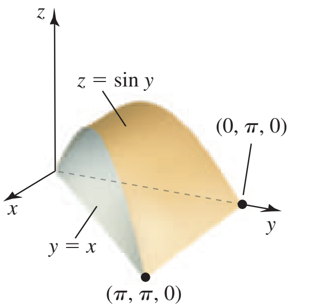
14. $\displaystyle\iiint_D xyze^{-x^2-y^2}\, dV$; $D = \{(x, y, z): 0 \le x \le \sqrt{\ln 2}, 0 \le y \le \sqrt{\ln 4}, 0 \le z \le 1\}$

15–29. Volumes of solids *Use a triple integral to find the volume of the following solids.*

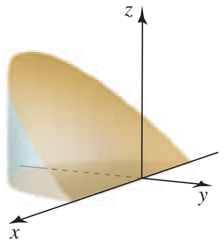
15. The solid in the first octant bounded by the plane $2x + 3y + 6z = 12$ and the coordinate planes



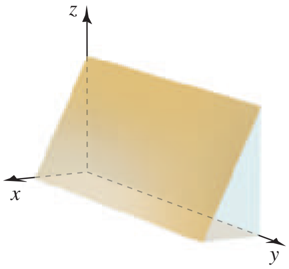
16. The solid in the first octant formed when the cylinder $z = \sin y$, for $0 \le y \le \pi$, is sliced by the planes $y = x$ and $x = 0$



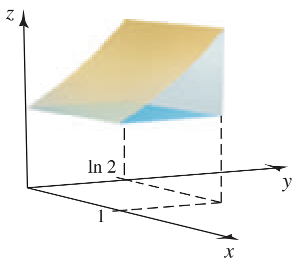
17. The wedge above the xy-plane formed when the cylinder $x^2 + y^2 = 4$ is cut by the planes $z = 0$ and $y = -z$



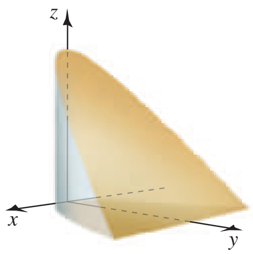
18. The prism in the first octant bounded by $z = 2 - 4x$ and $y = 8$



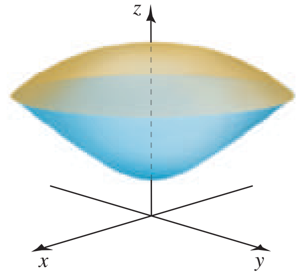
19. The solid bounded by the surfaces $z = e^y$ and $z = 1$ over the rectangle $\{(x, y): 0 \le x \le 1, 0 \le y \le \ln 2\}$



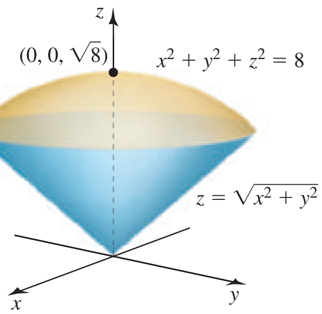
20. The wedge bounded by the parabolic cylinder $y = x^2$ and the planes $z = 3 - y$ and $z = 0$



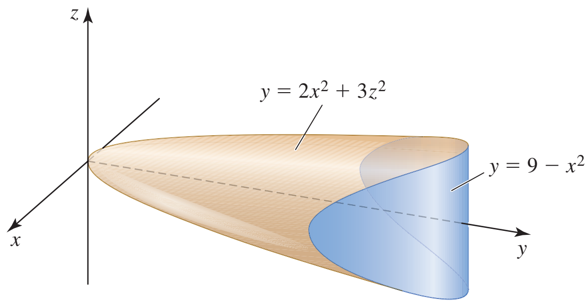
21. The solid between the sphere $x^2 + y^2 + z^2 = 19$ and the hyperboloid $z^2 - x^2 - y^2 = 1$, for $z > 0$



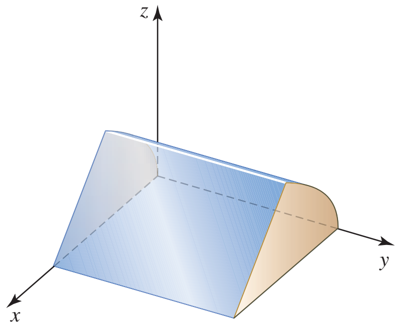
22. The solid bounded below by the cone $z = \sqrt{x^2 + y^2}$ and bounded above by the sphere $x^2 + y^2 + z^2 = 8$



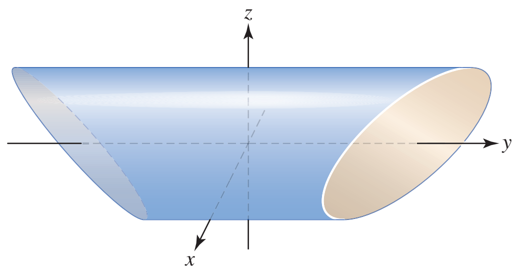
23. The solid bounded by the cylinder $y = 9 - x^2$ and the paraboloid $y = 2x^2 + 3z^2$



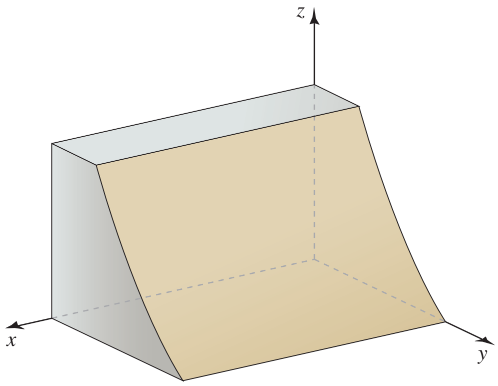
24. The wedge in the first octant bounded by the cylinder $x = z^2$ and the planes $z = 2 - x$, $y = 2$, $y = 0$, and $z = 0$



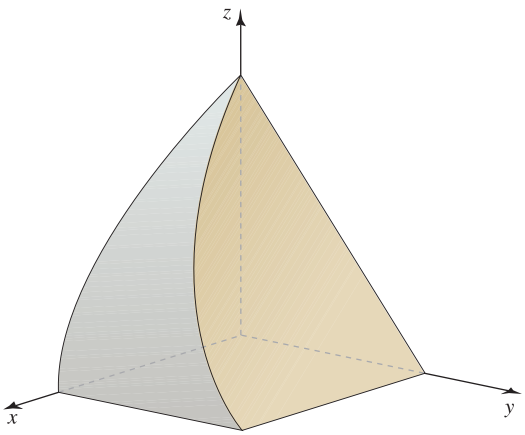
25. The wedge of the cylinder $x^2 + 4z^2 = 4$ created by the planes $y = 3 - x$ and $y = x - 3$



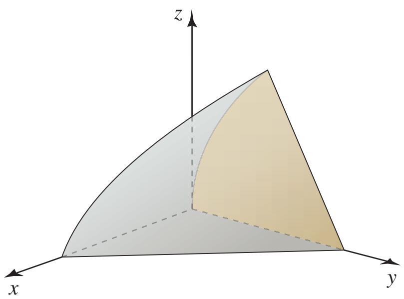
26. The solid bounded by $x = 0, x = 2, y = 0, y = e^{-z}, z = 0$, and $z = 1$



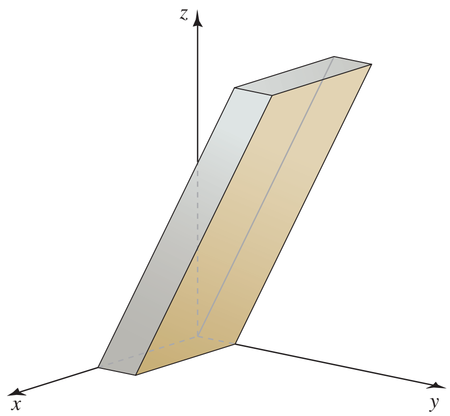
27. The solid bounded by $x = 0, x = 1 - z^2, y = 0, z = 0$, and $z = 1 - y$



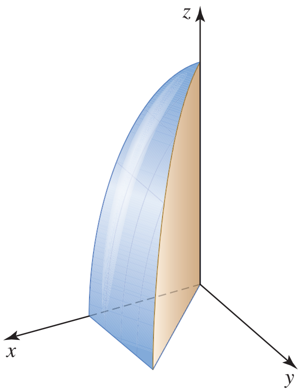
28. The solid bounded by $x = 0, y = z^2, z = 0$, and $z = 2 - x - y$



29. The solid bounded by $x = 0, x = 2, y = z, y = z + 1, z = 0$, and $z = 4$



30–35. Six orderings *Let D be the solid in the first octant bounded by the planes $y = 0$, $z = 0$, and $y = x$, and the cylinder $4x^2 + z^2 = 4$. Write the triple integral of $f(x, y, z)$ over D in the given order of integration.*



30. $dz\,dy\,dx$	**31.** $dz\,dx\,dy$	**32.** $dy\,dx\,dz$
33. $dy\,dz\,dx$	**34.** $dx\,dy\,dz$	**35.** $dx\,dz\,dy$

36. All six orders Let D be the solid bounded by $y = x, z = 1 - y^2$, $x = 0$, and $z = 0$. Write triple integrals over D in all six possible orders of integration.

37. Changing order of integration Write the integral $\int_0^2\int_0^1\int_0^{1-y} dz\,dy\,dx$ in the five other possible orders of integration.

38–46. Triple integrals *Evaluate the following integrals.*

38. $$\int_0^{\pi}\int_0^{\pi}\int_0^{\sin x} \sin y\,dz\,dx\,dy$$

39. $$\int_0^2\int_0^4\int_{y^2}^4 \sqrt{x}\,dz\,dx\,dy$$

40. $$\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} 2xz\,dz\,dy\,dx$$

41. $$\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2}} dz\,dy\,dx$$

42. $$\int_1^6\int_0^{4-2y/3}\int_0^{12-2y-3z} \frac{1}{y}\,dx\,dz\,dy$$

43. $\displaystyle\int_0^3 \int_0^{\sqrt{9-z^2}} \int_0^{\sqrt{1+x^2+z^2}} dy\, dx\, dz$

44. $\displaystyle\int_0^1 \int_y^{2-y} \int_0^{2-x-y} 15xy\, dz\, dx\, dy$

45. $\displaystyle\int_0^{\ln 8} \int_1^{\sqrt{z}} \int_{\ln y}^{\ln 2y} e^{x+y^2-z}\, dx\, dy\, dz$

46. $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{2-x} 4yz\, dz\, dy\, dx$

47–50. Changing the order of integration *Rewrite the following integrals using the indicated order of integration, and then evaluate the resulting integral.*

47. $\displaystyle\int_0^5 \int_{-1}^0 \int_0^{4x+4} dy\, dx\, dz$ in the order $dz\, dx\, dy$

48. $\displaystyle\int_0^1 \int_{-2}^2 \int_0^{\sqrt{4-y^2}} dz\, dy\, dx$ in the order $dy\, dz\, dx$

49. $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2}} dy\, dz\, dx$ in the order $dz\, dy\, dx$

50. $\displaystyle\int_0^4 \int_0^{\sqrt{16-x^2}} \int_0^{\sqrt{16-x^2-z^2}} dy\, dz\, dx$ in the order $dx\, dy\, dz$

51–54. Average value *Find the following average values.*

51. The average value of $f(x, y, z) = 8xy \cos z$ over the points inside the box $D = \{(x, y, z): 0 \le x \le 1, 0 \le y \le 2, 0 \le z \le \pi/2\}$

T 52. The average temperature in the box $D = \{(x, y, z): 0 \le x \le \ln 2, 0 \le y \le \ln 4, 0 \le z \le \ln 8\}$ with a temperature distribution of $T(x, y, z) = 128e^{-x-y-z}$

T 53. The average of the *squared* distance between the origin and points in the solid cylinder $D = \{(x, y, z): x^2 + y^2 \le 4, 0 \le z \le 2\}$

54. The average z-coordinate of points on and within a hemisphere of radius 4 centered at the origin with its base in the xy-plane

55. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. An iterated integral of a function over the box $D = \{(x, y, z): 0 \le x \le a, 0 \le y \le b, 0 \le z \le c\}$ can be expressed in eight different ways.

b. One possible iterated integral of f over the prism $D = \{(x, y, z): 0 \le x \le 1, 0 \le y \le 3x - 3, 0 \le z \le 5\}$ is $\int_0^{3x-3} \int_0^1 \int_0^5 f(x, y, z)\, dz\, dx\, dy$.

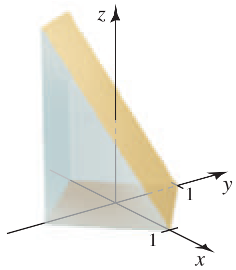
c. The region $D = \{(x, y, z): 0 \le x \le 1, 0 \le y \le \sqrt{1 - x^2}, 0 \le z \le \sqrt{1 - x^2}\}$ is a sphere.

Explorations and Challenges

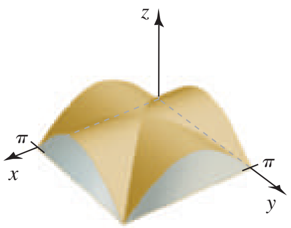
56. Changing the order of integration Use another order of integration to evaluate $\displaystyle\int_1^4 \int_z^{4z} \int_0^{\pi^2} \frac{\sin \sqrt{yz}}{x^{3/2}}\, dy\, dx\, dz$.

57–62. Miscellaneous volumes *Use a triple integral to compute the volume of the following regions.*

57. The wedge of the square column $|x| + |y| = 1$ created by the planes $z = 0$ and $x + y + z = 1$



58. The solid common to the cylinders $z = \sin x$ and $z = \sin y$ over the square $R = \{(x, y): 0 \le x \le \pi, 0 \le y \le \pi\}$ (The figure shows the cylinders, but not the common region.)

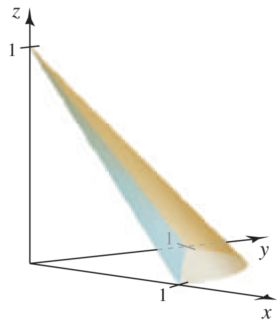


59. The parallelepiped (slanted box) with vertices (0, 0, 0), (1, 0, 0), (0, 1, 0), (1, 1, 0), (0, 1, 1), (1, 1, 1), (0, 2, 1), and (1, 2, 1) (Use integration and find the best order of integration.)

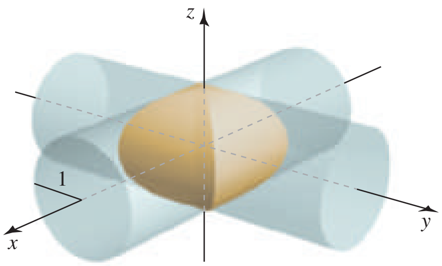
60. The larger of two solids formed when the parallelepiped (slanted box) with vertices (0, 0, 0), (2, 0, 0), (0, 2, 0), (2, 2, 0), (0, 1, 1), (2, 1, 1), (0, 3, 1), and (2, 3, 1) is sliced by the plane $y = 2$

61. The pyramid with vertices (0, 0, 0), (2, 0, 0), (2, 2, 0), (0, 2, 0), and (0, 0, 4)

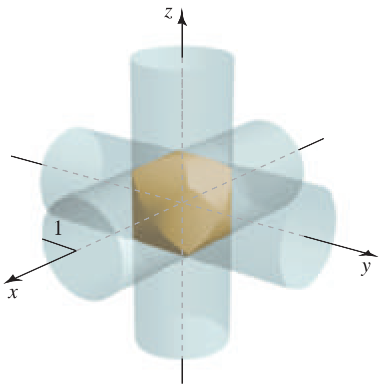
62. The solid in the first octant bounded by the cone $z = 1 - \sqrt{x^2 + y^2}$ and the plane $x + y + z = 1$



63. Two cylinders The x- and y-axes form the axes of two right circular cylinders with radius 1 (see figure). Find the volume of the solid that is common to the two cylinders.



64. Three cylinders The coordinate axes form the axes of three right circular cylinders with radius 1 (see figure). Find the volume of the solid that is common to the three cylinders.



65. Dividing the cheese Suppose a wedge of cheese fills the region in the first octant bounded by the planes $y = z$, $y = 4$, and $x = 4$. You could divide the wedge into two pieces of equal volume by slicing the wedge with the plane $x = 2$. Instead find a with $0 < a < 4$ such that slicing the wedge with the plane $y = a$ divides the wedge into two pieces of equal volume.

66. Partitioning a cube Consider the region $D_1 = \{(x, y, z): 0 \le x \le y \le z \le 1\}$.

a. Find the volume of D_1.

b. Let $D_{2,\ldots,}D_6$ be the "cousins" of D_1 formed by rearranging x, y, and z in the inequality $0 \le x \le y \le z \le 1$. Show that the volumes of $D_1, \ldots, D_6$ are equal.

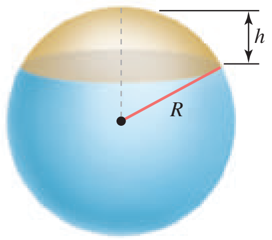
c. Show that the union of $D_1, \ldots, D_6$ is a unit cube.

67–71. General volume formulas *Find equations for the bounding surfaces, set up a volume integral, and evaluate the integral to obtain a volume formula for each region. Assume a, b, c, r, R, and h are positive constants.*

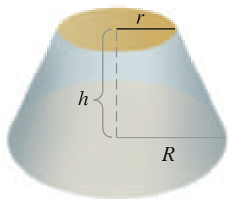
67. Cone Find the volume of a right circular cone with height h and base radius r.

68. Tetrahedron Find the volume of a tetrahedron whose vertices are located at $(0, 0, 0)$, $(a, 0, 0)$, $(0, b, 0)$, and $(0, 0, c)$.

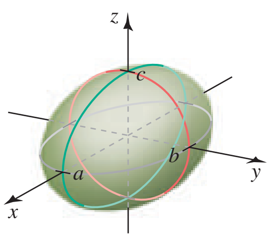
69. Spherical cap Find the volume of the cap of a sphere of radius R with height h.



70. Frustum of a cone Find the volume of a truncated cone of height h whose ends have radii r and R.



71. Ellipsoid Find the volume of an ellipsoid with axes of lengths $2a$, $2b$, and $2c$.



72. Exponential distribution The occurrence of random events (such as phone calls or e-mail messages) is often idealized using an exponential distribution. If λ is the average rate of occurrence of such an event, assumed to be constant over time, then the average time between occurrences is λ^{-1} (for example, if phone calls arrive at a rate of $\lambda = 2/\text{min}$, then the mean time between phone calls is $\lambda^{-1} = 1/2$ min). The exponential distribution is given by $f(t) = \lambda e^{-\lambda t}$, for $0 \le t < \infty$.

a. Suppose you work at a customer service desk and phone calls arrive at an average rate of $\lambda_1 = 0.8/\text{min}$ (meaning the average time between phone calls is $1/0.8 = 1.25$ min). The probability that a phone call arrives during the interval $[0, T]$ is $p(T) = \int_0^T \lambda_1 e^{-\lambda_1 t}\,dt$. Find the probability that a phone call arrives during the first 45 s (0.75 min) that you work at the desk.

b. Now suppose walk-in customers also arrive at your desk at an average rate of $\lambda_2 = 0.1/\text{min}$. The probability that a phone call *and* a customer arrive during the interval $[0, T]$ is

$$p(T) = \int_0^T \int_0^T \lambda_1 e^{-\lambda_1 t} \lambda_2 e^{-\lambda_2 s}\,dt\,ds.$$

Find the probability that a phone call and a customer arrive during the first 45 s that you work at the desk.

c. E-mail messages also arrive at your desk at an average rate of $\lambda_3 = 0.05/\text{min}$. The probability that a phone call *and* a customer *and* an e-mail message arrive during the interval $[0, T]$ is

$$p(T) = \int_0^T \int_0^T \int_0^T \lambda_1 e^{-\lambda_1 t} \lambda_2 e^{-\lambda_2 s} \lambda_3 e^{-\lambda_3 u}\,dt\,ds\,du.$$

Find the probability that a phone call and a customer and an e-mail message arrive during the first 45 s that you work at the desk.

73. Hypervolume Find the "volume" of the four-dimensional pyramid bounded by $w + x + y + z + 1 = 0$ and the coordinate planes $w = 0$, $x = 0$, $y = 0$, $z = 0$.

74. An identity (Putnam Exam 1941) Let f be a continuous function on $[0, 1]$. Prove that

$$\int_0^1 \int_x^1 \int_x^y f(x)f(y)f(z)\,dz\,dy\,dx = \frac{1}{6}\left(\int_0^1 f(x)\,dx\right)^3.$$

QUICK CHECK ANSWERS

1. $dx\,dy\,dz$, $dx\,dz\,dy$, $dy\,dx\,dz$, $dy\,dz\,dx$, $dz\,dx\,dy$, $dz\,dy\,dx$

2. $\int_0^1 \int_0^2 \int_0^3 (2 - z)\,dx\,dy\,dz$, $\int_0^2 \int_0^1 \int_0^3 (2 - z)\,dx\,dz\,dy$

3. $\int_0^2 \int_0^{4-2x} \int_0^6 dz\,dy\,dx$, $\int_0^6 \int_0^4 \int_0^{2-y/2} dx\,dy\,dz$

4. 0 ($\sin x$, $\sin y$, and $\sin z$ are odd functions.) ◄

16.5 Triple Integrals in Cylindrical and Spherical Coordinates

When evaluating triple integrals, you may have noticed that some regions (such as spheres, cones, and cylinders) have awkward descriptions in Cartesian coordinates. In this section, we examine two other coordinate systems in $\mathbb{R}^3$ that are easier to use when working with certain types of regions. These coordinate systems are helpful not only for integration, but also for general problem solving.

Cylindrical Coordinates

➤ In cylindrical coordinates, r and θ are the usual polar coordinates, with the additional restriction that $r \geq 0$. Adding the z-coordinate lifts points in the polar plane into $\mathbb{R}^3$.

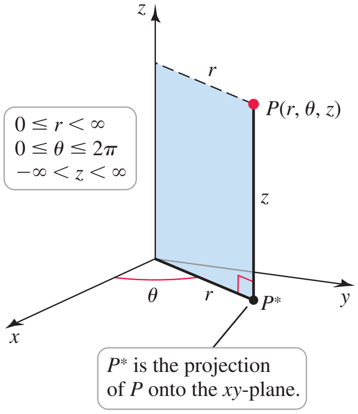


Figure 16.47

When we extend polar coordinates from $\mathbb{R}^2$ to $\mathbb{R}^3$, the result is *cylindrical coordinates*. In this coordinate system, a point P in $\mathbb{R}^3$ has coordinates (r, θ, z), where r and θ are polar coordinates for the point P^*, which is the projection of P onto the xy-plane (**Figure 16.47**). As in Cartesian coordinates, the z-coordinate is the signed vertical distance between P and the xy-plane. Any point in $\mathbb{R}^3$ can be represented by cylindrical coordinates using the intervals $0 \leq r < \infty$, $0 \leq \theta \leq 2\pi$, and $-\infty < z < \infty$.

Many sets of points have simple representations in cylindrical coordinates. For example, the set $\{(r, \theta, z): r = a\}$ is the set of points whose distance from the z-axis is a, which is a right circular cylinder of radius a. The set $\{(r, \theta, z): \theta = \theta_0\}$ is the set of points with a constant θ coordinate; it is a vertical half-plane emanating from the z-axis in the direction $\theta = \theta_0$. Table 16.4 summarizes these and other sets that are ideal for integration in cylindrical coordinates.

Table 16.4

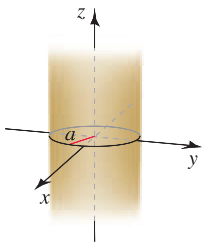
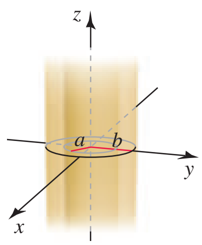
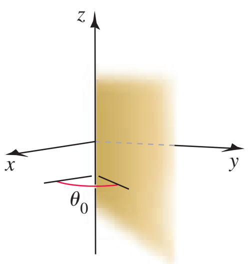
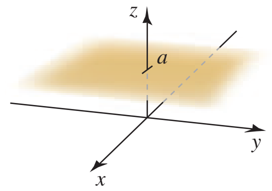
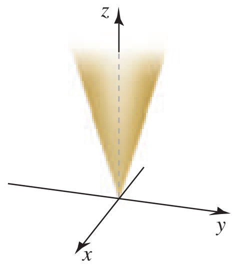
Name	Description	Example
Cylinder	$\{(r, \theta, z): r = a\}, a > 0$	
Cylindrical shell	$\{(r, \theta, z): 0 < a \leq r \leq b\}$	
Vertical half-plane	$\{(r, \theta, z): \theta = \theta_0\}$	

Table 16.4 (Continued)

Name	Description	Example
Horizontal plane	$\{(r, \theta, z): z = a\}$	
Cone	$\{(r, \theta, z): z = ar\}, a \neq 0$	

EXAMPLE 1 **Sets in cylindrical coordinates** Identify and sketch the following sets in cylindrical coordinates.

a. $Q = \{(r, \theta, z): 1 \leq r \leq 3, z \geq 0\}$

b. $S = \{(r, \theta, z): z = 1 - r, 0 \leq r \leq 1\}$

SOLUTION

a. The set Q is a cylindrical shell with inner radius 1 and outer radius 3 that extends indefinitely along the positive z-axis (Figure 16.48a). Because θ is unspecified, it takes on all values.

b. To identify this surface, it helps to work in steps. The set $S_1 = \{(r, \theta, z): z = r\}$ is a cone that opens *upward* with its vertex at the origin. Similarly, the set $S_2 = \{(r, \theta, z): z = -r\}$ is a cone that opens *downward* with its vertex at the origin. Therefore, S is S_2 shifted vertically upward by 1 unit; it is a cone that opens downward with its vertex at $(0, 0, 1)$. Because $0 \leq r \leq 1$, the base of the cone is on the xy-plane (Figure 16.48b).

Related Exercise 11 ◄

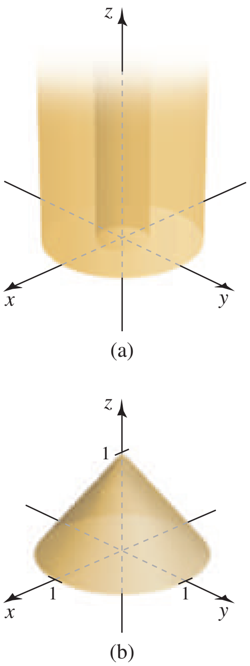


Figure 16.48

Equations for transforming Cartesian coordinates to cylindrical coordinates, and vice versa, are often needed for integration. We simply use the rules for polar coordinates (Section 12.2) with no change in the z-coordinate (Figure 16.49).

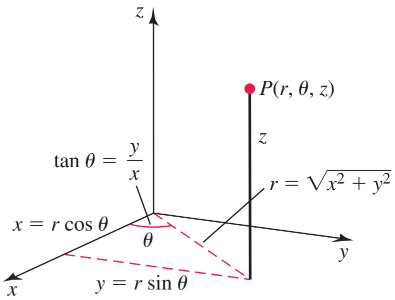


Figure 16.49

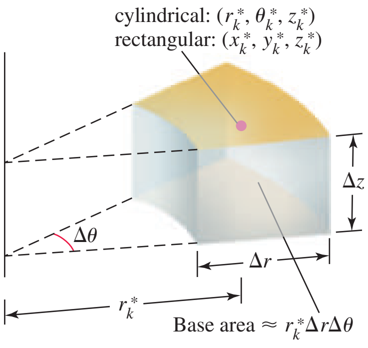
QUICK CHECK 1 Find the cylindrical coordinates of the point with rectangular coordinates $(1, -1, 5)$. Find the rectangular coordinates of the point with cylindrical coordinates $(2, \pi/3, 5)$. ◄

Transformations Between Cylindrical and Rectangular Coordinates

Rectangular → Cylindrical	Cylindrical → Rectangular
$r^2 = x^2 + y^2$	$x = r\cos\theta$
$\tan\theta = y/x$	$y = r\sin\theta$
$z = z$	$z = z$

Integration in Cylindrical Coordinates

Among the uses of cylindrical coordinates is the evaluation of triple integrals of the form $\iiint_D f(x, y, z)\, dV$. We begin with a region D in $\mathbb{R}^3$ and partition it into cylindrical wedges formed by changes of Δr, $\Delta\theta$, and Δz in the coordinate directions (Figure 16.50). Those wedges that lie entirely within D are labeled from $k = 1$ to $k = n$ in some convenient order. We let $(r_k^*, \theta_k^*, z_k^*)$ be the cylindrical coordinates of an arbitrary point in the kth wedge. This point also has Cartesian coordinates $(x_k^*, y_k^*, z_k^*) = (r_k^* \cos\theta_k^*, r_k^* \sin\theta_k^*, z_k^*)$.



Approximate volume $\Delta V_k = r_k^* \Delta r \Delta\theta \Delta z$

Figure 16.50

As shown in Figure 16.50, the base of the kth wedge is a polar rectangle with an approximate area of $r_k^* \Delta r \Delta\theta$ (Section 16.3). The height of the wedge is Δz. Multiplying these dimensions together, the approximate volume of the wedge is $\Delta V_k = r_k^* \Delta r \Delta\theta \Delta z$, for $k = 1, \ldots, n$.

We now assume $f(x, y, z)$ is continuous on D and form a Riemann sum over the region by adding function values multiplied by the corresponding approximate volumes:

$$\sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\Delta V_k = \sum_{k=1}^{n} f(r_k^* \cos\theta_k^*, r_k^* \sin\theta_k^*, z_k^*)\Delta V_k.$$

Let Δ be the maximum value of Δr, $\Delta\theta$, and Δz, for $k = 1, \ldots, n$. As $n \to \infty$ and $\Delta \to 0$, the Riemann sums approach a limit called the **triple integral of f over D in cylindrical coordinates**:

$$\iiint_D f(x, y, z)\, dV = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(r_k^* \cos\theta_k^*, r_k^* \sin\theta_k^*, z_k^*)\underbrace{r_k^* \Delta r \Delta\theta \Delta z}_{\Delta V_k}.$$

The rightmost sum tells us how to write a triple integral in x, y, and z as an iterated integral of $f(r\cos\theta, r\sin\theta, z)\,r$ in cylindrical coordinates.

Finding Limits of Integration We show how to find the limits of integration in one common situation involving cylindrical coordinates. Suppose D is a region in $\mathbb{R}^3$ consisting of points between the surfaces $z = G(x, y)$ and $z = H(x, y)$, where x and y belong to a region R in the xy-plane and $G(x, y) \le H(x, y)$ on R (Figure 16.51). Assuming f is continuous on D, the triple integral of f over D may be expressed as the iterated integral

$$\iiint_D f(x, y, z)\, dV = \iint_R \left(\int_{G(x, y)}^{H(x, y)} f(x, y, z)\, dz \right) dA.$$

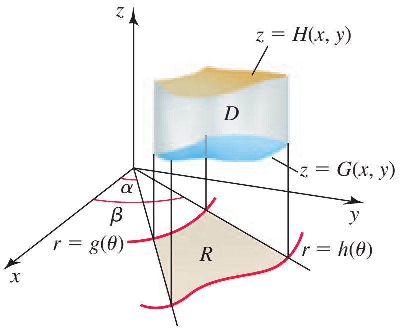


Figure 16.51

The inner integral with respect to z runs from the lower surface $z = G(x, y)$ to the upper surface $z = H(x, y)$, leaving an outer double integral over R.

If the region R is described in polar coordinates by

$$\{(r, \theta): g(\theta) \le r \le h(\theta), \alpha \le \theta \le \beta\},$$

then we evaluate the double integral over R in polar coordinates (Section 16.3). The effect is a change of variables from rectangular to cylindrical coordinates. Letting $x = r\cos\theta$ and $y = r\sin\theta$, we have the following result, which is another change of variables formula.

> The order of the differentials specifies the order in which the integrals are evaluated, so we write the volume element dV as $dz\, r\, dr\, d\theta$. Do not lose sight of the factor of r in the integrand. It plays the same role as it does in the area element $dA = r\, dr\, d\theta$ in polar coordinates.

THEOREM 16.6 Change of Variables for Triple Integrals in Cylindrical Coordinates

Let f be continuous over the region D, expressed in cylindrical coordinates as

$$D = \{(r, \theta, z): 0 \le g(\theta) \le r \le h(\theta), \alpha \le \theta \le \beta, G(x, y) \le z \le H(x, y)\}.$$

Then f is integrable over D, and the triple integral of f over D is

$$\iiint_D f(x, y, z)\, dV = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} \int_{G(r\cos\theta,\, r\sin\theta)}^{H(r\cos\theta,\, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, dz\, r\, dr\, d\theta.$$

Notice that the integrand and the limits of integration are converted from Cartesian to cylindrical coordinates. As with triple integrals in Cartesian coordinates, there are two immediate interpretations of this integral. If $f = 1$, then the triple integral $\iiint_D dV$ equals the volume of the region D. Also, if f describes the density of an object occupying the region D, the triple integral equals the mass of the object.

EXAMPLE 2 **Switching coordinate systems** Evaluate the integral

$$I = \int_0^{2\sqrt{2}} \int_{-\sqrt{8-x^2}}^{\sqrt{8-x^2}} \int_{-1}^{2} \sqrt{1 + x^2 + y^2}\, dz\, dy\, dx.$$

SOLUTION Evaluating this integral as it is given in Cartesian coordinates requires a tricky trigonometric substitution in the middle integral, followed by an even more difficult integral. Notice that z varies between the planes $z = -1$ and $z = 2$, while x and y vary over half of a disk in the xy-plane. Therefore, D is half of a solid cylinder (Figure 16.52a), which suggests a change to cylindrical coordinates.

The limits of integration in cylindrical coordinates are determined as follows:

Inner integral with respect to z A line through the half cylinder parallel to the z-axis enters at $z = -1$ and leaves at $z = 2$, so we integrate over the interval $-1 \le z \le 2$ (Figure 16.52b).

Middle integral with respect to r The projection of the half cylinder onto the xy-plane is the half disk R of radius $2\sqrt{2}$ centered at the origin, so r varies over the interval $0 \le r \le 2\sqrt{2}$ (Figure 16.52c).

Outer integral with respect to θ The half disk R is swept out by letting θ vary over the interval $-\pi/2 \le \theta \le \pi/2$ (Figure 16.52c).

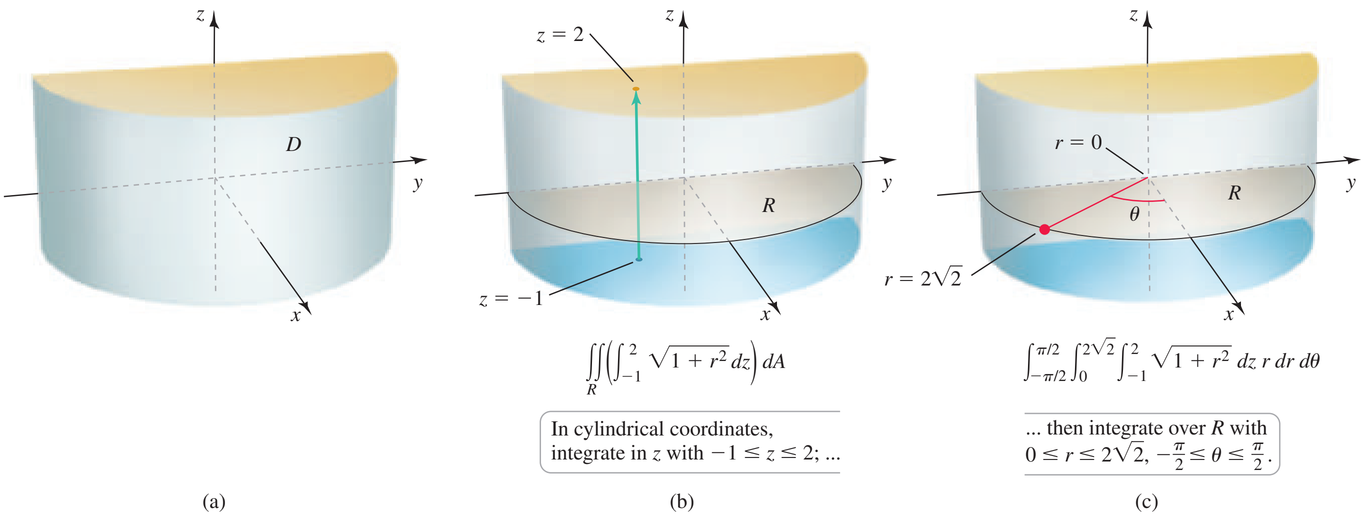


Figure 16.52

We also convert the integrand to cylindrical coordinates:

$$f(x, y, z) = \sqrt{1 + \underbrace{x^2 + y^2}_{r^2}} = \sqrt{1 + r^2}.$$

The evaluation of the integral in cylindrical coordinates now follows:

$$\begin{aligned} I &= \int_{-\pi/2}^{\pi/2} \int_0^{2\sqrt{2}} \int_{-1}^{2} \sqrt{1 + r^2}\, dz\, r\, dr\, d\theta && \text{Convert to cylindrical coordinates.} \\ &= 3 \int_{-\pi/2}^{\pi/2} \int_0^{2\sqrt{2}} \sqrt{1 + r^2}\, r\, dr\, d\theta && \text{Evaluate inner integral with respect to } z. \\ &= \int_{-\pi/2}^{\pi/2} (1 + r^2)^{3/2} \Big|_0^{2\sqrt{2}}\, d\theta && \text{Evaluate middle integral with respect to } r. \\ &= \int_{-\pi/2}^{\pi/2} 26\, d\theta = 26\pi. && \text{Evaluate outer integral with respect to } \theta. \end{aligned}$$

Related Exercises 17–18 ◄

QUICK CHECK 2 Find the limits of integration for a triple integral in cylindrical coordinates that gives the volume of a cylinder with height 20 and a circular base of radius 10 centered at the origin in the xy-plane. ◄

As illustrated in Example 2, triple integrals given in rectangular coordinates may be more easily evaluated after converting to cylindrical coordinates. Answering the following questions may help you choose the best coordinate system for a particular integral.

- In which coordinate system is the region of integration most easily described?
- In which coordinate system is the integrand most easily expressed?
- In which coordinate system is the triple integral most easily evaluated?

In general, if an integral in one coordinate system is difficult to evaluate, consider using a different coordinate system.

EXAMPLE 3 **Mass of a solid paraboloid** Find the mass of the solid D bounded by the paraboloid $z = 4 - r^2$ and the plane $z = 0$ (Figure 16.53a), where the density of the solid, given in cylindrical coordinates, is $f(r, \theta, z) = 5 - z$ (heavy near the base and light near the vertex).

SOLUTION The z-coordinate runs from the base $z = 0$ to the surface $z = 4 - r^2$ (Figure 16.53b). The projection R of the region D onto the xy-plane is found by setting $z = 0$ in the equation of the surface, $z = 4 - r^2$. The positive value of r satisfying the equation $4 - r^2 = 0$ is $r = 2$, so in polar coordinates $R = \{(r, \theta): 0 \le r \le 2, 0 \le \theta \le 2\pi\}$, which is a disk of radius 2 (Figure 16.53c).

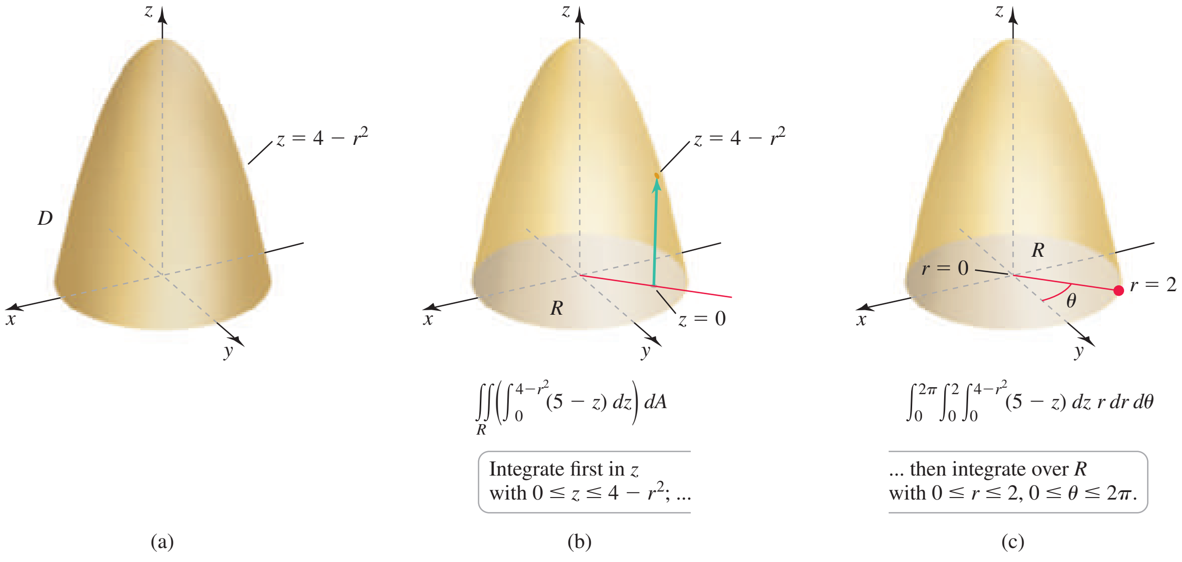


Figure 16.53

➤ In Example 3, the integrand is independent of θ, so the integral with respect to θ could have been done first, producing a factor of 2π.

The mass is computed by integrating the density function over D:

$$\iiint_D f(r, \theta, z)\, dV = \int_0^{2\pi}\int_0^2\int_0^{4-r^2} (5 - z)\, dz\, r\, dr\, d\theta \quad \text{Integrate density.}$$

$$= \int_0^{2\pi}\int_0^2 \left(5z - \frac{z^2}{2}\right)\Bigg|_0^{4-r^2} r\, dr\, d\theta \quad \text{Evaluate inner integral with respect to } z.$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^2 (24r - 2r^3 - r^5)\, dr\, d\theta \quad \text{Simplify.}$$

$$= \int_0^{2\pi} \frac{44}{3}\, d\theta \quad \text{Evaluate middle integral with respect to } r.$$

$$= \frac{88\pi}{3}. \quad \text{Evaluate outer integral with respect to } \theta.$$

Related Exercises 25–26 ◄

➤ Recall that to find the volume of a region D using a triple integral, we set $f = 1$ and evaluate

$$V = \iiint_D dV.$$

EXAMPLE 4 **Volume between two surfaces** Find the volume of the solid D between the cone $z = \sqrt{x^2 + y^2}$ and the inverted paraboloid $z = 12 - x^2 - y^2$ (Figure 16.54a).

SOLUTION Because $x^2 + y^2 = r^2$, the equation of the cone in cylindrical coordinates becomes $z = r$, and the equation of the paraboloid becomes $z = 12 - r^2$. The inner integral in z runs from the cone $z = r$ (the lower surface) to the paraboloid $z = 12 - r^2$ (the upper surface) (Figure 16.54b). We project D onto the xy-plane to produce the region R, whose boundary is determined by the intersection of the two surfaces. Equating the z-coordinates in the equations of the two surfaces, we have $12 - r^2 = r$, or $(r - 3)(r + 4) = 0$. Because $r \geq 0$, the relevant root is $r = 3$. Therefore, the projection of D onto the xy-plane is the polar region $R = \{(r, \theta): 0 \leq r \leq 3, 0 \leq \theta \leq 2\pi\}$, which is a disk of radius 3 centered at $(0, 0)$ (Figure 16.54c).

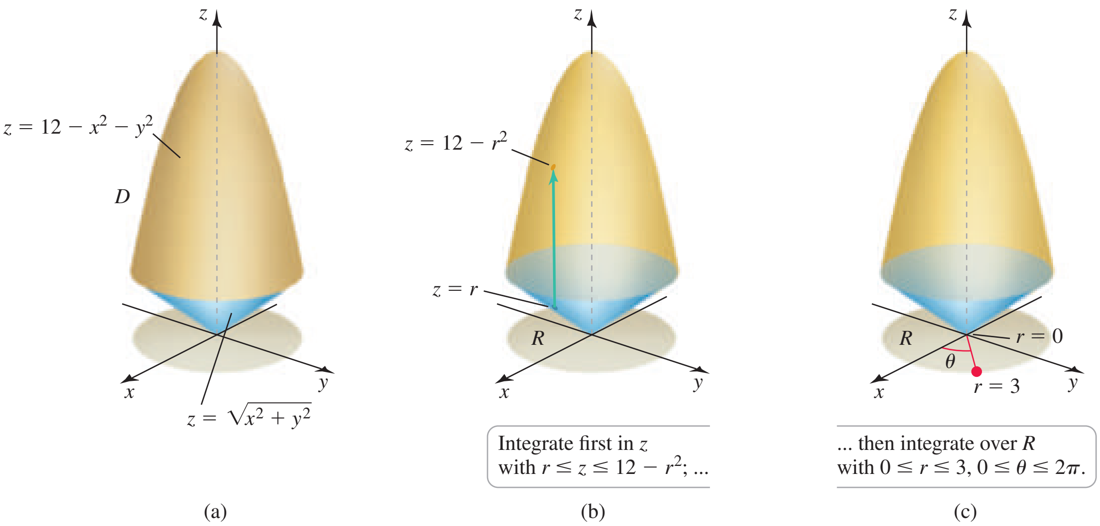


Figure 16.54

The volume of the region is

$$\begin{aligned}
\iiint_D dV &= \int_0^{2\pi}\int_0^3\int_r^{12-r^2} dz\, r\, dr\, d\theta \\
&= \int_0^{2\pi}\int_0^3 (12 - r^2 - r)\, r\, dr\, d\theta && \text{Evaluate inner integral with respect to } z. \\
&= \int_0^{2\pi} \frac{99}{4}\, d\theta && \text{Evaluate middle integral with respect to } r. \\
&= \frac{99\pi}{2}. && \text{Evaluate outer integral with respect to } \theta.
\end{aligned}$$

Related Exercises 30–31 ◄

Spherical Coordinates

➤ The coordinate ρ (pronounced "rho") in spherical coordinates should not be confused with r in cylindrical coordinates, which is the distance from P to the z-axis.

➤ The coordinate φ is called the *colatitude* because it is $\pi/2$ minus the latitude of points in the Northern Hemisphere. Physicists may reverse the roles of θ and φ; that is, θ is the colatitude and φ is the polar angle.

In spherical coordinates, a point P in $\mathbb{R}^3$ is represented by three coordinates (ρ, φ, θ) (Figure 16.55).

- ρ is the distance from the origin to P.
- φ is the angle between the positive z-axis and the line OP.
- θ is the same angle as in cylindrical coordinates; it measures rotation about the z-axis relative to the positive x-axis.

All points in $\mathbb{R}^3$ can be represented by spherical coordinates using the intervals $0 \leq \rho < \infty$, $0 \leq \varphi \leq \pi$, and $0 \leq \theta \leq 2\pi$.

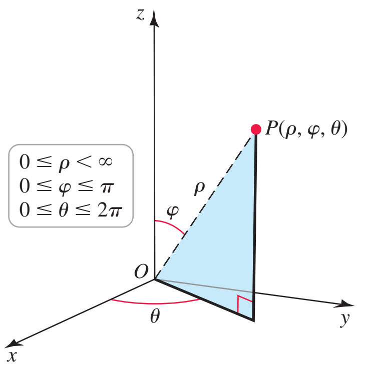


Figure 16.55

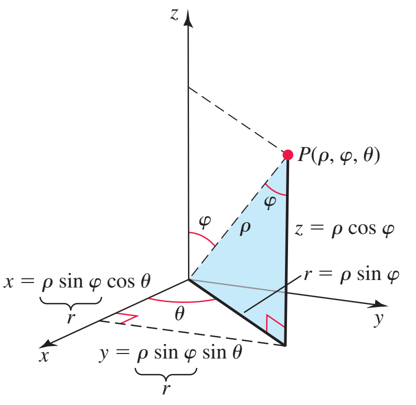


Figure 16.56

QUICK CHECK 3 Find the spherical coordinates of the point with rectangular coordinates $(1, \sqrt{3}, 2)$. Find the rectangular coordinates of the point with spherical coordinates $(2, \pi/4, \pi/4)$. ◄

Figure 16.56 allows us to find the relationships among rectangular and spherical coordinates. Given the spherical coordinates (ρ, φ, θ) of a point P, the distance from P to the z-axis is $r = \rho \sin \varphi$. We also see from Figure 16.56 that $x = r \cos \theta = \rho \sin \varphi \cos \theta$, $y = r \sin \theta = \rho \sin \varphi \sin \theta$, and $z = \rho \cos \varphi$.

Transformations Between Spherical and Rectangular Coordinates

Rectangular → Spherical	**Spherical → Rectangular**
$\rho^2 = x^2 + y^2 + z^2$	$x = \rho \sin \varphi \cos \theta$
Use trigonometry to find	$y = \rho \sin \varphi \sin \theta$
φ and θ.	$z = \rho \cos \varphi$

In spherical coordinates, some sets of points have simple representations. For instance, the set $\{(\rho, \varphi, \theta): \rho = a\}$ is the set of points whose ρ-coordinate is constant, which is a sphere of radius a centered at the origin. The set $\{(\rho, \varphi, \theta): \varphi = \varphi_0\}$ is the set of points with a constant φ-coordinate; it is a cone with its vertex at the origin and whose sides make an angle φ_0 with the positive z-axis.

EXAMPLE 5 **Sets in spherical coordinates** Express the following sets in rectangular coordinates and identify the set. Assume a is a positive real number.

a. $\{(\rho, \varphi, \theta): \rho = 2a \cos \varphi, 0 \le \varphi \le \pi/2, 0 \le \theta \le 2\pi\}$

b. $\{(\rho, \varphi, \theta): \rho = 4 \sec \varphi, 0 \le \varphi < \pi/2, 0 \le \theta \le 2\pi\}$

SOLUTION

a. To avoid working with square roots, we multiply both sides of $\rho = 2a \cos \varphi$ by ρ to obtain $\rho^2 = 2a\,\rho \cos \varphi$. Substituting rectangular coordinates, we have $x^2 + y^2 + z^2 = 2az$. Completing the square results in the equation

$$x^2 + y^2 + (z - a)^2 = a^2.$$

This is the equation of a sphere centered at $(0, 0, a)$ with radius a (Figure 16.57a). With the limits $0 \le \varphi \le \pi/2$ and $0 \le \theta \le 2\pi$, the set describes a full sphere.

b. The equation $\rho = 4 \sec \varphi$ is first written $\rho \cos \varphi = 4$. Noting that $z = \rho \cos \varphi$, the set consists of all points with $z = 4$, which is a horizontal plane (Figure 16.57b).

Related Exercises 37–38 ◄

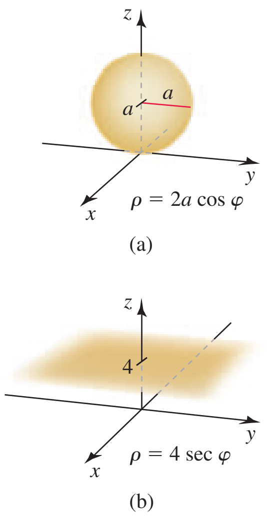
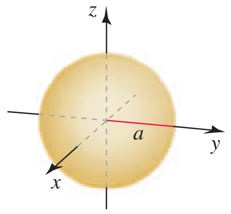
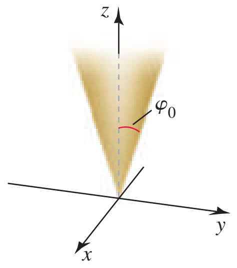
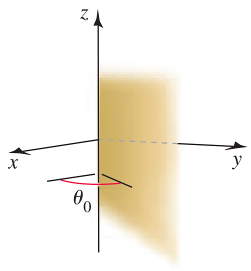
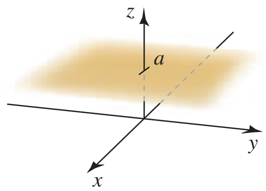
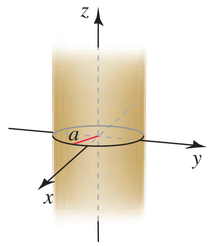
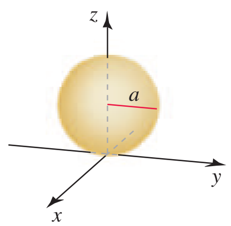


Figure 16.57

Table 16.5 summarizes some sets that have simple descriptions in spherical coordinates.

▶ Notice that the set of points (ρ, φ, θ) with $\varphi = \pi/2$ is the xy-plane, and if $\pi/2 < \varphi_0 < \pi$, the set of points with $\varphi = \varphi_0$ is a cone that opens downward.

Table 16.5

Name	Description	Example
Sphere, radius a, center $(0, 0, 0)$	$\{(\rho, \varphi, \theta): \rho = a\}, a > 0$	
Cone	$\{(\rho, \varphi, \theta): \varphi = \varphi_0\}, \varphi_0 \neq 0, \pi/2, \pi$	
Vertical half-plane	$\{(\rho, \varphi, \theta): \theta = \theta_0\}$	
Horizontal plane, $z = a$	$a > 0$: $\{(\rho, \varphi, \theta): \rho = a \sec \varphi, 0 \leq \varphi < \pi/2\}$ $a < 0$: $\{(\rho, \varphi, \theta): \rho = a \sec \varphi, \pi/2 < \varphi \leq \pi\}$	
Cylinder, radius $a > 0$	$\{(\rho, \varphi, \theta): \rho = a \csc \varphi, 0 < \varphi < \pi\}$	
Sphere, radius $a > 0$, center $(0, 0, a)$	$\{(\rho, \varphi, \theta): \rho = 2a \cos \varphi, 0 \leq \varphi \leq \pi/2\}$	

Integration in Spherical Coordinates

We now investigate triple integrals in spherical coordinates over a region D in $\mathbb{R}^3$. The region D is partitioned into "spherical boxes" that are formed by changes of $\Delta\rho$, $\Delta\varphi$, and $\Delta\theta$ in the coordinate directions (Figure 16.58). Those boxes that lie entirely within D are labeled from $k = 1$ to $k = n$. We let $(\rho_k^*, \varphi_k^*, \theta_k^*)$ be the spherical coordinates for an arbitrary point in the kth box. This point also has Cartesian coordinates

$$(x_k^*, y_k^*, z_k^*) = (\rho_k^* \sin \varphi_k^* \cos \theta_k^*, \rho_k^* \sin \varphi_k^* \sin \theta_k^*, \rho_k^* \cos \varphi_k^*).$$

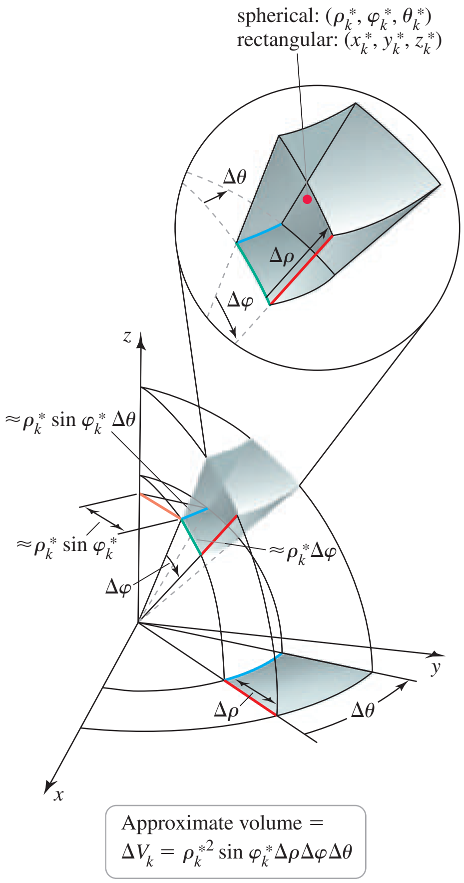


Figure 16.58

➤ Recall that the length s of a circular arc of radius r subtended by an angle θ is $s = r\theta$.

To approximate the volume of a typical box, note that the length of the box in the ρ-direction is $\Delta\rho$ (Figure 16.58). The approximate length of the kth box in the θ-direction is the length of an arc of a circle of radius $\rho_k^* \sin \varphi_k^*$ subtended by an angle $\Delta\theta$; this length is $\rho_k^* \sin \varphi_k^* \Delta\theta$. The approximate length of the box in the φ-direction is the length of an arc of radius ρ_k^* subtended by an angle $\Delta\varphi$; this length is $\rho_k^* \Delta\varphi$. Multiplying these dimensions together, the approximate volume of the kth spherical box is $\Delta V_k = \rho_k^{*2} \sin \varphi_k^* \Delta\rho\Delta\varphi\Delta\theta$, for $k = 1, \ldots, n$.

We now assume $f(x, y, z)$ is continuous on D and form a Riemann sum over the region by adding function values multiplied by the corresponding approximate volumes:

$$\sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\Delta V_k = \sum_{k=1}^{n} f(\rho_k^* \sin \varphi_k^* \cos \theta_k^*, \rho_k^* \sin \varphi_k^* \sin \theta_k^*, \rho_k^* \cos \varphi_k^*)\Delta V_k.$$

We let Δ denote the maximum value of $\Delta\rho$, $\Delta\varphi$, and $\Delta\theta$. As $n \to \infty$ and $\Delta \to 0$, the Riemann sums approach a limit called the **triple integral of f over D in spherical coordinates**:

$$\iiint_D f(x, y, z)\, dV$$

$$= \lim_{\Delta\to 0} \sum_{k=1}^{n} f(\rho_k^* \sin \varphi_k^* \cos \theta_k^*, \rho_k^* \sin \varphi_k^* \sin \theta_k^*, \rho_k^* \cos \varphi_k^*)\underbrace{\rho_k^{*2} \sin \varphi_k^* \Delta\rho\Delta\varphi\Delta\theta}_{\Delta V_k}.$$

The rightmost sum tells us how to write a triple integral in x, y, and z as an iterated integral of $f(\rho \sin \varphi \cos \theta, \rho \sin \varphi \sin \theta, \rho \cos \varphi)\rho^2 \sin \varphi$ in spherical coordinates.

Finding Limits of Integration We consider a common situation in which the region of integration D, expressed in spherical coordinates, has the form

$$D = \{(\rho, \varphi, \theta): 0 \le g(\varphi, \theta) \le \rho \le h(\varphi, \theta), a \le \varphi \le b, \alpha \le \theta \le \beta\}.$$

In other words, D is bounded in the ρ-direction by two surfaces given by g and h. In the angular directions, the region lies between two cones $(a \le \varphi \le b)$ and two half-planes $(\alpha \le \theta \le \beta)$ (Figure 16.59).

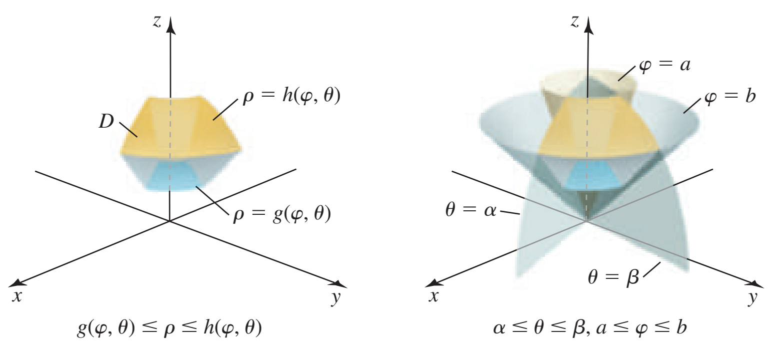


Figure 16.59

For this type of region, the inner integral is with respect to ρ, which varies from $\rho = g(\varphi, \theta)$ to $\rho = h(\varphi, \theta)$. As ρ varies between these limits, imagine letting θ and φ vary over the intervals $a \le \varphi \le b$ and $\alpha \le \theta \le \beta$. The effect is to sweep out all points of D. Notice that the middle and outer integrals, with respect to θ and φ, may be done in either order (Figure 16.60).

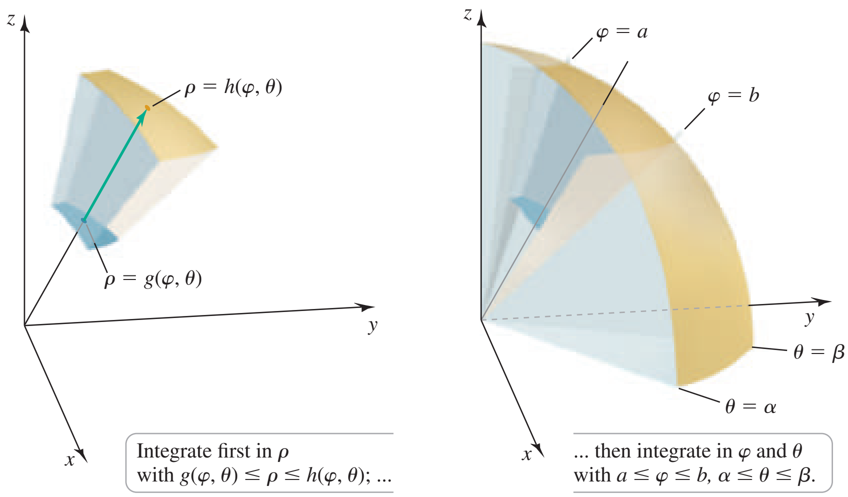


Figure 16.60

In summary, to integrate over all points of D, we carry out the following steps.

1. Integrate with respect to ρ from $\rho = g(\varphi, \theta)$ to $\rho = h(\varphi, \theta)$; the result (in general) is a function of φ and θ.
2. Integrate with respect to φ from $\varphi = a$ to $\varphi = b$; the result (in general) is a function of θ.
3. Integrate with respect to θ from $\theta = \alpha$ to $\theta = \beta$; the result is (always) a real number.

Another change of variables expresses the triple integral as an iterated integral in spherical coordinates.

➤ The element of volume in spherical coordinates is $dV = \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$.

THEOREM 16.7 Change of Variables for Triple Integrals in Spherical Coordinates

Let f be continuous over the region D, expressed in spherical coordinates as

$$D = \{(\rho, \varphi, \theta): 0 \le g(\varphi, \theta) \le \rho \le h(\varphi, \theta), a \le \varphi \le b, \alpha \le \theta \le \beta\}.$$

Then f is integrable over D, and the triple integral of f over D is

$$\iiint_D f(x, y, z)\, dV$$
$$= \int_\alpha^\beta \int_a^b \int_{g(\varphi,\theta)}^{h(\varphi,\theta)} f(\rho \sin\varphi \cos\theta, \rho \sin\varphi \sin\theta, \rho\cos\varphi)\rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta.$$

If the integrand is given in terms of Cartesian coordinates x, y, and z, it must be expressed in spherical coordinates before integrating. As with other triple integrals, if $f = 1$, then the triple integral equals the volume of D. If f is a density function for an object occupying the region D, then the triple integral equals the mass of the object.

EXAMPLE 6 A triple integral Evaluate $\iiint_D (x^2 + y^2 + z^2)^{-3/2}\, dV$, where D is the region in the first octant between two spheres of radius 1 and 2 centered at the origin.

SOLUTION Both the integrand f and the region D are greatly simplified when expressed in spherical coordinates. The integrand becomes

$$(x^2 + y^2 + z^2)^{-3/2} = (\rho^2)^{-3/2} = \rho^{-3},$$

while the region of integration (**Figure 16.61**) is

$$D = \{(\rho, \varphi, \theta): 1 \le \rho \le 2, 0 \le \varphi \le \pi/2, 0 \le \theta \le \pi/2\}.$$

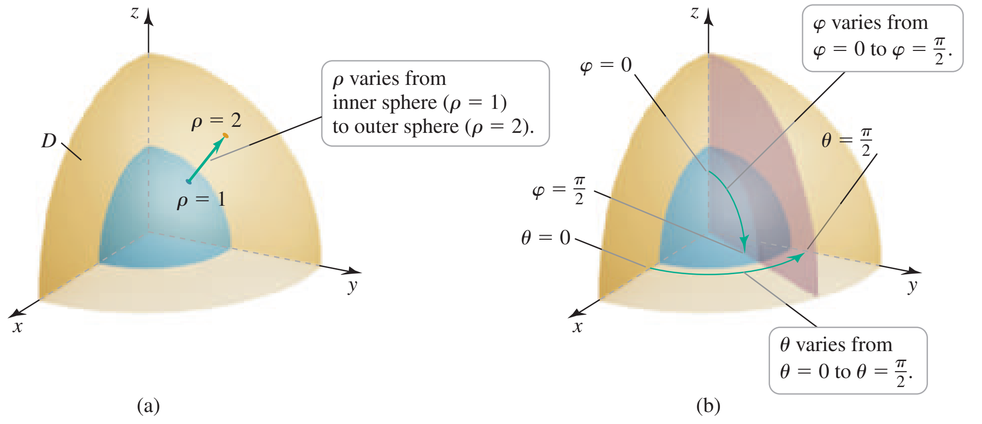


Figure 16.61

The integral is evaluated as follows:

$$\iiint_D f(x, y, z)\, dV = \int_0^{\pi/2}\int_0^{\pi/2}\int_1^2 \rho^{-3}\, \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta \qquad \text{Convert to spherical coordinates.}$$

$$= \int_0^{\pi/2}\int_0^{\pi/2}\int_1^2 \rho^{-1} \sin\varphi\, d\rho\, d\varphi\, d\theta \qquad \text{Simplify.}$$

$$= \int_0^{\pi/2}\int_0^{\pi/2} \ln|\rho|\Big|_1^2 \sin\varphi\, d\varphi\, d\theta \qquad \text{Evaluate inner integral with respect to } \rho.$$

$$= \ln 2 \int_0^{\pi/2}\int_0^{\pi/2} \sin\varphi\, d\varphi\, d\theta \qquad \text{Simplify.}$$

$$= \ln 2 \int_0^{\pi/2} (-\cos\varphi)\Big|_0^{\pi/2} d\theta \qquad \text{Evaluate middle integral with respect to } \varphi.$$

$$= \ln 2 \int_0^{\pi/2} d\theta = \frac{\pi \ln 2}{2}. \qquad \text{Evaluate outer integral with respect to } \theta.$$

Related Exercises 41, 43 ◀

EXAMPLE 7 **Ice cream cone** Find the volume of the solid region D that lies inside the cone $\varphi = \pi/6$ and inside the sphere $\rho = 4$ (**Figure 16.62a**).

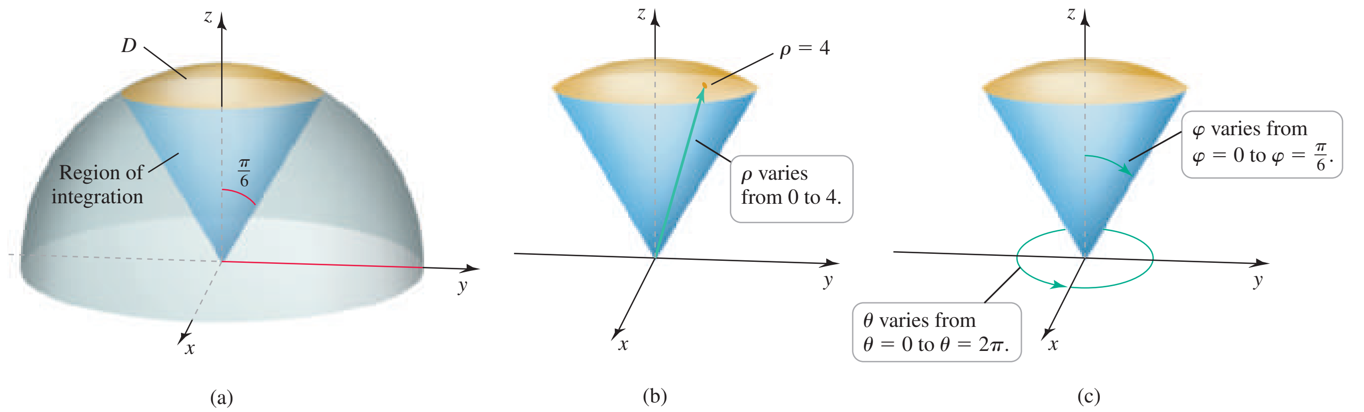


Figure 16.62

SOLUTION To find the volume, we evaluate a triple integral with $f = 1$. In the radial direction, the region extends from the origin $\rho = 0$ to the sphere $\rho = 4$ (Figure 16.62b). To sweep out all points of D, φ varies from 0 to $\pi/6$, and θ varies from 0 to 2π (Figure 16.62c). The volume of the region is

$$\iiint_D dV = \int_0^{2\pi}\int_0^{\pi/6}\int_0^4 \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta \qquad \text{Convert to an iterated integral.}$$

$$= \int_0^{2\pi}\int_0^{\pi/6} \frac{\rho^3}{3}\Big|_0^4 \sin\varphi \, d\varphi \, d\theta \qquad \text{Evaluate inner integral with respect to } \rho.$$

$$= \frac{64}{3}\int_0^{2\pi}\int_0^{\pi/6} \sin\varphi \, d\varphi \, d\theta \qquad \text{Simplify.}$$

$$= \frac{64}{3}\int_0^{2\pi} \underbrace{(-\cos\varphi)\Big|_0^{\pi/6}}_{1-\sqrt{3}/2} d\theta \qquad \text{Evaluate middle integral with respect to } \varphi.$$

$$= \frac{32}{3}(2-\sqrt{3})\int_0^{2\pi} d\theta \qquad \text{Simplify.}$$

$$= \frac{64\pi}{3}(2-\sqrt{3}). \qquad \text{Evaluate outer integral with respect to } \theta.$$

Related Exercises 49, 51 ◄

SECTION 16.5 EXERCISES

Getting Started

1. Explain how cylindrical coordinates are used to describe a point in $\mathbb{R}^3$.

2. Explain how spherical coordinates are used to describe a point in $\mathbb{R}^3$.

3. Describe the set $\{(r, \theta, z): r = 4z\}$ in cylindrical coordinates.

4. Describe the set $\{(\rho, \varphi, \theta): \varphi = \pi/4\}$ in spherical coordinates.

5. Explain why $dz\, r\, dr\, d\theta$ is the volume of a small "box" in cylindrical coordinates.

6. Explain why $\rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$ is the volume of a small "box" in spherical coordinates.

7. Write the integral $\iiint_D w(r, \theta, z)\, dV$ as an iterated integral, where the region D, expressed in cylindrical coordinates, is $D = \{(r, \theta, z): G(r, \theta) \le z \le H(r, \theta), g(\theta) \le r \le h(\theta), \alpha \le \theta \le \beta\}$.

8. Write the integral $\iiint_D w(\rho, \varphi, \theta)\, dV$ as an iterated integral, where the region D, expressed in spherical coordinates, is $D = \{(\rho, \varphi, \theta): g(\varphi, \theta) \le \rho \le h(\varphi, \theta), a \le \varphi \le b, \alpha \le \theta \le \beta\}$.

9. What coordinate system is *suggested* if the integrand of a triple integral involves $x^2 + y^2$?

10. What coordinate system is *suggested* if the integrand of a triple integral involves $x^2 + y^2 + z^2$?

Practice Exercises

11–14. Sets in cylindrical coordinates *Identify and sketch the following sets in cylindrical coordinates.*

11. $\{(r, \theta, z): 0 \le r \le 3, 0 \le \theta \le \pi/3, 1 \le z \le 4\}$

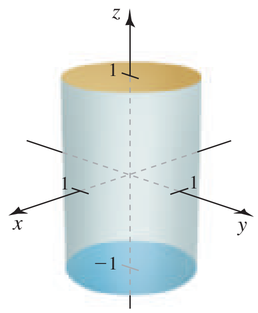
12. $\{(r, \theta, z): 0 \le \theta \le \pi/2, z = 1\}$

13. $\{(r, \theta, z): 2r \le z \le 4\}$

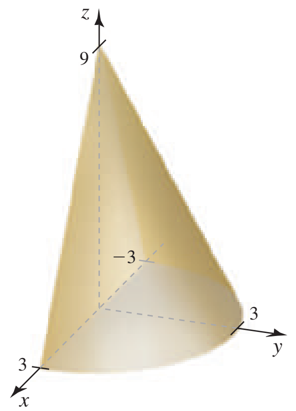
14. $\{(r, \theta, z): 0 \le z \le 8 - 2r\}$

15–22. Integrals in cylindrical coordinates *Evaluate the following integrals in cylindrical coordinates. The figures, if given, illustrate the region of integration.*

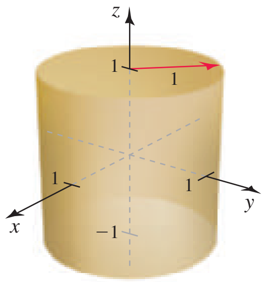
15. $\displaystyle\int_0^{2\pi}\int_0^1\int_{-1}^1 dz\, r\, dr\, d\theta$



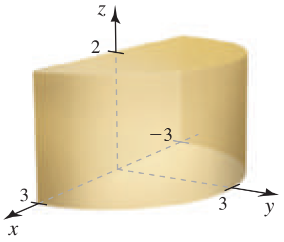
16. $\displaystyle\int_0^3\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}}\int_0^{9-3\sqrt{x^2+y^2}} dz\, dx\, dy$



17. $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{-1}^{1}(x^2+y^2)^{3/2}\,dz\,dx\,dy$



18. $\displaystyle\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{2}\frac{1}{1+x^2+y^2}\,dz\,dy\,dx$



19. $\displaystyle\int_{0}^{4}\int_{0}^{\sqrt{2}/2}\int_{x}^{\sqrt{1-x^2}}e^{-x^2-y^2}\,dy\,dx\,dz$

20. $\displaystyle\int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}}\int_{\sqrt{x^2+y^2}}^{4}dz\,dy\,dx$

21. $\displaystyle\int_{0}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{\sqrt{x^2+y^2}}(x^2+y^2)^{-1/2}\,dz\,dy\,dx$

22. $\displaystyle\int_{-1}^{1}\int_{0}^{1/2}\int_{\sqrt{3}y}^{\sqrt{1-y^2}}(x^2+y^2)^{1/2}\,dx\,dy\,dz$

23–26. Mass from density *Find the mass of the following objects with the given density functions. Assume (r, θ, z) are cylindrical coordinates.*

23. The solid cylinder $D = \{(r, \theta, z): 0 \le r \le 4, 0 \le z \le 10\}$ with density $\rho(r, \theta, z) = 1 + z/2$

24. The solid cylinder $D = \{(r, \theta, z): 0 \le r \le 3, 0 \le z \le 2\}$ with density $\rho(r, \theta, z) = 5e^{-r^2}$

25. The solid cone $D = \{(r, \theta, z): 0 \le z \le 6 - r, 0 \le r \le 6\}$ with density $\rho(r, \theta, z) = 7 - z$

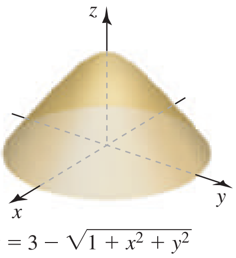
26. The solid paraboloid $D = \{(r, \theta, z): 0 \le z \le 9 - r^2, 0 \le r \le 3\}$ with density $\rho(r, \theta, z) = 1 + z/9$

27. Which weighs more? For $0 \le r \le 1$, the solid bounded by the cone $z = 4 - 4r$ and the solid bounded by the paraboloid $z = 4 - 4r^2$ have the same base in the xy-plane and the same height. Which object has the greater mass if the density of both objects is $\rho(r, \theta, z) = 10 - 2z$?

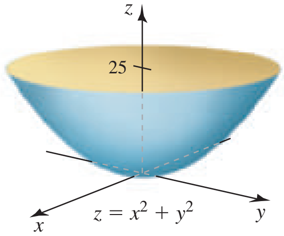
28. Which weighs more? Which of the objects in Exercise 27 weighs more if the density of both objects is $\rho(r, \theta, z) = \dfrac{8}{\pi}e^{-z}$?

29–34. Volumes in cylindrical coordinates *Use cylindrical coordinates to find the volume of the following solids.*

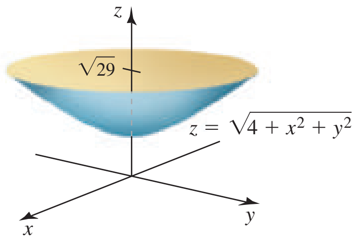
29. The solid bounded by the plane $z = 0$ and the hyperboloid $z = 3 - \sqrt{1 + x^2 + y^2}$



30. The solid bounded by the plane $z = 25$ and the paraboloid $z = x^2 + y^2$



31. The solid bounded by the plane $z = \sqrt{29}$ and the hyperboloid $z = \sqrt{4 + x^2 + y^2}$



32. The solid cylinder whose height is 4 and whose base is the disk $\{(r, \theta): 0 \le r \le 2\cos\theta\}$

33. The solid in the first octant bounded by the cylinder $r = 1$, and the planes $z = x$ and $z = 0$

34. The solid bounded by the cylinders $r = 1$ and $r = 2$, and the planes $z = 4 - x - y$ and $z = 0$

35–38. Sets in spherical coordinates *Identify and sketch the following sets in spherical coordinates.*

35. $\{(\rho, \varphi, \theta): 1 \le \rho \le 3\}$

36. $\{(\rho, \varphi, \theta): \rho = 2\csc\varphi, 0 < \varphi < \pi\}$

37. $\{(\rho, \varphi, \theta): \rho = 4\cos\varphi, 0 \le \varphi \le \pi/2\}$

38. $\{(\rho, \varphi, \theta): \rho = 2\sec\varphi, 0 \le \varphi < \pi/2\}$

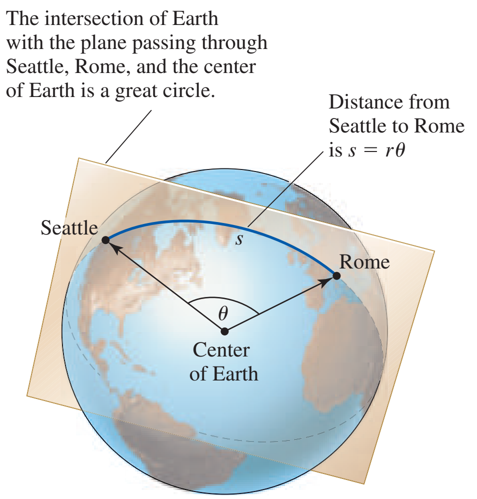
T **39–40. Latitude, longitude, and distances** *Assume Earth is a sphere with radius $r = 3960$ miles, oriented in xyz-space so that its center passes through the origin O, the positive z-axis passes through the North Pole, and the xz-plane passes through Greenwich, England (the intersection of Earth's surface and the xz-plane is called the prime meridian). The location of a point on Earth is given by its latitude* *(degrees north or south of the equator) and its longitude (degrees east or west of the prime meridian).*

39. Seattle has a latitude of 47.6° North and a longitude of 122.3° West; Rome, Italy, has a latitude of 41.9° North and a longitude of 12.5° East.

a. Find the approximate spherical and rectangular coordinates of Seattle. Express the angular coordinates in radians.

b. Find the approximate spherical and rectangular coordinates of Rome.

c. Consider the intersection curve of a sphere, and a plane passing through the center of the sphere and two points A and B on the sphere. It can be shown that the arc length of the segment of the intersection curve from A to B is the shortest distance on the sphere from A to B. Find the approximate shortest distance from Seattle to Rome. (*Hint:* Recall that $\cos\theta = \dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}$, where $0 \le \theta \le \pi$ is the angle between $\mathbf{u}$ and $\mathbf{v}$; use the arc length formula $s = r\theta$ to find the distance.)



40. Los Angeles has a latitude of 34.05° North and a longitude of 118.24° West, and New York City has a latitude of 40.71° North and a longitude of 74.01° West. Find the approximate shortest distance from Los Angeles to New York City.

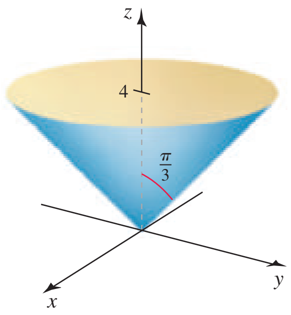
41–47. Integrals in spherical coordinates *Evaluate the following integrals in spherical coordinates.*

41. $\iiint_D (x^2 + y^2 + z^2)^{5/2}\,dV$; D is the unit ball.

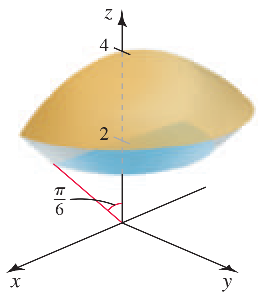
42. $\iiint_D e^{-(x^2+y^2+z^2)^{3/2}}\,dV$; D is the unit ball.

43. $\iiint_D \dfrac{dV}{(x^2 + y^2 + z^2)^{3/2}}$; D is the solid between the spheres of radius 1 and 2 centered at the origin.

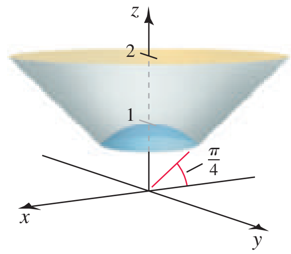
44. $\displaystyle\int_0^{2\pi}\int_0^{\pi/3}\int_0^{4\sec\varphi} \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$



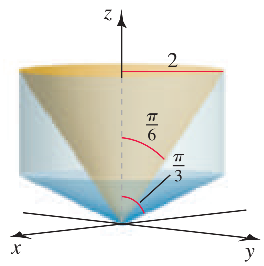
45. $\displaystyle\int_0^{\pi}\int_0^{\pi/6}\int_{2\sec\varphi}^{4} \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$



46. $\displaystyle\int_0^{2\pi}\int_0^{\pi/4}\int_1^{2\sec\varphi} (\rho^{-3})\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$



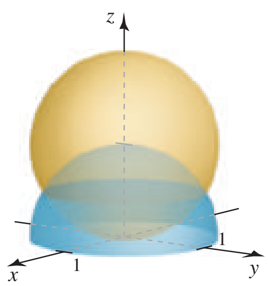
47. $\displaystyle\int_0^{2\pi}\int_{\pi/6}^{\pi/3}\int_0^{2\csc\varphi} \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$



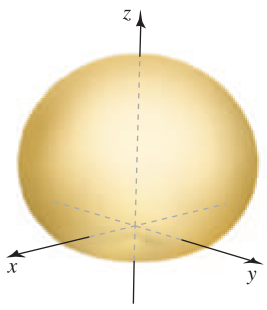
48–54. Volumes in spherical coordinates *Use spherical coordinates to find the volume of the following solids.*

48. A ball of radius $a > 0$

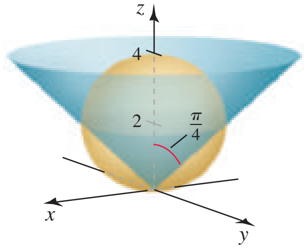
49. The solid bounded by the sphere $\rho = 2\cos\varphi$ and the hemisphere $\rho = 1$, $z \ge 0$



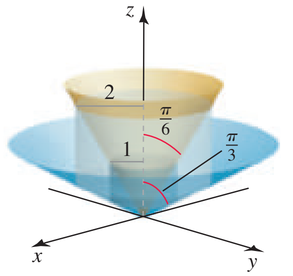
50. The solid cardioid of revolution $D = \{(\rho, \varphi, \theta)\colon 0 \le \rho \le 1 + \cos\varphi, 0 \le \varphi \le \pi, 0 \le \theta \le 2\pi\}$



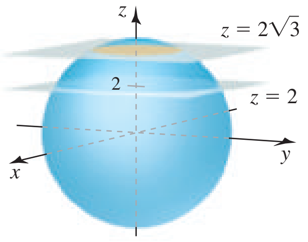
51. The solid outside the cone $\varphi = \pi/4$ and inside the sphere $\rho = 4\cos\varphi$



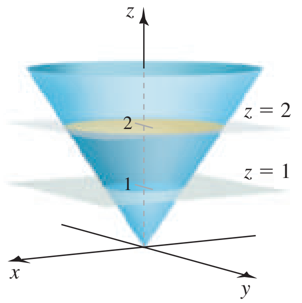
52. The solid bounded by the cylinders $r = 1$ and $r = 2$, and the cones $\varphi = \pi/6$ and $\varphi = \pi/3$



53. That part of the ball $\rho \le 4$ that lies between the planes $z = 2$ and $z = 2\sqrt{3}$



54. The solid lying between the planes $z = 1$ and $z = 2$ that is bounded by the cone $z = (x^2 + y^2)^{1/2}$



55. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. Any point on the z-axis has more than one representation in both cylindrical and spherical coordinates.

b. The sets $\{(r, \theta, z): r = z\}$ in cylindrical coordinates and the set $\{(\rho, \varphi, \theta): \varphi = \pi/4\}$ in spherical coordinates describe the same set of points.

56. Spherical to rectangular Convert the equation $\rho^2 = \sec 2\varphi$, where $0 \le \varphi < \pi/4$, to rectangular coordinates and identify the surface.

57. Spherical to rectangular Convert the equation $\rho^2 = -\sec 2\varphi$, where $\pi/4 < \varphi \le \pi/2$, to rectangular coordinates and identify the surface.

58–61. Mass from density *Find the mass of the following solids with the given density functions. Note that density is described by the function f to avoid confusion with the radial spherical coordinate ρ.*

58. The ball of radius 4 centered at the origin with a density $f(\rho, \varphi, \theta) = 1 + \rho$

59. The ball of radius 8 centered at the origin with a density $f(\rho, \varphi, \theta) = 2e^{-\rho^3}$

60. The solid cone $\{(r, \theta, z): 0 \le z \le 4, 0 \le r \le \sqrt{3}z, 0 \le \theta \le 2\pi\}$ with a density $f(r, \theta, z) = 5 - z$

61. The solid cylinder $\{(r, \theta, z): 0 \le r \le 2, 0 \le \theta \le 2\pi, -1 \le z \le 1\}$ with a density $f(r, \theta, z) = (2 - |z|)(4 - r)$

62–63. Changing order of integration *If possible, write an iterated integral in cylindrical coordinates of a function $g(r, \theta, z)$ for the following regions in the specified orders. Sketch the region of integration.*

62. The solid outside the cylinder $r = 1$ and inside the sphere $\rho = 5$, for $z \ge 0$, in the orders $dz\,dr\,d\theta$, $dr\,dz\,d\theta$, and $d\theta\,dz\,dr$

63. The solid above the cone $z = r$ and below the sphere $\rho = 2$, for $z \ge 0$, in the orders $dz\,dr\,d\theta$, $dr\,dz\,d\theta$, and $d\theta\,dz\,dr$

64–65. Changing order of integration *If possible, write iterated integrals in spherical coordinates for the following regions in the specified orders. Sketch the region of integration. Assume g is continuous on the region.*

64. $\int_0^{2\pi}\int_0^{\pi/2}\int_0^{4\sec\varphi} g(\rho, \varphi, \theta)\rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$ in the orders $d\rho\,d\theta\,d\varphi$ and $d\theta\,d\rho\,d\varphi$

65. $\int_0^{2\pi}\int_{\pi/6}^{\pi/2}\int_{\csc\varphi}^{2} g(\rho, \varphi, \theta)\rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$ in the orders $d\rho\,d\theta\,d\varphi$ and $d\theta\,d\rho\,d\varphi$

66–71. Miscellaneous volumes *Choose the best coordinate system and find the volume of the following solids. Surfaces are specified using the coordinates that give the simplest description, but the simplest integration may be with respect to different variables.*

66. The solid inside the sphere $\rho = 1$ and below the cone $\varphi = \pi/4$, for $z \ge 0$

67. That part of the solid cylinder $r \le 2$ that lies between the cones $\varphi = \pi/3$ and $\varphi = 2\pi/3$

68. That part of the ball $\rho \le 2$ that lies between the cones $\varphi = \pi/3$ and $\varphi = 2\pi/3$

69. The solid bounded by the cylinder $r = 1$, for $0 \le z \le x + y$

70. The solid inside the cylinder $r = 2\cos\theta$, for $0 \le z \le 4 - x$

71. The wedge cut from the cardioid cylinder $r = 1 + \cos\theta$ by the planes $z = 2 - x$ and $z = x - 2$

72. Volume of a drilled hemisphere Find the volume of material remaining in a hemisphere of radius 2 after a cylindrical hole of radius 1 is drilled through the center of the hemisphere perpendicular to its base.

73. Density distribution A right circular cylinder with height 8 cm and radius 2 cm is filled with water. A heated filament running along its axis produces a variable density in the water given by $\rho(r) = 1 - 0.05e^{-0.01r^2}\,\mathrm{g/cm^3}$ (ρ stands for density here, not for the radial spherical coordinate). Find the mass of the water in the cylinder. Neglect the volume of the filament.

74. Charge distribution A spherical cloud of electric charge has a known charge density $Q(\rho)$, where ρ is the spherical coordinate. Find the total charge in the cloud in the following cases.

a. $Q(\rho) = \dfrac{2 \times 10^{-4}}{\rho^4}$, $1 \le \rho < \infty$

b. $Q(\rho) = (2 \times 10^{-4})e^{-0.01\rho^3}$, $0 \le \rho < \infty$

Explorations and Challenges

75. Gravitational field due to spherical shell A point mass m is a distance d from the center of a thin spherical shell of mass M and radius R. The magnitude of the gravitational force on the point mass is given by the integral

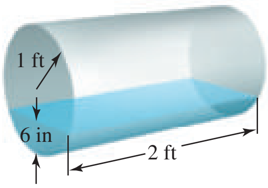
$$F(d) = \frac{GMm}{4\pi}\int_0^{2\pi}\int_0^{\pi}\frac{(d - R\cos\varphi)\sin\varphi}{(R^2 + d^2 - 2Rd\cos\varphi)^{3/2}}\,d\varphi\,d\theta,$$

where G is the gravitational constant.

a. Use the change of variable $x = \cos\varphi$ to evaluate the integral and show that if $d > R$, then $F(d) = GMm/d^2$, which means the force is the same as it would be if the mass of the shell were concentrated at its center.

b. Show that if $d < R$ (the point mass is inside the shell), then $F = 0$.

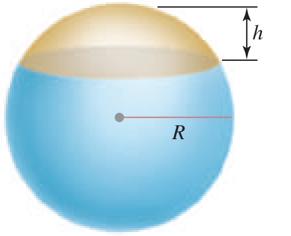
76. Water in a gas tank Before a gasoline-powered engine is started, water must be drained from the bottom of the fuel tank. Suppose the tank is a right circular cylinder on its side with a length of 2 ft and a radius of 1 ft. If the water level is 6 in above the lowest part of the tank, determine how much water must be drained from the tank.



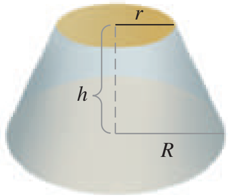
77–80. General volume formulas *Use integration to find the volume of the following solids. In each case, choose a convenient coordinate system, find equations for the bounding surfaces, set up a triple integral, and evaluate the integral. Assume a, b, c, r, R, and h are positive constants.*

77. Cone Find the volume of a solid right circular cone with height h and base radius r.

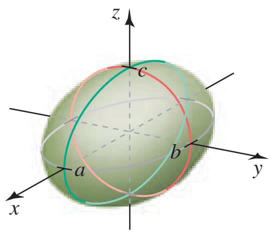
78. Spherical cap Find the volume of the cap of a sphere of radius R with thickness h.



79. Frustum of a cone Find the volume of a truncated solid cone of height h whose ends have radii r and R.



80. Ellipsoid Find the volume of a solid ellipsoid with axes of lengths $2a$, $2b$, and $2c$.



81. Intersecting spheres One sphere is centered at the origin and has a radius of R. Another sphere is centered at $(0, 0, r)$ and has a radius of r, where $r > R/2$. What is the volume of the region common to the two spheres?

QUICK CHECK ANSWERS

1. $(\sqrt{2}, 7\pi/4, 5), (1, \sqrt{3}, 5)$
2. $0 \le r \le 10, 0 \le \theta \le 2\pi, 0 \le z \le 20$
3. $(2\sqrt{2}, \pi/4, \pi/3), (1, 1, \sqrt{2})$ ◀

16.6 Integrals for Mass Calculations

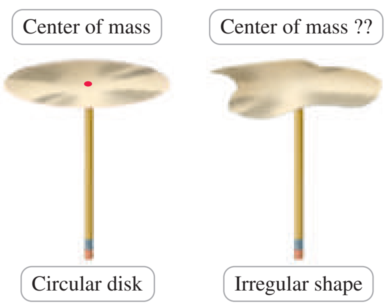


Figure 16.63

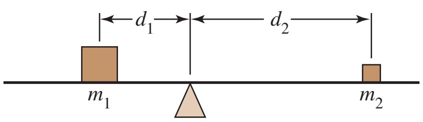


Figure 16.64

Intuition says that a thin circular disk (such as a DVD without a hole) should balance on a pencil placed at the center of the disk (**Figure 16.63**). If, however, you were given a thin plate with an irregular shape, at what point would it balance? This question asks about the *center of mass* of a thin object (thin enough that it can be treated as a two-dimensional region). Similarly, given a solid object with an irregular shape and variable density, where is the point at which all of the mass of the object would be located if it were treated as a point mass? In this section, we use integration to compute the center of mass of one-, two-, and three-dimensional objects.

Sets of Individual Objects

Methods for finding the center of mass of an object are ultimately based on a well-known playground principle: If two people with masses m_1 and m_2 sit at distances d_1 and d_2 from the pivot point of a seesaw (with no mass), then the seesaw balances, provided $m_1d_1 = m_2d_2$ (**Figure 16.64**).

QUICK CHECK 1 A 90-kg person sits 2 m from the balance point of a seesaw. How far from that point must a 60-kg person sit to balance the seesaw? Assume the seesaw has no mass. ◄

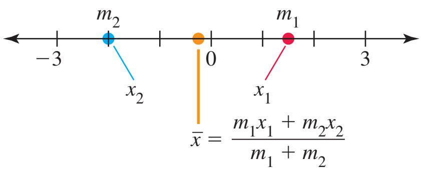


Figure 16.65

➤ The center of mass may be viewed as the weighted average of the x-coordinates, with the masses serving as the weights. Notice how the units work out: If x_1 and x_2 have units of meters and m_1 and m_2 have units of kilograms, then $\bar{x}$ has units of meters.

To generalize the problem, we introduce a coordinate system with the origin at $x = 0$ (**Figure 16.65**). Suppose the location of the balance point $\bar{x}$ is unknown. The coordinates of the two masses m_1 and m_2 are denoted x_1 and x_2, respectively, with $x_1 > x_2$. The mass m_1 is a distance $x_1 - \bar{x}$ from the balance point (because distance is positive and $x_1 > \bar{x}$). The mass m_2 is a distance $\bar{x} - x_2$ from the balance point (because $\bar{x} > x_2$). The playground principle becomes

$$m_1\underbrace{(x_1 - \bar{x})}_{\substack{\text{distance from}\\ \text{balance point}\\ \text{to } m_1}} = m_2\underbrace{(\bar{x} - x_2)}_{\substack{\text{distance from}\\ \text{balance point}\\ \text{to } m_2}},$$

or $m_1(x_1 - \bar{x}) + m_2(x_2 - \bar{x}) = 0$.

Solving this equation for $\bar{x}$, we find that the balance point or *center of mass* of the two-mass system is located at

$$\bar{x} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}.$$

The quantities m_1x_1 and m_2x_2 are called *moments about the origin* (or just *moments*). The location of the center of mass is the sum of the moments divided by the sum of the masses.

For example, an 80-kg man standing 2 m to the right of the origin will balance a 160-kg gorilla sitting 4 m to the left of the origin, provided the pivot on their seesaw is placed at

$$\bar{x} = \frac{80(2) + 160(-4)}{80 + 160} = -2,$$

or 2 m to the left of the origin (**Figure 16.66**).

QUICK CHECK 2 Solve the equation $m_1(x_1 - \bar{x}) + m_2(x_2 - \bar{x}) = 0$ for $\bar{x}$ to verify the preceding expression for the center of mass. ◄

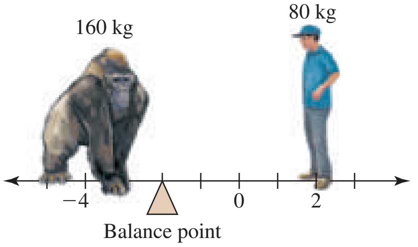


Figure 16.66

Several Objects on a Line Generalizing the preceding argument to n objects having masses $m_1, m_2, \ldots,$ and m_n with coordinates $x_1, x_2, \ldots,$ and x_n, respectively, the balance condition becomes

$$m_1(x_1 - \bar{x}) + m_2(x_2 - \bar{x}) + \cdots + m_n(x_n - \bar{x}) = \sum_{k=1}^{n} m_k(x_k - \bar{x}) = 0.$$

Solving this equation for the location of the center of mass, we find that

$$\bar{x} = \frac{m_1x_1 + m_2x_2 + \cdots + m_nx_n}{m_1 + m_2 + \cdots + m_n} = \frac{\sum_{k=1}^{n} m_kx_k}{\sum_{k=1}^{n} m_k}.$$

Again, the location of the center of mass is the sum of the moments $m_1x_1, m_2x_2, \ldots,$ and m_nx_n divided by the sum of the masses.

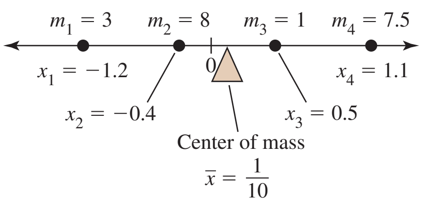


Figure 16.67

EXAMPLE 1 **Center of mass for four objects** Find the point at which the system shown in **Figure 16.67** balances.

SOLUTION The center of mass is

$$\begin{aligned}\bar{x} &= \frac{m_1x_1 + m_2x_2 + m_3x_3 + m_4x_4}{m_1 + m_2 + m_3 + m_4}\\ &= \frac{3(-1.2) + 8(-0.4) + 1(0.5) + 7.5(1.1)}{3 + 8 + 1 + 7.5}\\ &= \frac{1}{10}.\end{aligned}$$

The balancing point is slightly to the right of the origin.

Related Exercises 7–8 ◄

➤ Density is usually measured in units of *mass per volume*. However, for thin, narrow objects such as rods and wires, linear density with units of *mass per length* is used. For thin, flat objects such as plates and sheets, area density with units of *mass per area* is used.

QUICK CHECK 3 In Figure 16.68, suppose $a = 0$, $b = 3$, and the density of the rod in g/cm is $\rho(x) = 4 - x$. Where is the rod lightest? Heaviest? ◀

Continuous Objects in One Dimension

Now consider a thin rod or wire with density ρ that varies along the length of the rod (Figure 16.68). The density in this case has units of mass per length (for example, g/cm). As before, we want to determine the location $\bar{x}$ at which the rod balances on a pivot.

Using the slice-and-sum strategy, we divide the rod, which corresponds to the interval $a \le x \le b$, into n subintervals, each with a width of $\Delta x = \dfrac{b-a}{n}$ (Figure 16.69). The corresponding grid points are

$$x_0 = a, x_1 = a + \Delta x, \ldots, x_k = a + k\,\Delta x, \ldots, \text{and } x_n = b.$$

The mass of the kth segment of the rod is approximately the density at x_k multiplied by the length of the interval, or $m_k \approx \rho(x_k)\Delta x$.

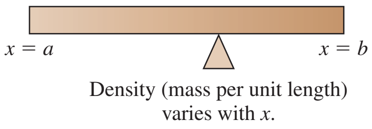


Figure 16.68

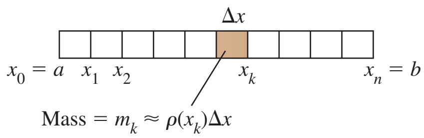


Figure 16.69

We now use the center-of-mass formula for several distinct objects to write the approximate center of mass of the rod as

$$\bar{x} = \frac{\sum_{k=1}^{n} m_k x_k}{\sum_{k=1}^{n} m_k} \approx \frac{\sum_{k=1}^{n} (\rho(x_k)\Delta x)\, x_k}{\sum_{k=1}^{n} \rho(x_k)\Delta x}.$$

➤ An object consisting of two different materials that meet at an interface has a discontinuous density function. Physical density functions either are continuous or have a finite number of discontinuities.

➤ We assume the rod has positive mass and the limits in the numerator and denominator exist, so the limit of the quotient is the quotient of the limits.

To model a rod with a continuous density, we let $\Delta x \to 0$ and $n \to \infty$; the center of mass of the rod is

$$\bar{x} = \lim_{\Delta x \to 0} \frac{\sum_{k=1}^{n} (\rho(x_k)\Delta x)\, x_k}{\sum_{k=1}^{n} \rho(x_k)\Delta x} = \frac{\lim_{\Delta x \to 0} \sum_{k=1}^{n} x_k\, \rho(x_k)\Delta x}{\lim_{\Delta x \to 0} \sum_{k=1}^{n} \rho(x_k)\Delta x} = \frac{\int_a^b x\rho(x)\,dx}{\int_a^b \rho(x)\,dx}.$$

As discussed in Section 6.7, the denominator of the last fraction, $\int_a^b \rho(x)\,dx$, is the mass of the rod. The numerator is the "sum" of the moments of the individual pieces of the rod, which is called the *total moment*.

➤ The units of a moment are mass × length. The center of mass is a moment divided by a mass, which has units of length. Notice that if the density is constant, then ρ effectively does not enter the calculation of $\bar{x}$.

DEFINITION Center of Mass in One Dimension

Let ρ be an integrable density function on the interval $[a, b]$ (which represents a thin rod or wire). The **center of mass** is located at the point $\bar{x} = \dfrac{M}{m}$, where the **total moment** M and mass m are

$$M = \int_a^b x\rho(x)\,dx \quad \text{and} \quad m = \int_a^b \rho(x)\,dx.$$

Observe the parallels between the discrete and continuous cases:

$$n \text{ individual objects: } \quad \bar{x} = \frac{\sum_{k=1}^{n} x_k m_k}{\sum_{k=1}^{n} m_k}; \qquad \text{continuous object: } \quad \bar{x} = \frac{\int_a^b x\rho(x)\,dx}{\int_a^b \rho(x)\,dx}.$$

EXAMPLE 2 **Center of mass of a one-dimensional object** Suppose a thin 2-m bar is made of an alloy whose density in kg/m is $\rho(x) = 1 + x^2$, where $0 \le x \le 2$. Find the center of mass of the bar.

SOLUTION The total mass of the bar in kilograms is

$$m = \int_a^b \rho(x)\,dx = \int_0^2 (1 + x^2)\,dx = \left(x + \frac{x^3}{3}\right)\bigg|_0^2 = \frac{14}{3}.$$

➤ Notice that the density of the bar increases with x. As a consistency check, our calculation must result in a center of mass to the right of the midpoint of the bar.

The total moment of the bar, with units kg-m, is

$$M = \int_a^b x\rho(x)\,dx = \int_0^2 x(1 + x^2)\,dx = \left(\frac{x^2}{2} + \frac{x^4}{4}\right)\bigg|_0^2 = 6.$$

Therefore, the center of mass is located at $\bar{x} = \dfrac{M}{m} = \dfrac{9}{7} \approx 1.29$ m.

Related Exercises 10–11 ◄

Two-Dimensional Objects

In two dimensions, we start with an integrable density function $\rho(x, y)$ defined over a closed bounded region R in the xy-plane. The density is now an *area density* with units of mass per area (for example, kg/m^2). The region represents a thin plate (or *lamina*). The center of mass is the point at which a pivot must be located to balance the plate. If the density is constant, the location of the center of mass depends only on the shape of the plate, in which case the center of mass is called the *centroid*.

For a two- or three-dimensional object, the coordinates for the center of mass are computed independently by applying the one-dimensional argument in each coordinate direction (Figure 16.70). The mass of the plate is the integral of the density function over R:

$$m = \iint_R \rho(x, y)\,dA.$$

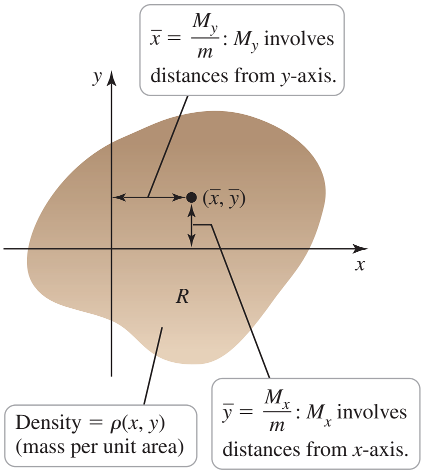


Figure 16.70

In analogy with the moment calculation in the one-dimensional case, we now define two moments.

➤ The moment with respect to the y-axis M_y is a weighted average of distances from the y-axis, so it has x in the integrand (the distance between a point and the y-axis). Similarly, the moment with respect to the x-axis M_x is a weighted average of distances from the x-axis, so it has y in the integrand.

DEFINITION **Center of Mass in Two Dimensions**

Let ρ be an integrable area density function defined over a closed bounded region R in $\mathbb{R}^2$. The coordinates of the center of mass of the object represented by R are

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m}\iint_R x\rho(x, y)\,dA \quad \text{and} \quad \bar{y} = \frac{M_x}{m} = \frac{1}{m}\iint_R y\rho(x, y)\,dA,$$

where $m = \iint_R \rho(x, y)\,dA$ is the mass, and M_y and M_x are the moments with respect to the y-axis and x-axis, respectively. If ρ is constant, the center of mass is called the **centroid** and is independent of the density.

QUICK CHECK 4 Explain why the integral for M_y has x in the integrand. Explain why the density drops out of the center-of-mass calculation if it is constant. ◄

As before, the center-of-mass coordinates are weighted averages of the distances from the coordinate axes. For two- and three-dimensional objects, the center of mass need not lie within the object (Exercises 51, 61, and 62).

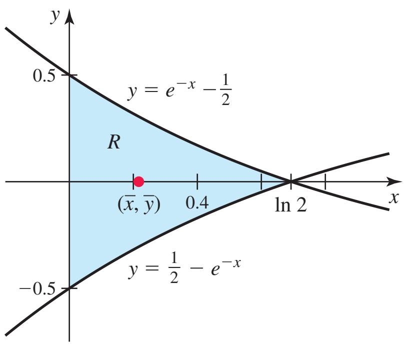


Figure 16.71

EXAMPLE 3 **Centroid calculation** Find the centroid (center of mass) of the unit-density, dart-shaped region bounded by the y-axis and the curves $y = e^{-x} - \frac{1}{2}$ and $y = \frac{1}{2} - e^{-x}$ (Figure 16.71).

SOLUTION Because the region is symmetric about the x-axis and the density is constant, the y-coordinate of the center of mass is $\bar{y} = 0$. This leaves the integrals for m and M_y to evaluate.

The first task is to find the point at which the curves intersect. Solving $e^{-x} - \frac{1}{2} = \frac{1}{2} - e^{-x}$, we find that $x = \ln 2$, from which it follows that $y = 0$. Therefore, the intersection point is $(\ln 2, 0)$. The moment M_y (with $\rho = 1$) is given by

$$\begin{aligned} M_y &= \int_0^{\ln 2}\int_{1/2-e^{-x}}^{e^{-x}-1/2} x\,dy\,dx && \text{Definition of } M_y \\ &= \int_0^{\ln 2} x\left(\left(e^{-x} - \frac{1}{2}\right) - \left(\frac{1}{2} - e^{-x}\right)\right)dx && \text{Evaluate inner integral.} \\ &= \int_0^{\ln 2} x(2e^{-x} - 1)\,dx. && \text{Simplify.} \end{aligned}$$

➤ The density does not enter the center-of-mass calculation when the density is constant. So it is easiest to set $\rho = 1$.

➤ If possible, try to arrange the coordinate system so that at least one of the integrations in the center-of-mass calculation can be avoided by using symmetry. Often the mass (or area) can be found using geometry if the density is constant.

Using integration by parts for this integral, we find that

$$\begin{aligned} M_y &= \int_0^{\ln 2} \underbrace{x}_{u}\,\underbrace{(2e^{-x} - 1)\,dx}_{dv} \\ &= -x(2e^{-x} + x)\Big|_0^{\ln 2} + \int_0^{\ln 2}(2e^{-x} + x)\,dx && \text{Integration by parts} \\ &= 1 - \ln 2 - \frac{1}{2}\ln^2 2 \approx 0.067. && \text{Evaluate and simplify.} \end{aligned}$$

With $\rho = 1$, the mass of the region is given by

$$\begin{aligned} m &= \int_0^{\ln 2}\int_{1/2-e^{-x}}^{e^{-x}-1/2} dy\,dx && \text{Definition of } m \\ &= \int_0^{\ln 2}(2e^{-x} - 1)\,dx && \text{Evaluate inner integral.} \\ &= (-2e^{-x} - x)\Big|_0^{\ln 2} && \text{Evaluate outer integral.} \\ &= 1 - \ln 2 \approx 0.307. && \text{Simplify.} \end{aligned}$$

Therefore, the x-coordinate of the center of mass is $\bar{x} = \frac{M_y}{m} \approx 0.217$. The center of mass is located approximately at $(0.217, 0)$.

Related Exercise 18 ◄

EXAMPLE 4 **Variable-density plate** Find the center of mass of the rectangular plate $R = \{(x, y): -1 \le x \le 1, 0 \le y \le 1\}$ with a density of $\rho(x, y) = 2 - y$ (heavy at the lower edge and light at the top edge; Figure 16.72).

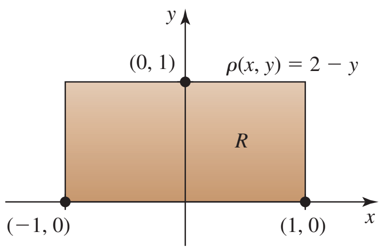


Figure 16.72

SOLUTION Because the plate is symmetric with respect to the y-axis and because the density is independent of x, we have $\bar{x} = 0$. We must still compute m and M_x.

$$m = \iint_R \rho(x, y)\,dA = \int_{-1}^{1}\int_0^1 (2 - y)\,dy\,dx = \frac{3}{2}\int_{-1}^{1} dx = 3$$

$$M_x = \iint_R y\rho(x, y)\,dA = \int_{-1}^{1}\int_0^1 y(2 - y)\,dy\,dx = \frac{2}{3}\int_{-1}^{1} dx = \frac{4}{3}$$

➤ To verify that $\bar{x} = 0$, notice that to find M_y, we integrate an odd function in x over $-1 \le x \le 1$; the result is zero.

Therefore, the center-of-mass coordinates are

$$\bar{x} = \frac{M_y}{m} = 0 \quad \text{and} \quad \bar{y} = \frac{M_x}{m} = \frac{4/3}{3} = \frac{4}{9}.$$

Related Exercise 21 ◄

Three-Dimensional Objects

We now extend the preceding arguments to compute the center of mass of three-dimensional solids. Assume D is a closed bounded region in $\mathbb{R}^3$, on which an integrable density function ρ is defined. The units of the density are mass per volume (for example, g/cm^3). The coordinates of the center of mass depend on the mass of the region, which by

Section 16.4 is the integral of the density function over D. Three moments enter the picture: M_{yz} involves distances from the yz-plane; therefore, it has an x in the integrand. Similarly, M_{xz} involves distances from the xz-plane, so it has a y in the integrand, and M_{xy} involves distances from the xy-plane, so it has a z in the integrand. As before, the coordinates of the center of mass are the total moments divided by the total mass (**Figure 16.73**).

QUICK CHECK 5 Explain why the integral for the moment M_{xy} has z in the integrand. ◄

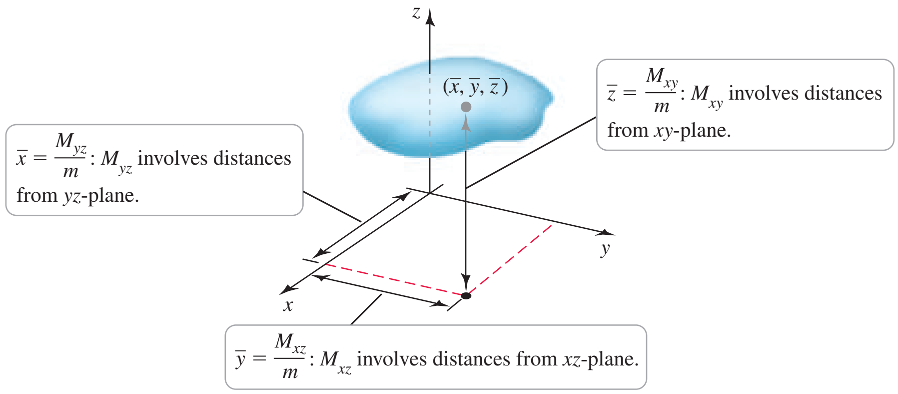


Figure 16.73

DEFINITION Center of Mass in Three Dimensions

Let ρ be an integrable density function on a closed bounded region D in $\mathbb{R}^3$. The coordinates of the center of mass of the region are

$$\bar{x} = \frac{M_{yz}}{m} = \frac{1}{m}\iiint_D x\rho(x, y, z)\,dV, \quad \bar{y} = \frac{M_{xz}}{m} = \frac{1}{m}\iiint_D y\rho(x, y, z)\,dV, \text{ and}$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{1}{m}\iiint_D z\rho(x, y, z)\,dV,$$

where $m = \iiint_D \rho(x, y, z)\,dV$ is the mass, and M_{yz}, M_{xz}, and M_{xy} are the moments with respect to the coordinate planes.

EXAMPLE 5 Center of mass with constant density Find the center of mass of the constant-density solid cone D bounded by the surface $z = 4 - \sqrt{x^2 + y^2}$ and $z = 0$ (**Figure 16.74**).

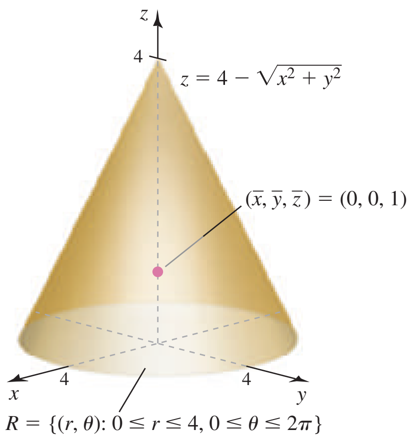


Figure 16.74

SOLUTION Because the cone is symmetric about the z-axis and has uniform density, the center of mass lies on the z-axis; that is, $\bar{x} = 0$ and $\bar{y} = 0$. Setting $z = 0$, the base of the cone in the xy-plane is the disk of radius 4 centered at the origin. Therefore, the cone has height 4 and radius 4; by the volume formula, its volume is $\pi r^2 h/3 = 64\pi/3$. The cone has a constant density, so we assume $\rho = 1$ and its mass is $m = 64\pi/3$.

To obtain the value of $\bar{z}$, only M_{xy} needs to be calculated, which is most easily done in cylindrical coordinates. The cone is described by the equation $z = 4 - \sqrt{x^2 + y^2} = 4 - r$. The projection of the cone onto the xy-plane, which is the region of integration in the xy-plane, is the disk $R = \{(r, \theta): 0 \le r \le 4, 0 \le \theta \le 2\pi\}$. The integration for M_{xy} now follows:

$$\begin{aligned}
M_{xy} &= \iiint_D z\,dV && \text{Definition of } M_{xy} \text{ with } \rho = 1\\
&= \int_0^{2\pi}\int_0^4\int_0^{4-r} z\,dz\,r\,dr\,d\theta && \text{Convert to an iterated integral.}\\
&= \int_0^{2\pi}\int_0^4 \frac{z^2}{2}\bigg|_0^{4-r} r\,dr\,d\theta && \text{Evaluate inner integral with respect to } z.\\
&= \frac{1}{2}\int_0^{2\pi}\int_0^4 r(4-r)^2\,dr\,d\theta && \text{Simplify.}
\end{aligned}$$

$$= \frac{1}{2}\int_0^{2\pi} \frac{64}{3}\, d\theta \qquad \text{Evaluate middle integral with respect to } r.$$

$$= \frac{64\pi}{3}. \qquad \text{Evaluate outer integral with respect to } \theta.$$

The z-coordinate of the center of mass is $\bar{z} = \dfrac{M_{xy}}{m} = \dfrac{64\pi/3}{64\pi/3} = 1$, and the center of mass is located at $(0, 0, 1)$. It can be shown (Exercise 55) that the center of mass of a constant-density cone of height h is located $h/4$ units from the base on the axis of the cone, independent of the radius.

Related Exercise 28 ◄

EXAMPLE 6 **Center of mass with variable density** Find the center of mass of the interior of the hemisphere D of radius a with its base on the xy-plane. The density of the object given in spherical coordinates is $f(\rho, \varphi, \theta) = 2 - \rho/a$ (heavy near the center and light near the outer surface; **Figure 16.75**).

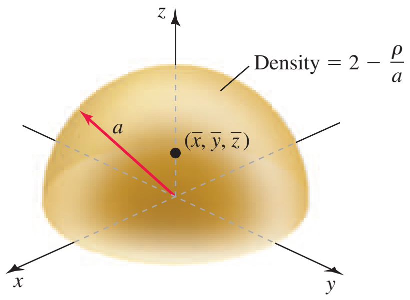


Figure 16.75

SOLUTION The center of mass lies on the z-axis because of the symmetry of both the solid and the density function; therefore, $\bar{x} = \bar{y} = 0$. Only the integrals for m and M_{xy} need to be evaluated, and they should be done in spherical coordinates.

The integral for the mass is

$$m = \iiint_D f(\rho, \varphi, \theta)\, dV \qquad \text{Definition of } m$$

$$= \int_0^{2\pi}\int_0^{\pi/2}\int_0^a \left(2 - \frac{\rho}{a}\right)\rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta \qquad \text{Convert to an iterated integral.}$$

$$= \int_0^{2\pi}\int_0^{\pi/2}\left(\frac{2\rho^3}{3} - \frac{\rho^4}{4a}\right)\Bigg|_0^a \sin\varphi\, d\varphi\, d\theta \qquad \text{Evaluate inner integral with respect to } \rho.$$

$$= \int_0^{2\pi}\int_0^{\pi/2} \frac{5a^3}{12}\sin\varphi\, d\varphi\, d\theta \qquad \text{Simplify.}$$

$$= \frac{5a^3}{12}\int_0^{2\pi} \underbrace{(-\cos\varphi)\Big|_0^{\pi/2}}_{1} d\theta \qquad \text{Evaluate middle integral with respect to } \varphi.$$

$$= \frac{5a^3}{12}\int_0^{2\pi} d\theta \qquad \text{Simplify.}$$

$$= \frac{5\pi a^3}{6}. \qquad \text{Evaluate outer integral with respect to } \theta.$$

In spherical coordinates, $z = \rho\cos\varphi$, so the integral for the moment M_{xy} is

$$M_{xy} = \iiint_D z\, f(\rho, \varphi, \theta)\, dV \qquad \text{Definition of } M_{xy}$$

$$= \int_0^{2\pi}\int_0^{\pi/2}\int_0^a \underbrace{\rho\cos\varphi}_{z}\left(2 - \frac{\rho}{a}\right)\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta \qquad \text{Convert to an iterated integral.}$$

$$= \int_0^{2\pi}\int_0^{\pi/2}\left(\frac{\rho^4}{2} - \frac{\rho^5}{5a}\right)\Bigg|_0^a \sin\varphi\cos\varphi\, d\varphi\, d\theta \qquad \text{Evaluate inner integral with respect to } \rho.$$

$$= \int_0^{2\pi}\int_0^{\pi/2} \frac{3a^4}{10}\underbrace{\sin\varphi\cos\varphi}_{(\sin 2\varphi)/2}\, d\varphi\, d\theta \qquad \text{Simplify.}$$

$$= \frac{3a^4}{10}\int_0^{2\pi}\underbrace{\left(-\frac{\cos 2\varphi}{4}\right)\Bigg|_0^{\pi/2}}_{1/2} d\theta \qquad \text{Evaluate middle integral with respect to } \varphi.$$

$$= \frac{3a^4}{20}\int_0^{2\pi} d\theta \qquad \text{Simplify.}$$

$$= \frac{3\pi a^4}{10}. \qquad \text{Evaluate outer integral with respect to } \theta.$$

The z-coordinate of the center of mass is $\bar{z} = \frac{M_{xy}}{m} = \frac{3\pi a^4/10}{5\pi a^3/6} = \frac{9a}{25} = 0.36a$. It can be shown (Exercise 56) that the center of mass of a uniform-density hemispherical solid of radius a is $3a/8 = 0.375a$ units above the base. In this case, the variable density lowers the center of mass toward the base.

Related Exercise 35 ◄

SECTION 16.6 EXERCISES

Getting Started

1. Explain how to find the balance point for two people on opposite ends of a (massless) plank that rests on a pivot.

2. If a thin 1-m cylindrical rod has a density of $\rho = 1$ g/cm for its left half and a density of $\rho = 2$ g/cm for its right half, what is its mass and where is its center of mass?

3. Explain how to find the center of mass of a thin plate with a variable density.

4. In the integral for the moment M_x of a thin plate, why does y appear in the integrand?

5. Explain how to find the center of mass of a three-dimensional object with a variable density.

6. In the integral for the moment M_{xz} with respect to the xz-plane of a solid, why does y appear in the integrand?

Practice Exercises

7–8. Individual masses on a line *Sketch the following systems on a number line and find the location of the center of mass.*

7. $m_1 = 10$ kg located at $x = 3$ m; $m_2 = 3$ kg located at $x = -1$ m

8. $m_1 = 8$ kg located at $x = 2$ m; $m_2 = 4$ kg located at $x = -4$ m; $m_3 = 1$ kg located at $x = 0$ m

9–14. One-dimensional objects *Find the mass and center of mass of the thin rods with the following density functions.*

9. $\rho(x) = 1 + \sin x$, for $0 \le x \le \pi$

10. $\rho(x) = 1 + x^3$, for $0 \le x \le 1$

11. $\rho(x) = 2 - \frac{x^2}{16}$, for $0 \le x \le 4$

12. $\rho(x) = 2 + \cos x$, for $0 \le x \le \pi$

13. $\rho(x) = \begin{cases} 1 & \text{if } 0 \le x \le 2 \\ 1 + x & \text{if } 2 < x \le 4 \end{cases}$

14. $\rho(x) = \begin{cases} x^2 & \text{if } 0 \le x \le 1 \\ x(2 - x) & \text{if } 1 < x \le 2 \end{cases}$

15–20. Centroid calculations *Find the mass and centroid (center of mass) of the following thin plates, assuming constant density. Sketch the region corresponding to the plate and indicate the location of the center of mass. Use symmetry when possible to simplify your work.*

15. The region bounded by $y = \sin x$ and $y = 1 - \sin x$ between $x = \pi/4$ and $x = 3\pi/4$

16. The region in the first quadrant bounded by $x^2 + y^2 = 16$

17. The region bounded by $y = 1 - |x|$ and the x-axis

18. The region bounded by $y = e^x$, $y = e^{-x}$, $x = 0$, and $x = \ln 2$

19. The region bounded by $y = \ln x$, the x-axis, and $x = e$

20. The region bounded by $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$, for $y \ge 0$

21–26. Variable-density plates *Find the center of mass of the following plane regions with variable density. Describe the distribution of mass in the region.*

21. $R = \{(x, y): 0 \le x \le 4, 0 \le y \le 2\}$; $\rho(x, y) = 1 + x/2$

22. $R = \{(x, y): 0 \le x \le 1, 0 \le y \le 5\}$; $\rho(x, y) = 2e^{-y/2}$

23. The triangular plate in the first quadrant bounded by $x + y = 4$ with $\rho(x, y) = 1 + x + y$

24. The upper half ($y \ge 0$) of the disk bounded by the circle $x^2 + y^2 = 4$ with $\rho(x, y) = 1 + y/2$

25. The upper half ($y \ge 0$) of the plate bounded by the ellipse $x^2 + 9y^2 = 9$ with $\rho(x, y) = 1 + y$

26. The quarter disk in the first quadrant bounded by $x^2 + y^2 = 4$ with $\rho(x, y) = 1 + x^2 + y^2$

27–32. Center of mass of constant-density solids *Find the center of mass of the following solids, assuming a constant density of 1. Sketch the region and indicate the location of the centroid. Use symmetry when possible and choose a convenient coordinate system.*

27. The upper half of the ball $x^2 + y^2 + z^2 \le 16$ (for $z \ge 0$)

28. The solid bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 25$

29. The tetrahedron in the first octant bounded by $z = 1 - x - y$ and the coordinate planes

30. The solid bounded by the cone $z = 16 - r$ and the plane $z = 0$

31. The sliced solid cylinder bounded by $x^2 + y^2 = 1$, $z = 0$, and $y + z = 1$

32. The solid bounded by the upper half ($z \ge 0$) of the ellipsoid $4x^2 + 4y^2 + z^2 = 16$

33–38. Variable-density solids *Find the coordinates of the center of mass of the following solids with variable density.*

33. $R = \{(x, y, z): 0 \le x \le 4, 0 \le y \le 1, 0 \le z \le 1\}$; $\rho(x, y, z) = 1 + x/2$

34. The region bounded by the paraboloid $z = 4 - x^2 - y^2$ and $z = 0$ with $\rho(x, y, z) = 5 - z$

35. The solid bounded by the upper half of the sphere $\rho = 6$ and $z = 0$ with density $f(\rho, \varphi, \theta) = 1 + \rho/4$

36. The interior of the cube in the first octant formed by the planes $x = 1$, $y = 1$, and $z = 1$, with $\rho(x, y, z) = 2 + x + y + z$

37. The interior of the prism formed by the planes $z = x$, $x = 1$, and $y = 4$, and the coordinate planes, with $\rho(x, y, z) = 2 + y$

38. The solid bounded by the cone $z = 9 - r$ and $z = 0$ with $\rho(r, \theta, z) = 1 + z$

39. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

- **a.** A thin plate of constant density that is symmetric about the x-axis has a center of mass with an x-coordinate of zero.
- **b.** A thin plate of constant density that is symmetric about both the x-axis and the y-axis has its center of mass at the origin.
- **c.** The center of mass of a thin plate must lie on the plate.
- **d.** The center of mass of a connected solid region (all in one piece) must lie within the region.

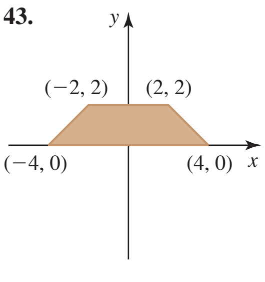
40. Limiting center of mass A thin rod of length L has a linear density given by $\rho(x) = 2e^{-x/3}$ on the interval $0 \le x \le L$. Find the mass and center of mass of the rod. How does the center of mass change as $L \to \infty$?

41. Limiting center of mass A thin rod of length L has a linear density given by $\rho(x) = \dfrac{10}{1 + x^2}$ on the interval $0 \le x \le L$. Find the mass and center of mass of the rod. How does the center of mass change as $L \to \infty$?

42. Limiting center of mass A thin plate is bounded by the graphs of $y = e^{-x}$, $y = -e^{-x}$, $x = 0$, and $x = L$. Find its center of mass. How does the center of mass change as $L \to \infty$?

43–44. Two-dimensional plates *Find the mass and center of mass of the thin constant-density plates shown in the figure.*

43.



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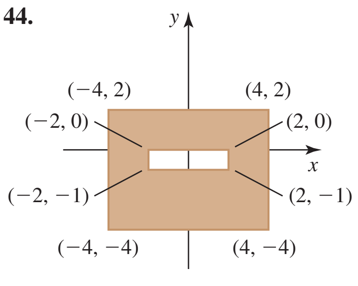


45–50. Centroids *Use polar coordinates to find the centroid of the following constant-density plane regions.*

45. The semicircular disk $R = \{(r, \theta): 0 \le r \le 2, 0 \le \theta \le \pi\}$

46. The quarter-circular disk $R = \{(r, \theta): 0 \le r \le 2, 0 \le \theta \le \pi/2\}$

47. The region bounded by the cardioid $r = 1 + \cos\theta$

48. The region bounded by the cardioid $r = 3 - 3\cos\theta$

49. The region bounded by one leaf of the rose $r = \sin 2\theta$, for $0 \le \theta \le \pi/2$

50. The region bounded by the limaçon $r = 2 + \cos\theta$

51. Semicircular wire A thin (one-dimensional) wire of constant density is bent into the shape of a semicircle of radius r. Find the location of its center of mass. (*Hint:* Treat the wire as a thin half-annulus with width Δa, and then let $\Delta a \to 0$.)

52. Parabolic region A thin plate of unit density occupies the region between the parabola $y = ax^2$ and the horizontal line $y = b$, where $a > 0$ and $b > 0$. Show that the center of mass is $\left(0, \dfrac{3b}{5}\right)$, independent of a.

53. Circular crescent Find the center of mass of the region in the first quadrant bounded by the circle $x^2 + y^2 = a^2$ and the lines $x = a$ and $y = a$, where $a > 0$.

54–57. Centers of mass for general objects *Consider the following two- and three-dimensional regions with variable dimensions. Specify the surfaces and curves that bound the region, choose a convenient coordinate system, and compute the center of mass assuming constant density. All parameters are positive real numbers.*

54. A solid rectangular box has sides of lengths a, b, and c. Where is the center of mass relative to the faces of the box?

55. A solid cone has a base with a radius of a and a height of h. How far from the base is the center of mass?

56. A solid is enclosed by a hemisphere of radius a. How far from the base is the center of mass?

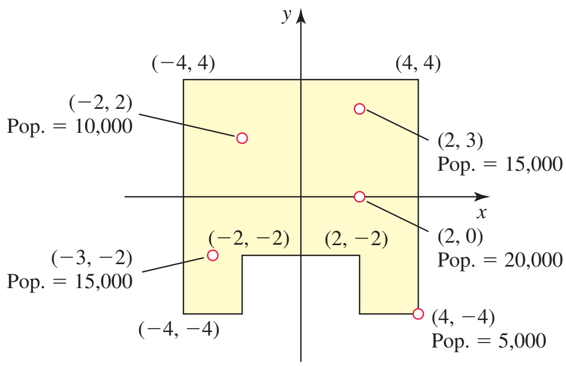
57. A region is enclosed by an isosceles triangle with two sides of length s and a base of length b. How far from the base is the center of mass?

Explorations and Challenges

58. A tetrahedron is bounded by the coordinate planes and the plane $x/a + y/a + z/a = 1$. What are the coordinates of the center of mass?

59. A solid is enclosed by the upper half of an ellipsoid with a circular base of radius r and a height of a. How far from the base is the center of mass?

60. Geographic vs. population center Geographers measure the *geographical center* of a country (which is the centroid) and the *population center* of a country (which is the center of mass computed with the population density). A hypothetical country is shown in the figure with the location and population of five towns. Assuming no one lives outside the towns, find the geographical center of the country and the population center of the country.



61. Center of mass on the edge Consider the thin constant-density plate $\{(r, \theta): 0 < a \leq r \leq 1, 0 \leq \theta \leq \pi\}$ bounded by two semicircles and the x-axis.

a. Find and graph the y-coordinate of the center of mass of the plate as a function of a.

b. For what value of a is the center of mass on the edge of the plate?

62. Center of mass on the edge Consider the constant-density solid $\{(\rho, \varphi, \theta): 0 < a \leq \rho \leq 1, 0 \leq \varphi \leq \pi/2, 0 \leq \theta \leq 2\pi\}$ bounded by two hemispheres and the xy-plane.

a. Find and graph the z-coordinate of the center of mass of the plate as a function of a.

b. For what value of a is the center of mass on the edge of the solid?

63. Draining a soda can A cylindrical soda can has a radius of 4 cm and a height of 12 cm. When the can is full of soda, the center of mass of the contents of the can is 6 cm above the base on the axis of the can (halfway along the axis of the can). As the can is drained, the center of mass descends for a while. However, when the can is empty (filled only with air), the center of mass is once again 6 cm above the base on the axis of the can. Find the depth of soda in the can for which the center of mass is at its lowest point. Neglect the mass of the can, and assume the density of the soda is $1\ \text{g/cm}^3$ and the density of air is $0.001\ \text{g/cm}^3$.

64. Triangle medians A triangular region has a base that connects the vertices $(0, 0)$ and $(b, 0)$, and a third vertex at (a, h), where $a > 0$, $b > 0$, and $h > 0$.

a. Show that the centroid of the triangle is $\left(\frac{a+b}{3}, \frac{h}{3}\right)$.

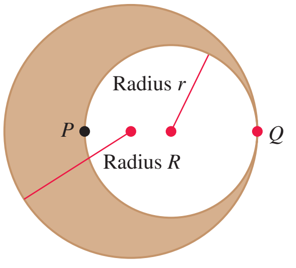
b. Recall that the three medians of a triangle extend from each vertex to the midpoint of the opposite side. Knowing that the medians of a triangle intersect in a point M and that each median bisects the triangle, conclude that the centroid of the triangle is M.

65. The golden earring A disk of radius r is removed from a larger disk of radius R to form an earring (see figure). Assume the earring is a thin plate of uniform density.

a. Find the center of mass of the earring in terms of r and R. (*Hint:* Place the origin of a coordinate system either at the center of the large disk or at Q; either way, the earring is symmetric about the x-axis.)

b. Show that the ratio $\frac{R}{r}$ such that the center of mass lies at the point P (on the edge of the inner disk) is the golden mean $\frac{1 + \sqrt{5}}{2} \approx 1.618$.

(*Source:* P. Glaister, *Golden Earrings*, *Mathematical Gazette*, 80, 1996)



QUICK CHECK ANSWERS

1. 3 m **3.** It is heaviest at $x = 0$ and lightest at $x = 3$. **4.** The distance from the point (x, y) to the y-axis is x. The constant density appears in the integral for the moment, and it appears in the integral for the mass. Therefore, the density cancels when we divide the two integrals. **5.** The distance from the xy-plane to a point (x, y, z) is z. ➤

16.7 Change of Variables in Multiple Integrals

Converting double integrals from rectangular coordinates to polar coordinates (Section 16.3) and converting triple integrals from rectangular coordinates to cylindrical or spherical coordinates (Section 16.5) are examples of a general procedure known as a *change of variables*. The idea is not new: The Substitution Rule introduced in Chapter 5 with single-variable integrals is also a change of variables. The aim of this section is to show you how to change variables in double and triple integrals.

Recap of Change of Variables

Recall how a change of variables is used to simplify a single-variable integral. For example, to simplify the integral $\int_0^1 2\sqrt{2x + 1}\, dx$, we choose a new variable $u = 2x + 1$, which means that $du = 2\, dx$. Therefore,

$$\int_0^1 2\sqrt{2x + 1}\, dx = \int_1^3 \sqrt{u}\, du.$$

This equality means that the area under the curve $y = 2\sqrt{2x + 1}$ from $x = 0$ to $x = 1$ equals the area under the curve $y = \sqrt{u}$ from $u = 1$ to $u = 3$ (Figure 16.76). The relation $du = 2\,dx$ relates the length of a small interval on the u-axis to the length of the corresponding interval on the x-axis.

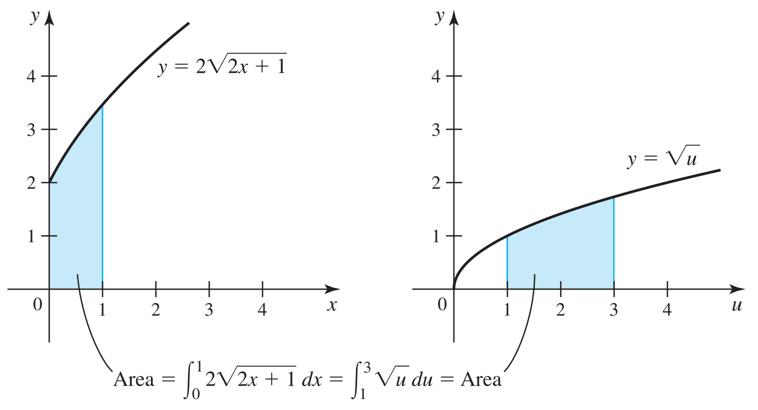


Figure 16.76

Similarly, some double and triple integrals can be simplified through a change of variables. For example, the region of integration for

$$\int_0^1 \int_0^{\sqrt{1-x^2}} e^{1-x^2-y^2}\,dy\,dx$$

is the quarter disk $R = \{(x, y): x \geq 0, y \geq 0, x^2 + y^2 \leq 1\}$. Changing variables to polar coordinates with $x = r\cos\theta$, $y = r\sin\theta$, and $dy\,dx = r\,dr\,d\theta$, we have

$$\int_0^1 \int_0^{\sqrt{1-x^2}} e^{1-x^2-y^2}\,dy\,dx \overset{\substack{x = r\cos\theta \\ y = r\sin\theta}}{=} \int_0^{\pi/2} \int_0^1 e^{1-r^2}\,r\,dr\,d\theta.$$

In this case, the original region of integration R is transformed into a new region $S = \{(r, \theta): 0 \leq r \leq 1, 0 \leq \theta \leq \pi/2\}$, which is a rectangle in the $r\theta$-plane.

Transformations in the Plane

A change of variables in a double integral is a *transformation* that relates two sets of variables, (u, v) and (x, y). It is written compactly as $(x, y) = T(u, v)$. Because it relates pairs of variables, T has two components,

$$T\colon x = g(u, v) \quad \text{and} \quad y = h(u, v).$$

Geometrically, T takes a region S in the uv-plane and "maps" it point by point to a region R in the xy-plane (Figure 16.77). We write the outcome of this process as $R = T(S)$ and call R the **image** of S under T.

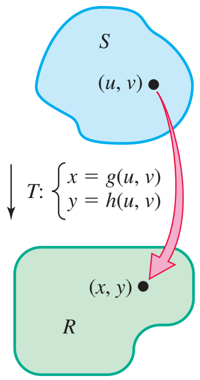


Figure 16.77

EXAMPLE 1 **Image of a transformation** Consider the transformation from polar to rectangular coordinates given by

$$T\colon \quad x = g(r, \theta) = r\cos\theta \quad \text{and} \quad y = h(r, \theta) = r\sin\theta.$$

Find the image under this transformation of the rectangle

$$S = \{(r, \theta): 0 \leq r \leq 1, 0 \leq \theta \leq \pi/2\}.$$

➤ In Example 1, we have replaced the coordinates u and v with the familiar polar coordinates r and θ.

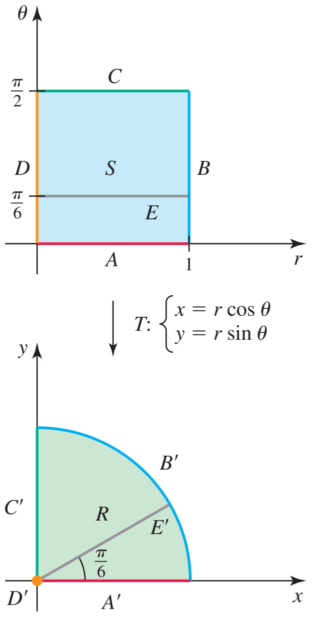


Figure 16.78

SOLUTION If we apply T to every point of S (**Figure 16.78**), what is the resulting set R in the xy-plane? One way to answer this question is to walk around the boundary of S, let's say counterclockwise, and determine the corresponding path in the xy-plane. In the $r\theta$-plane, we let the horizontal axis be the r-axis and the vertical axis be the θ-axis. Starting at the origin, we denote the edges of the rectangle S as follows.

$$A = \{(r, \theta): 0 \le r \le 1, \theta = 0\} \quad \text{Lower boundary}$$

$$B = \left\{(r, \theta): r = 1, 0 \le \theta \le \frac{\pi}{2}\right\} \quad \text{Right boundary}$$

$$C = \left\{(r, \theta): 0 \le r \le 1, \theta = \frac{\pi}{2}\right\} \quad \text{Upper boundary}$$

$$D = \left\{(r, \theta): r = 0, 0 \le \theta \le \frac{\pi}{2}\right\} \quad \text{Left boundary}$$

Table 16.6 shows the effect of the transformation on the four boundaries of S; the corresponding boundaries of R in the xy-plane are denoted A', B', C', and D' (Figure 16.78).

Table 16.6

Boundary of S in $r\theta$-plane	Transformation equations	Boundary of R in xy-plane
A: $0 \le r \le 1, \theta = 0$	$x = r\cos\theta = r$, $y = r\sin\theta = 0$	A': $0 \le x \le 1, y = 0$
B: $r = 1, 0 \le \theta \le \pi/2$	$x = r\cos\theta = \cos\theta$, $y = r\sin\theta = \sin\theta$	B': quarter unit circle
C: $0 \le r \le 1, \theta = \pi/2$	$x = r\cos\theta = 0$, $y = r\sin\theta = r$	C': $x = 0, 0 \le y \le 1$
D: $r = 0, 0 \le \theta \le \pi/2$	$x = r\cos\theta = 0$, $y = r\sin\theta = 0$	D': single point $(0, 0)$

QUICK CHECK 1 How would the image of S change in Example 1 if $S = \{(r, \theta): 0 \le r \le 1, 0 \le \theta \le \pi\}$?◄

The image of the rectangular boundary of S is the boundary of R. Furthermore, it can be shown that every point in the interior of R is the image of one point in the interior of S. (For example, the horizontal line segment E in the $r\theta$-plane in Figure 16.78 is mapped to the line segment E' in the xy-plane.) Therefore, the image of S is the quarter disk R in the xy-plane.

Related Exercises 11–12 ◄

Recall that a function f is *one-to-one* on an interval I if $f(x_1) = f(x_2)$ only when $x_1 = x_2$, where x_1 and x_2 are points of I. We need an analogous property for transformations when changing variables.

DEFINITION One-to-One Transformation

A transformation T from a region S to a region R is one-to-one on S if $T(P) = T(Q)$ only when $P = Q$, where P and Q are points in S.

Notice that the polar coordinate transformation in Example 1 is not one-to-one on the rectangle $S = \{(r, \theta): 0 \le r \le 1, 0 \le \theta \le \pi/2\}$ (because all points with $r = 0$ map to the point $(0, 0)$). However, this transformation *is* one-to-one on the interior of S.

We can now anticipate how a transformation (change of variables) is used to simplify a double integral. Suppose we have the integral $\iint_R f(x, y)\, dA$. The goal is to find a transformation to a new set of coordinates (u, v) such that the new equivalent integral $\iint_S f(x(u, v), y(u, v))\, dA$ involves a simple region S (such as a rectangle), a simple integrand, or both. The next theorem allows us to do exactly that, but it first requires a new concept.

➤ The Jacobian is named after the German mathematician Carl Gustav Jacob Jacobi (1804–1851). In some books, the Jacobian is the matrix of partial derivatives. In others, as here, the Jacobian is the determinant of the matrix of partial derivatives. Both $J(u, v)$ and $\frac{\partial(x, y)}{\partial(u, v)}$ are used to refer to the Jacobian.

DEFINITION Jacobian Determinant of a Transformation of Two Variables

Given a transformation T: $x = g(u, v)$, $y = h(u, v)$, where g and h are differentiable on a region of the uv-plane, the **Jacobian determinant** (or **Jacobian**) of T is

$$J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

QUICK CHECK 2 Find $J(u, v)$ if $x = u + v$, $y = 2v$. ◄

The Jacobian is easiest to remember as the determinant of a 2×2 matrix of partial derivatives. With the Jacobian in hand, we can state the change-of-variables rule for double integrals.

➤ The condition that g and h have continuous first partial derivatives ensures that the new integrand is integrable.

THEOREM 16.8 Change of Variables for Double Integrals

Let T: $x = g(u, v)$, $y = h(u, v)$ be a transformation that maps a closed bounded region S in the uv-plane to a region R in the xy-plane. Assume T is one-to-one on the interior of S and g and h have continuous first partial derivatives there. If f is continuous on R, then

$$\iint_R f(x, y)\, dA = \iint_S f(g(u, v), h(u, v))\, |J(u, v)|\, dA.$$

➤ In the integral over R, dA corresponds to $dx\,dy$. In the integral over S, dA corresponds to $du\,dv$. The relation $dx\,dy = |J|\,du\,dv$ is the analog of $du = g'(x)\,dx$ in a change of variables with one variable.

The proof of this result is technical and is found in advanced texts. The factor $|J(u, v)|$ that appears in the second integral is the absolute value of the Jacobian. Matching the area elements in the two integrals of Theorem 16.8, we see that $dx\,dy = |J(u, v)|\,du\,dv$. This expression shows that the Jacobian is a magnification (or reduction) factor: It relates the area of a small region $dx\,dy$ in the xy-plane to the area of the corresponding region $du\,dv$ in the uv-plane. If the transformation equations are linear, then this relationship is exact in the sense that area$(T(S)) = |J(u, v)| \cdot$ area of S (see Exercise 60). The way in which the Jacobian arises is explored in Exercise 61.

EXAMPLE 2 Jacobian of the polar-to-rectangular transformation Compute the Jacobian of the transformation

$$T: \quad x = g(r, \theta) = r\cos\theta \quad \text{and} \quad y = h(r, \theta) = r\sin\theta.$$

SOLUTION The necessary partial derivatives are

$$\frac{\partial x}{\partial r} = \cos\theta, \quad \frac{\partial x}{\partial \theta} = -r\sin\theta, \quad \frac{\partial y}{\partial r} = \sin\theta, \quad \text{and} \quad \frac{\partial y}{\partial \theta} = r\cos\theta.$$

Therefore,

$$J(r, \theta) = \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r.$$

This determinant calculation confirms the change-of-variables formula for polar coordinates: $dx\,dy$ becomes $r\,dr\,d\theta$.

Related Exercise 20 ◄

We are now ready for a change of variables. To transform the integral $\iint_R f(x, y)\,dA$ into $\iint_S f(x(u, v), y(u, v))\,|J(u, v)|\,dA$, we must find the transformation $x = g(u, v)$ and $y = h(u, v)$, and then use it to find the new region of integration S. The next example illustrates how the region S is found, assuming the transformation is given.

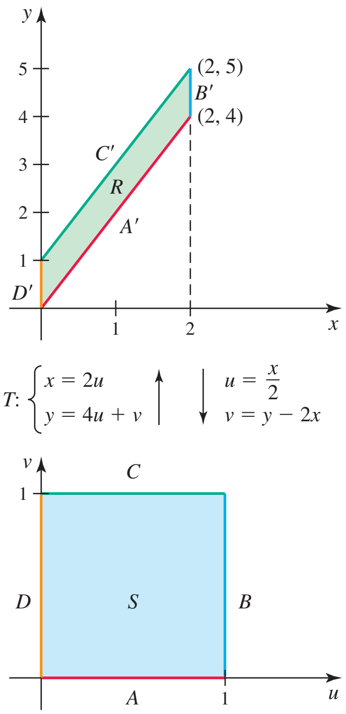


Figure 16.79

➤ The relations that "go the other direction" make up the inverse transformation, usually denoted T^{-1}.

EXAMPLE 3 Double integral with a change of variables given Evaluate the integral $\iint_R \sqrt{2x(y - 2x)}\,dA$, where R is the parallelogram in the xy-plane with vertices $(0, 0)$, $(0, 1)$, $(2, 4)$, and $(2, 5)$ (Figure 16.79). Use the transformation

$$T\colon x = 2u \quad \text{and} \quad y = 4u + v.$$

SOLUTION To what region S in the uv-plane is R mapped? Because T takes points in the uv-plane and assigns them to points in the xy-plane, we must reverse the process by solving $x = 2u$, $y = 4u + v$ for u and v.

$$\text{First equation: } x = 2u \implies u = \frac{x}{2}$$

$$\text{Second equation: } y = 4u + v \implies v = y - 4u = y - 2x$$

Rather than walk around the boundary of R in the xy-plane to determine the resulting region S in the uv-plane, it suffices to find the images of the vertices of R. You should confirm that the vertices map as shown in Table 16.7.

Table 16.7

(x, y)	(u, v)
$(0, 0)$	$(0, 0)$
$(0, 1)$	$(0, 1)$
$(2, 5)$	$(1, 1)$
$(2, 4)$	$(1, 0)$

Connecting the points in the uv-plane in order, we see that S is the unit square $\{(u, v)\colon 0 \le u \le 1, 0 \le v \le 1\}$ (Figure 16.79). These inequalities determine the limits of integration in the uv-plane.

Replacing $2x$ with $4u$ and $y - 2x$ with v, the original integrand becomes $\sqrt{2x(y - 2x)} = \sqrt{4uv}$. The Jacobian is

$$J(u, v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 4 & 1 \end{vmatrix} = 2.$$

➤ T is an example of a *shearing transformation*. The greater the u-coordinate of a point, the more that point is displaced in the v-direction. It also involves a uniform stretch in the u-direction.

The integration now follows:

$$\iint_R \sqrt{2x(y - 2x)}\,dA = \iint_S \sqrt{4uv}\,\underbrace{|J(u, v)|}_{2}\,dA \qquad \text{Change variables.}$$

$$= \int_0^1 \int_0^1 \sqrt{4uv}\,2\,du\,dv \qquad \text{Convert to an iterated integral.}$$

$$= 4\int_0^1 \frac{2}{3}\sqrt{v}\,(u^{3/2})\Big|_0^1\,dv \qquad \text{Evaluate inner integral.}$$

$$= \frac{8}{3}\cdot\frac{2}{3}\,(v^{3/2})\Big|_0^1 = \frac{16}{9}. \qquad \text{Evaluate outer integral.}$$

The effect of the change of variables is illustrated in Figure 16.80, where we see the surface $z = \sqrt{2x(y - 2x)}$ over the region R and the surface $w = 2\sqrt{4uv}$ over the region S. The volumes of the solids beneath the two surfaces are equal, but the integral over S is easier to evaluate.

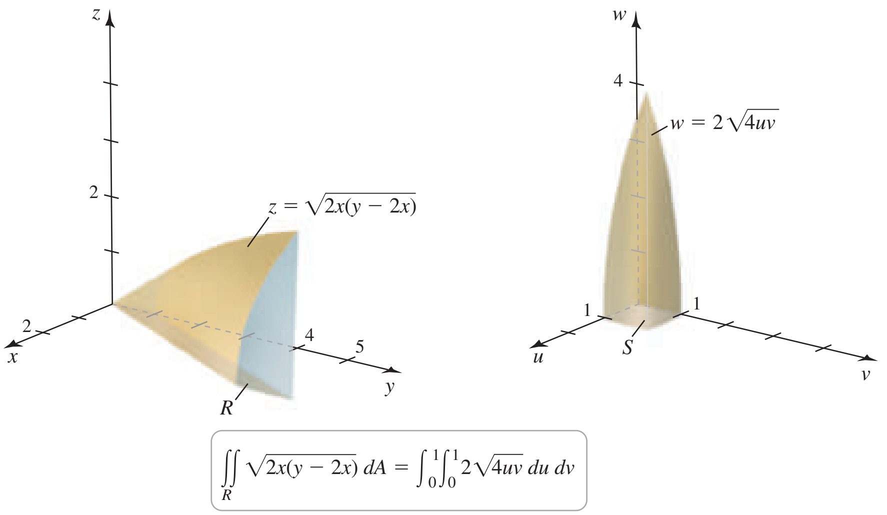


Figure 16.80

Related Exercise 29 ◄

QUICK CHECK 3 Solve the equations $u = x + y$, $v = -x + 2y$ for x and y. ◄

In Example 3, the required transformation was given. More practically, we must deduce an appropriate transformation from the form of either the integrand or the region of integration.

EXAMPLE 4 Change of variables determined by the integrand Evaluate $\iint_R \sqrt{\dfrac{x-y}{x+y+1}}\,dA$, where R is the square with vertices $(0, 0)$, $(1, -1)$, $(2, 0)$, and $(1, 1)$ (Figure 16.81).

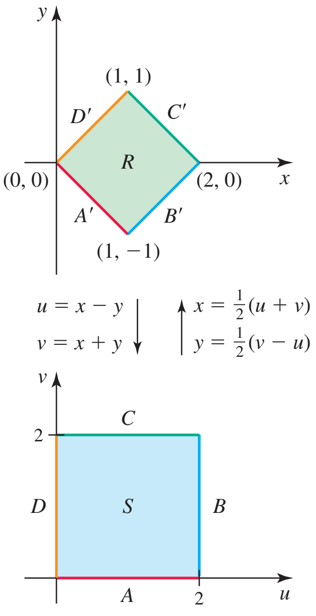


Figure 16.81

➤ The transformation in Example 4 is a *rotation*. It rotates the points of R about the origin 45° in the counterclockwise direction (it also increases lengths by a factor of $\sqrt{2}$). In this example, the change of variables $u = x + y$ and $v = x - y$ would work just as well.

SOLUTION Evaluating the integral as it stands requires splitting the region R into two subregions; furthermore, the integrand presents difficulties. The terms $x + y$ and $x - y$ in the integrand suggest the new variables

$$u = x - y \quad \text{and} \quad v = x + y.$$

To determine the region S in the uv-plane that corresponds to R under this transformation, we find the images of the vertices of R in the uv-plane and connect them in order. The result is the square $S = \{(u, v): 0 \le u \le 2, 0 \le v \le 2\}$ (Figure 16.81). Before computing the Jacobian, we express x and y in terms of u and v. Adding the two equations and solving for x, we have $x = (u + v)/2$. Subtracting the two equations and solving for y gives $y = (v - u)/2$. The Jacobian now follows:

$$J(u, v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{1}{2} & \dfrac{1}{2} \\ -\dfrac{1}{2} & \dfrac{1}{2} \end{vmatrix} = \frac{1}{2}.$$

With the choice of new variables, the original integrand $\sqrt{\dfrac{x-y}{x+y+1}}$ becomes $\sqrt{\dfrac{u}{v+1}}$.

The integration in the uv-plane may now be done:

$$\begin{aligned}
\iint_R \sqrt{\frac{x-y}{x+y+1}}\,dA &= \iint_S \sqrt{\frac{u}{v+1}}\,|J(u,v)|\,dA && \text{Change of variables} \\
&= \int_0^2\int_0^2 \sqrt{\frac{u}{v+1}}\,\frac{1}{2}\,du\,dv && \text{Convert to an iterated integral.} \\
&= \frac{1}{2}\int_0^2 (v+1)^{-1/2}\,\frac{2}{3}\,(u^{3/2})\Big|_0^2\,dv && \text{Evaluate inner integral.}
\end{aligned}$$

➤ An appropriate change of variables for a double integral is not always obvious. Some trial and error is often needed to come up with a transformation that simplifies the integrand and/or the region of integration. Strategies are discussed at the end of this section.

$$= \frac{2^{3/2}}{3} 2(v + 1)^{1/2}\Big|_0^2 \qquad \text{Evaluate outer integral.}$$

$$= \frac{4\sqrt{2}}{3}(\sqrt{3} - 1). \qquad \text{Simplify.}$$

Related Exercises 32, 36 ◀

QUICK CHECK 4 In Example 4, what is the ratio of the area of S to the area of R? How is this ratio related to J? ◀

EXAMPLE 5 Change of variables determined by the region Let R be the region in the first quadrant bounded by the parabolas $x = y^2$, $x = y^2 - 4$, $x = 9 - y^2$, and $x = 16 - y^2$ (Figure 16.82). Evaluate $\iint_R y^2\, dA$.

SOLUTION Notice that the bounding curves may be written as $x - y^2 = 0$, $x - y^2 = -4$, $x + y^2 = 9$, and $x + y^2 = 16$. The first two parabolas have the form $x - y^2 = C$, where C is a constant, which suggests the new variable $u = x - y^2$. The last two parabolas have the form $x + y^2 = C$, which suggests the new variable $v = x + y^2$. Therefore, the new variables are

$$u = x - y^2 \quad \text{and} \quad v = x + y^2.$$

The boundary curves of S are $u = -4$, $u = 0$, $v = 9$, and $v = 16$. Therefore, the new region is $S = \{(u, v): -4 \le u \le 0, 9 \le v \le 16\}$ (Figure 16.82). To compute the Jacobian, we must find the transformation T by writing x and y in terms of u and v. Solving for x and y, and observing that $y \ge 0$ for all points in R, we find that

$$T:\quad x = \frac{u + v}{2} \quad \text{and} \quad y = \sqrt{\frac{v - u}{2}}.$$

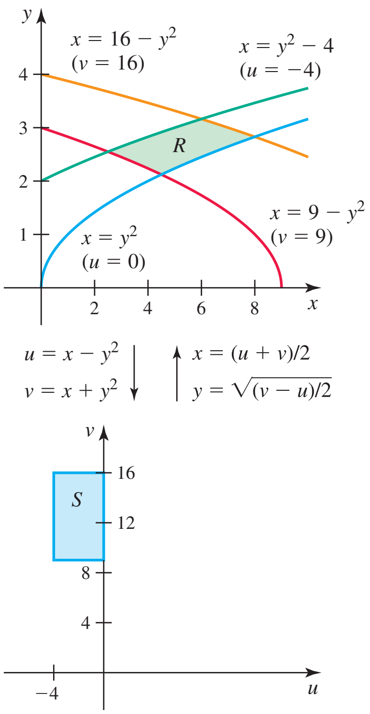


Figure 16.82

The points of S satisfy $v > u$, so $\sqrt{v - u}$ is defined. Now the Jacobian may be computed:

$$J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2\sqrt{2(v - u)}} & \frac{1}{2\sqrt{2(v - u)}} \end{vmatrix} = \frac{1}{2\sqrt{2(v - u)}}.$$

The change of variables proceeds as follows:

$$\iint_R y^2\, dA = \int_9^{16}\int_{-4}^0 \underbrace{\frac{v - u}{2}}_{y^2}\, \underbrace{\frac{1}{2\sqrt{2(v - u)}}}_{|J(u,v)|}\, du\, dv \qquad \text{Convert to an iterated integral.}$$

$$= \frac{1}{4\sqrt{2}} \int_9^{16}\int_{-4}^0 \sqrt{v - u}\, du\, dv \qquad \text{Simplify.}$$

$$= \frac{1}{4\sqrt{2}}\, \frac{2}{3} \int_9^{16} \left(-(v - u)^{3/2}\right)\Big|_{-4}^0 dv \qquad \text{Evaluate inner integral.}$$

$$= \frac{1}{6\sqrt{2}} \int_9^{16} \left((v + 4)^{3/2} - v^{3/2}\right) dv \qquad \text{Simplify.}$$

$$= \frac{1}{6\sqrt{2}}\, \frac{2}{5} \left((v + 4)^{5/2} - v^{5/2}\right)\Big|_9^{16} \qquad \text{Evaluate outer integral.}$$

$$= \frac{\sqrt{2}}{30}\left(32 \cdot 5^{5/2} - 13^{5/2} - 781\right) \qquad \text{Simplify.}$$

$$\approx 18.79.$$

Related Exercises 33–34 ◀

Change of Variables in Triple Integrals

With triple integrals, we work with a transformation T of the form

$$T:\qquad x = g(u, v, w), \quad y = h(u, v, w), \quad \text{and} \quad z = p(u, v, w).$$

In this case, T maps a region S in uvw-space to a region D in xyz-space. As before, the goal is to transform the integral $\iiint_D f(x, y, z)\, dV$ into a new integral over the region S that is easier to evaluate. First, we need a Jacobian.

> Recall that expanding about the first row yields
>
> $$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}).$$

DEFINITION Jacobian Determinant of a Transformation of Three Variables

Given a transformation T: $x = g(u, v, w)$, $y = h(u, v, w)$, and $z = p(u, v, w)$, where g, h, and p are differentiable on a region of uvw-space, the **Jacobian determinant** (or **Jacobian**) of T is

$$J(u, v, w) = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}.$$

The Jacobian is evaluated as a 3×3 determinant and is a function of u, v, and w. A change of variables with respect to three variables proceeds in analogy to the two-variable case.

> If we match the elements of volume in both integrals, then $dx\,dy\,dz = |J(u, v, w)|\,du\,dv\,dw$. As before, the Jacobian is a magnification (or reduction) factor, now relating the volume of a small region in xyz-space to the volume of the corresponding region in uvw-space.

THEOREM 16.9 Change of Variables for Triple Integrals

Let T: $x = g(u, v, w)$, $y = h(u, v, w)$, and $z = p(u, v, w)$ be a transformation that maps a closed bounded region S in uvw-space to a region $D = T(S)$ in xyz-space. Assume T is one-to-one on the interior of S and g, h, and p have continuous first partial derivatives there. If f is continuous on D, then

$$\iiint_D f(x, y, z)\,dV = \iiint_S f(g(u, v, w), h(u, v, w), p(u, v, w))\,|J(u, v, w)|\,dV.$$

> To see that triple integrals in cylindrical and spherical coordinates as derived in Section 16.5 are consistent with this change-of-variables formulation, see Exercises 46 and 47.

EXAMPLE 6 **A triple integral** Use a change of variables to evaluate $\iiint_D xz\,dV$, where D is a parallelepiped bounded by the planes

$$y = x, \quad y = x + 2, \quad z = x, \quad z = x + 3, \quad z = 0, \quad \text{and} \quad z = 4$$

(Figure 16.83a).

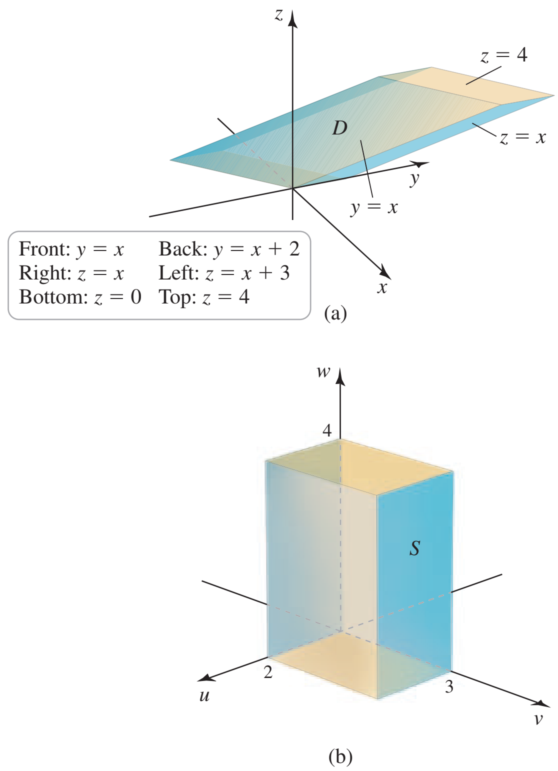


Figure 16.83

SOLUTION The key is to note that D is bounded by three pairs of parallel planes.

- $y - x = 0$ and $y - x = 2$
- $z - x = 0$ and $z - x = 3$
- $z = 0$ and $z = 4$

These combinations of variables suggest the new variables

$$u = y - x, \qquad v = z - x, \quad \text{and} \quad w = z.$$

With this choice, the new region of integration (**Figure 16.83b**) is the rectangular box

$$S = \{(u, v, w): 0 \le u \le 2, 0 \le v \le 3, 0 \le w \le 4\}.$$

To compute the Jacobian, we must express x, y, and z in terms of u, v, and w. A few steps of algebra lead to the transformation

$$T: \qquad x = w - v, \quad y = u - v + w, \quad \text{and} \quad z = w.$$

The resulting Jacobian is

$$J(u, v, w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 0 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1.$$

➤ It is easiest to expand the Jacobian determinant in Example 6 about the third row.

Noting that the integrand is $xz = (w - v)w = w^2 - vw$, the integral may now be evaluated:

$$\begin{aligned}
\iiint_D xz\, dV &= \iiint_S (w^2 - vw)\,|J(u, v, w)|\, dV && \text{Change variables.}\\
&= \int_0^4 \int_0^3 \int_0^2 (w^2 - vw) \underbrace{1}_{|J(u,v,w)|}\, du\, dv\, dw && \text{Convert to an iterated integral.}\\
&= \int_0^4 \int_0^3 2(w^2 - vw)\, dv\, dw && \text{Evaluate inner integral.}\\
&= 2\int_0^4 \left(vw^2 - \frac{v^2 w}{2} \right)\Big|_0^3 dw && \text{Evaluate middle integral.}\\
&= 2\int_0^4 \left(3w^2 - \frac{9w}{2} \right) dw && \text{Simplify.}\\
&= 2\left(w^3 - \frac{9w^2}{4} \right)\Big|_0^4 = 56. && \text{Evaluate outer integral.}
\end{aligned}$$

Related Exercises 40–41 ◄

QUICK CHECK 5 Interpret a Jacobian with a value of 1 (as in Example 6). ◄

Strategies for Choosing New Variables

Sometimes a change of variables simplifies the integrand but leads to an awkward region of integration. Conversely, the new region of integration may be simplified at the expense of additional complications in the integrand. Here are a few suggestions for finding new variables of integration. The observations are made with respect to double integrals, but they also apply to triple integrals. As before, R is the original region of integration in the xy-plane, and S is the new region in the uv-plane.

1. **Aim for simple regions of integration in the *uv*-plane** The new region of integration in the uv-plane should be as simple as possible. Double integrals are easiest to evaluate over rectangular regions with sides parallel to the coordinate axes.
2. **Is $(x, y) \to (u, v)$ or $(u, v) \to (x, y)$ better?** For some problems it is easier to write (x, y) as functions of (u, v); in other cases, the opposite is true. Depending on the

➤ Inverting the transformation means solving for x and y in terms of u and v, or vice versa.

problem, inverting the transformation (finding relations that go in the opposite direction) may be easy, difficult, or impossible.

- If you know (x, y) in terms of (u, v) (that is, $x = g(u, v)$ and $y = h(u, v)$), then computing the Jacobian is straightforward, as is sketching the region R given the region S. However, the transformation must be inverted to determine the shape of S.
- If you know (u, v) in terms of (x, y) (that is, $u = G(x, y)$ and $v = H(x, y)$), then sketching the region S is straightforward. However, the transformation must be inverted to compute the Jacobian.

3. **Let the integrand suggest new variables** New variables are often chosen to simplify the integrand. For example, the integrand $\sqrt{\frac{x - y}{x + y}}$ calls for new variables $u = x - y$ and $v = x + y$ (or $u = x + y$, $v = x - y$). There is, however, no guarantee that this change of variables will simplify the region of integration. In cases in which only one combination of variables appears, let one new variable be that combination and let the other new variable be unchanged. For example, if the integrand is $(x + 4y)^{3/2}$, try letting $u = x + 4y$ and $v = y$.
4. **Let the region suggest new variables** Example 5 illustrates an ideal situation. It occurs when the region R is bounded by two pairs of "parallel" curves in the families $g(x, y) = C_1$ and $h(x, y) = C_2$ (**Figure 16.84**). In this case, the new region of integration is a rectangle $S = \{(u, v): a_1 \le u \le a_2, b_1 \le v \le b_2\}$, where $u = g(x, y)$ and $v = h(x, y)$.

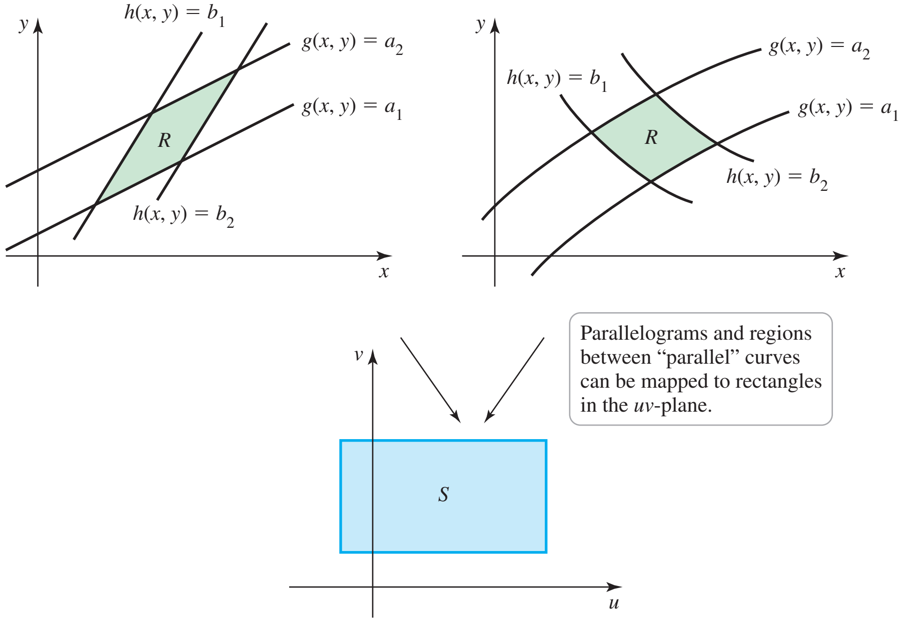


Figure 16.84

As another example, suppose the region is bounded by the lines $y = x$ (or $y/x = 1$) and $y = 2x$ (or $y/x = 2$) and by the hyperbolas $xy = 1$ and $xy = 3$. Then the new variables should be $u = xy$ and $v = y/x$ (or vice versa). The new region of integration is the rectangle $S = \{(u, v): 1 \le u \le 3, 1 \le v \le 2\}$.

SECTION 16.7 EXERCISES

Getting Started

1. Suppose S is the unit square in the first quadrant of the uv-plane. Describe the image of the transformation $T: x = 2u, y = 2v$.

2. Explain how to compute the Jacobian of the transformation $T: x = g(u, v), y = h(u, v)$.

3. Using the transformation $T: x = u + v, y = u - v$, the image of the unit square $S = \{(u, v): 0 \le u \le 1, 0 \le v \le 1\}$ is a region R in the xy-plane. Explain how to change variables in the integral $\iint_R f(x, y)\, dA$ to find a new integral over S.

4. Suppose S is the unit cube in the first octant of uvw-space with one vertex at the origin. What is the image of the transformation $T: x = u/2, y = v/2, z = w/2$?

Practice Exercises

5–12. Transforming a square *Let $S = \{(u, v): 0 \le u \le 1, 0 \le v \le 1\}$ be a unit square in the uv-plane. Find the image of S in the xy-plane under the following transformations.*

5. $T: x = 2u, y = v/2$

6. $T: x = -u, y = -v$

7. $T: x = (u + v)/2, y = (u - v)/2$

8. $T: x = 2u + v, y = 2u$

9. $T: x = u^2 - v^2, y = 2uv$

10. $T: x = 2uv, y = u^2 - v^2$

11. $T: x = u \cos \pi v, y = u \sin \pi v$

12. $T: x = v \sin \pi u, y = v \cos \pi u$

13–16. Images of regions *Find the image R in the xy-plane of the region S using the given transformation T. Sketch both R and S.*

13. $S = \{(u, v): v \le 1 - u, u \ge 0, v \ge 0\}; T: x = u, y = v^2$

14. $S = \{(u, v): u^2 + v^2 \le 1\}; T: x = 2u, y = 4v$

15. $S = \{(u, v): 1 \le u \le 3, 2 \le v \le 4\}; T: x = u/v, y = v$

16. $S = \{(u, v): 2 \le u \le 3, 3 \le v \le 6\}; T: x = u, y = v/u$

17–22. Computing Jacobians *Compute the Jacobian $J(u, v)$ for the following transformations.*

17. $T: x = 3u, y = -3v$

18. $T: x = 4v, y = -2u$

19. $T: x = 2uv, y = u^2 - v^2$

20. $T: x = u \cos \pi v, y = u \sin \pi v$

21. $T: x = (u + v)/\sqrt{2}, y = (u - v)/\sqrt{2}$

22. $T: x = u/v, y = v$

23–26. Solve and compute Jacobians *Solve the following relations for x and y, and compute the Jacobian $J(u, v)$.*

23. $u = x + y, v = 2x - y$

24. $u = xy, v = x$

25. $u = 2x - 3y, v = y - x$

26. $u = x + 4y, v = 3x + 2y$

27–30. Double integrals—transformation given *To evaluate the following integrals, carry out these steps.*

a. Sketch the original region of integration R in the xy-plane and the new region S in the uv-plane using the given change of variables.
b. Find the limits of integration for the new integral with respect to u and v.
c. Compute the Jacobian.
d. Change variables and evaluate the new integral.

27. $\displaystyle\iint_R xy\, dA$, where R is the square with vertices $(0, 0), (1, 1), (2, 0)$, and $(1, -1)$; use $x = u + v, y = u - v$.

28. $\displaystyle\iint_R x^2 y\, dA$, where $R = \{(x, y): 0 \le x \le 2, x \le y \le x + 4\}$; use $x = 2u, y = 4v + 2u$.

29. $\displaystyle\iint_R x^2\sqrt{x + 2y}\, dA$, where $R = \{(x, y): 0 \le x \le 2, -x/2 \le y \le 1 - x\}$; use $x = 2u, y = v - u$.

30. $\displaystyle\iint_R xy\, dA$, where R is bounded by the ellipse $9x^2 + 4y^2 = 36$; use $x = 2u, y = 3v$.

31–36. Double integrals—your choice of transformation *Evaluate the following integrals using a change of variables. Sketch the original and new regions of integration, R and S.*

31. $\displaystyle\int_0^1 \int_y^{y+2} \sqrt{x - y}\, dx\, dy$

32. $\displaystyle\iint_R \sqrt{y^2 - x^2}\, dA$, where R is the diamond bounded by $y - x = 0$, $y - x = 2$, $y + x = 0$, and $y + x = 2$

33. $\displaystyle\iint_R \left(\frac{y - x}{y + 2x + 1}\right)^4 dA$, where R is the parallelogram bounded by $y - x = 1$, $y - x = 2$, $y + 2x = 0$, and $y + 2x = 4$

34. $\displaystyle\iint_R e^{xy}\, dA$, where R is the region in the first quadrant bounded by the hyperbolas $xy = 1$ and $xy = 4$, and the lines $y/x = 1$ and $y/x = 3$

35. $\displaystyle\iint_R xy\, dA$, where R is the region bounded by the hyperbolas $xy = 1$ and $xy = 4$, and the lines $y = 1$ and $y = 3$

36. $\displaystyle\iint_R (x - y)\sqrt{x - 2y}\, dA$, where R is the triangular region bounded by $y = 0$, $x - 2y = 0$, and $x - y = 1$

37–40. Jacobians in three variables *Evaluate the Jacobians $J(u, v, w)$ for the following transformations.*

37. $x = v + w, y = u + w, z = u + v$

38. $x = u + v - w, y = u - v + w, z = -u + v + w$

39. $x = vw, y = uw, z = u^2 - v^2$

40. $u = x - y, v = x - z, w = y + z$ (*Hint:* Solve for x, y, and z first.)

41–44. Triple integrals *Use a change of variables to evaluate the following integrals.*

41. $\iiint_D xy\, dV$; D is bounded by the planes $y - x = 0$, $y - x = 2$, $z - y = 0$, $z - y = 1$, $z = 0$, and $z = 3$.

42. $\iiint_D dV$; D is bounded by the planes $y - 2x = 0$, $y - 2x = 1$, $z - 3y = 0$, $z - 3y = 1$, $z - 4x = 0$, and $z - 4x = 3$.

43. $\iiint_D z\, dV$; D is bounded by the paraboloid $z = 16 - x^2 - 4y^2$ and the xy-plane. Use $x = 4u \cos v$, $y = 2u \sin v$, $z = w$.

44. $\iiint_D dV$; D is bounded by the upper half of the ellipsoid $x^2/9 + y^2/4 + z^2 = 1$ and the xy-plane. Use $x = 3u$, $y = 2v$, $z = w$.

45. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. If the transformation $T: x = g(u, v)$, $y = h(u, v)$ is linear in u and v, then the Jacobian is a constant.

b. The transformation $x = au + bv$, $y = cu + dv$ generally maps triangular regions to triangular regions.

c. The transformation $x = 2v$, $y = -2u$ maps circles to circles.

46. Cylindrical coordinates Evaluate the Jacobian for the transformation from cylindrical coordinates (r, θ, Z) to rectangular coordinates (x, y, z): $x = r \cos \theta$, $y = r \sin \theta$, $z = Z$. Show that $J(r, \theta, Z) = r$.

47. Spherical coordinates Evaluate the Jacobian for the transformation from spherical to rectangular coordinates: $x = \rho \sin \varphi \cos \theta$, $y = \rho \sin \varphi \sin \theta$, $z = \rho \cos \varphi$. Show that $J(\rho, \varphi, \theta) = \rho^2 \sin \varphi$.

48–52. Ellipse problems *Let R be the region bounded by the ellipse $x^2/a^2 + y^2/b^2 = 1$, where $a > 0$ and $b > 0$ are real numbers. Let T be the transformation $x = au$, $y = bv$.*

48. Find the area of R.

49. Evaluate $\iint_R |xy|\, dA$.

50. Find the center of mass of the upper half of R $(y \geq 0)$ assuming it has a constant density.

51. Find the average square of the distance between points of R and the origin.

52. Find the average distance between points in the upper half of R and the x-axis.

53–56. Ellipsoid problems *Let D be the solid bounded by the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, where $a > 0$, $b > 0$, and $c > 0$ are real numbers. Let T be the transformation $x = au$, $y = bv$, $z = cw$.*

53. Find the volume of D.

54. Evaluate $\iiint_D |xyz|\, dV$.

55. Find the center of mass of the upper half of D $(z \geq 0)$ assuming it has a constant density.

56. Find the average square of the distance between points of D and the origin.

Explorations and Challenges

57. Parabolic coordinates Let T be the transformation $x = u^2 - v^2$, $y = 2uv$.

a. Show that the lines $u = a$ in the uv-plane map to parabolas in the xy-plane that open in the negative x-direction with vertices on the positive x-axis.

b. Show that the lines $v = b$ in the uv-plane map to parabolas in the xy-plane that open in the positive x-direction with vertices on the negative x-axis.

c. Evaluate $J(u, v)$.

d. Use a change of variables to find the area of the region bounded by $x = 4 - y^2/16$ and $x = y^2/4 - 1$.

e. Use a change of variables to find the area of the curved rectangle above the x-axis bounded by $x = 4 - y^2/16$, $x = 9 - y^2/36$, $x = y^2/4 - 1$, and $x = y^2/64 - 16$.

f. Describe the effect of the transformation $x = 2uv$, $y = u^2 - v^2$ on horizontal and vertical lines in the uv-plane.

58. Shear transformations in $\mathbb{R}^2$ The transformation T in $\mathbb{R}^2$ given by $x = au + bv$, $y = cv$, where a, b, and c are positive real numbers, is a *shear transformation*. Let S be the unit square $\{(u, v): 0 \leq u \leq 1, 0 \leq v \leq 1\}$. Let $R = T(S)$ be the image of S.

a. Explain with pictures the effect of T on S.

b. Compute the Jacobian of T.

c. Find the area of R and compare it to the area of S (which is 1).

d. Assuming a constant density, find the center of mass of R (in terms of a, b, and c) and compare it to the center of mass of S, which is $\left(\frac{1}{2}, \frac{1}{2}\right)$.

e. Find an analogous transformation that gives a shear in the y-direction.

59. Shear transformations in $\mathbb{R}^3$ The transformation T in $\mathbb{R}^3$ given by

$$x = au + bv + cw, \qquad y = dv + ew, \qquad z = w,$$

where a, b, c, d, and e are positive real numbers, is one of many possible shear transformations in $\mathbb{R}^3$. Let S be the unit cube $\{(u, v, w): 0 \leq u \leq 1, 0 \leq v \leq 1, 0 \leq w \leq 1\}$. Let $D = T(S)$ be the image of S.

a. Explain with pictures and words the effect of T on S.

b. Compute the Jacobian of T.

c. Find the volume of D and compare it to the volume of S (which is 1).

d. Assuming a constant density, find the center of mass of D and compare it to the center of mass of S, which is $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$.

60. Linear transformations Consider the linear transformation T in $\mathbb{R}^2$ given by $x = au + bv$, $y = cu + dv$, where a, b, c, and d are real numbers, with $ad \neq bc$.

a. Find the Jacobian of T.

b. Let S be the square in the uv-plane with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$, and $(1, 1)$, and let $R = T(S)$. Show that $\text{area}(R) = |J(u, v)|$.

c. Let ℓ be the line segment joining the points P and Q in the uv-plane. Show that $T(\ell)$ (the image of ℓ under T) is the line segment joining $T(P)$ and $T(Q)$ in the xy-plane. (*Hint:* Use vectors.)

d. Show that if S is a parallelogram in the uv-plane and $R = T(S)$, then $\text{area}(R) = |J(u, v)| \cdot \text{area of } S$. (*Hint:* Without loss of generality, assume the vertices of S are $(0, 0)$, $(A, 0)$, (B, C), and $(A + B, C)$, where A, B, and C are positive, and use vectors.)

61. Meaning of the Jacobian The Jacobian is a magnification (or reduction) factor that relates the area of a small region near the point (u, v) to the area of the image of that region near the point (x, y).

a. Suppose S is a rectangle in the uv-plane with vertices $O(0, 0)$, $P(\Delta u, 0)$, $(\Delta u, \Delta v)$, and $Q(0, \Delta v)$ (see figure). The image of S under the transformation $x = g(u, v)$, $y = h(u, v)$ is a region R in the xy-plane. Let O', P', and Q' be the images of O, P, and Q, respectively, in the xy-plane, where O', P', and Q' do not all lie on the same line. Explain why the coordinates of O', P', and Q' are $(g(0, 0), h(0, 0))$, $(g(\Delta u, 0), h(\Delta u, 0))$, and $(g(0, \Delta v), h(0, \Delta v))$, respectively.

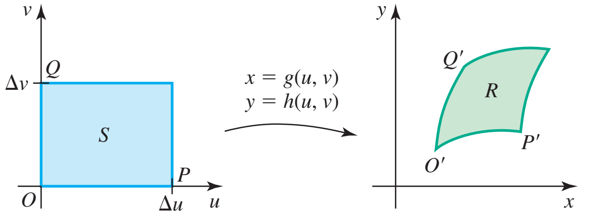
b. Use a Taylor series in both variables to show that

$$\begin{aligned} g(\Delta u, 0) &\approx g(0,0) + g_u(0,0)\Delta u, \\ g(0, \Delta v) &\approx g(0,0) + g_v(0,0)\Delta v, \\ h(\Delta u, 0) &\approx h(0,0) + h_u(0,0)\Delta u, \text{ and} \\ h(0, \Delta v) &\approx h(0,0) + h_v(0,0)\Delta v, \end{aligned}$$

where $g_u(0, 0)$ is $\dfrac{\partial x}{\partial u}$ evaluated at $(0, 0)$, with similar meanings for g_v, h_u, and h_v.

c. Consider the parallelogram determined by the vectors $\overrightarrow{O'P'}$ and $\overrightarrow{O'Q'}$. Use the cross product to show that the area of the parallelogram is approximately $|J(u, v)|\,\Delta u\,\Delta v$.

d. Explain why the ratio of the area of R to the area of S is approximately $|J(u, v)|$.



62. Open and closed boxes Consider the region R bounded by three pairs of parallel planes: $ax + by = 0$, $ax + by = 1$; $cx + dz = 0$, $cx + dz = 1$; and $ey + fz = 0$, $ey + fz = 1$, where a, b, c, d, e, and f are real numbers. For the purposes of evaluating triple integrals, when do these six planes bound a finite region? Carry out the following steps.

a. Find three vectors $\mathbf{n}_1$, $\mathbf{n}_2$, and $\mathbf{n}_3$ each of which is normal to one of the three pairs of planes.

b. Show that the three normal vectors lie in a plane if their triple scalar product $\mathbf{n}_1 \cdot (\mathbf{n}_2 \times \mathbf{n}_3)$ is zero.

c. Show that the three normal vectors lie in a plane if $ade + bcf = 0$.

d. Assuming $\mathbf{n}_1$, $\mathbf{n}_2$, and $\mathbf{n}_3$ lie in a plane P, find a vector $\mathbf{N}$ that is normal to P. Explain why a line in the direction of $\mathbf{N}$ does not intersect any of the six planes, and therefore the six planes do not form a bounded region.

e. Consider the change of variables $u = ax + by$, $v = cx + dz$, $w = ey + fz$. Show that

$$J(x, y, z) = \frac{\partial(u, v, w)}{\partial(x, y, z)} = -ade - bcf.$$

What is the value of the Jacobian if R is unbounded?

QUICK CHECK ANSWERS

1. The image is a semicircular disk of radius 1. **2.** $J(u, v) = 2$ **3.** $x = 2u/3 - v/3$, $y = u/3 + v/3$ **4.** The ratio is 2, which is $1/J(u, v)$. **5.** It means that the volume of a small region in xyz-space is unchanged when it is transformed by T into a small region in uvw-space. ◄

CHAPTER 16 REVIEW EXERCISES

1. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. Assuming g is integrable and a, b, c, and d are constants,
$$\int_c^d \int_a^b g(x, y)\,dx\,dy = \left(\int_a^b g(x, y)\,dx\right)\left(\int_c^d g(x, y)\,dy\right).$$

b. The spherical equation $\varphi = \pi/2$, the cylindrical equation $z = 0$, and the rectangular equation $z = 0$ all describe the same set of points.

c. Changing the order of integration in $\iiint_D f(x, y, z)\,dx\,dy\,dz$ from $dx\,dy\,dz$ to $dy\,dz\,dx$ requires also changing the integrand from $f(x, y, z)$ to $f(y, z, x)$.

d. The transformation T: $x = v$, $y = -u$ maps a square in the uv-plane to a triangle in the xy-plane.

2–4. Evaluating integrals *Evaluate the following integrals as they are written.*

2. $\displaystyle\int_1^2 \int_1^4 \frac{xy}{(x^2 + y^2)^2}\,dx\,dy$

3. $\displaystyle\int_1^3 \int_1^{e^x} \frac{x}{y}\,dy\,dx$

4. $\displaystyle\int_1^2 \int_0^{\ln x} x^3 e^y\,dy\,dx$

5–7. Changing the order of integration *Assuming f is integrable, change the order of integration in the following integrals.*

5. $\displaystyle\int_{-1}^1 \int_{x^2}^1 f(x, y)\,dy\,dx$

6. $\displaystyle\int_0^2 \int_{y-1}^1 f(x, y)\,dx\,dy$

7. $\displaystyle\int_0^1 \int_0^{\sqrt{1-y^2}} f(x, y)\,dx\,dy$

8–10. Area of plane regions *Use double integrals to compute the area of the following regions. Make a sketch of the region.*

8. The region bounded by the lines $y = -x - 4$, $y = x$, and $y = 2x - 4$

9. The region bounded by $y = |x|$ and $y = 20 - x^2$

10. The region between the curves $y = x^2$ and $y = 1 + x - x^2$

11–16. Miscellaneous double integrals *Choose a convenient method for evaluating the following integrals.*

11. $\iint_R \frac{2y}{\sqrt{x^4 + 1}}\, dA$; R is the region bounded by $x = 1$, $x = 2$, $y = x^{3/2}$, and $y = 0$.

12. $\iint_R x^{-1/2} e^y\, dA$; R is the region bounded by $x = 1$, $x = 4$, $y = \sqrt{x}$, and $y = 0$.

13. $\iint_R (x + y)\, dA$; R is the disk bounded by the circle $r = 4 \sin \theta$.

14. $\iint_R (x^2 + y^2)\, dA$; R is the region $\{(x, y): 0 \le x \le 2, 0 \le y \le x\}$.

15. $\int_0^1 \int_{y^{1/3}}^1 x^{10} \cos(\pi x^4 y)\, dx\, dy$

16. $\int_0^2 \int_{y^2}^4 x^8 y \sqrt{1 + x^4 y^2}\, dx\, dy$

17–18. Cartesian to polar coordinates *Evaluate the following integrals over the specified region. Assume (r, θ) are polar coordinates.*

17. $\iint_R 3x^2 y\, dA$; $R = \{(r, \theta): 0 \le r \le 1, 0 \le \theta \le \pi/2\}$

18. $\iint_R \frac{dA}{(1 + x^2 + y^2)^2}$; $R = \{(r, \theta): 1 \le r \le 4, 0 \le \theta \le \pi\}$

19–21. Computing areas *Sketch the following regions and use a double integral to find their areas.*

19. The region bounded by all leaves of the rose $r = 3 \cos 2\theta$

20. The region inside both of the circles $r = 2$ and $r = 4 \cos \theta$

21. The region that lies inside both of the cardioids $r = 2 - 2 \cos \theta$ and $r = 2 + 2 \cos \theta$

22–23. Average values

22. Find the average value of $z = \sqrt{16 - x^2 - y^2}$ over the disk in the xy-plane centered at the origin with radius 4.

23. Find the average distance from the points in the solid cone bounded by $z = 2\sqrt{x^2 + y^2}$ to the z-axis, for $0 \le z \le 8$.

24–26. Changing order of integration *Rewrite the following integrals using the indicated order of integration.*

24. $\int_0^1 \int_0^{\sqrt{1-z^2}} \int_0^{\sqrt{1-x^2}} f(x, y, z)\, dy\, dx\, dz$ in the order $dz\, dy\, dx$

25. $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{2\sqrt{x^2+y^2}}^2 f(x, y, z)\, dz\, dy\, dx$ in the order $dx\, dz\, dy$

26. $\int_0^2 \int_0^{9-x^2} \int_0^x f(x, y, z)\, dy\, dz\, dx$ in the order $dz\, dx\, dy$

27–31. Triple integrals *Evaluate the following integrals, changing the order of integration if needed.*

27. $\int_0^1 \int_{-z}^z \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy\, dx\, dz$

28. $\int_0^\pi \int_0^y \int_0^{\sin x} dz\, dx\, dy$

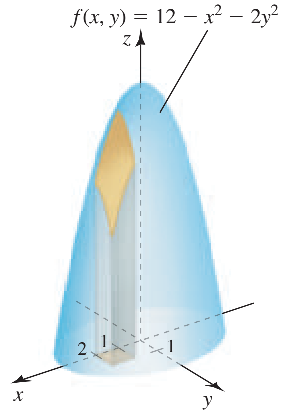
29. $\int_1^9 \int_0^1 \int_{2y}^2 \frac{4 \sin x^2}{\sqrt{z}}\, dx\, dy\, dz$

30. $\int_0^2 \int_{-\sqrt{2-x^2/2}}^{\sqrt{2-x^2/2}} \int_{x^2+3y^2}^{8-x^2-y^2} dz\, dy\, dx$

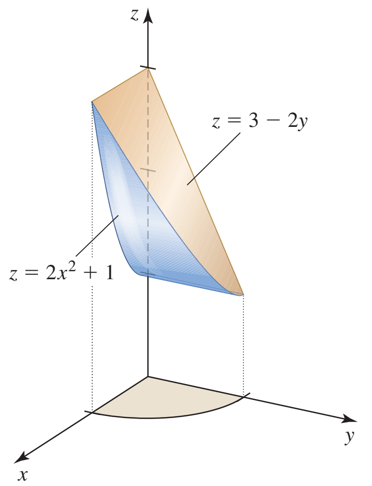
31. $\int_0^2 \int_0^{y^{1/3}} \int_0^{y^2} yz^5(1 + x + y^2 + z^6)^2\, dx\, dz\, dy$

32–38. Volumes of solids *Find the volume of the following solids.*

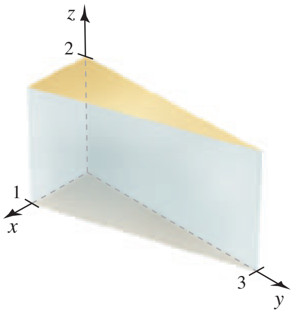
32. The solid beneath the paraboloid $f(x, y) = 12 - x^2 - 2y^2$ and above the region $R = \{(x, y): 1 \le x \le 2, 0 \le y \le 1\}$



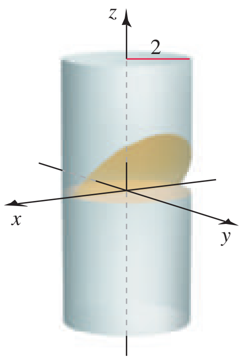
33. The solid bounded by the surfaces $x = 0$, $y = 0$, $z = 3 - 2y$, and $z = 2x^2 + 1$



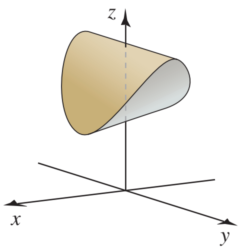
34. The prism in the first octant bounded by the planes $y = 3 - 3x$ and $z = 2$



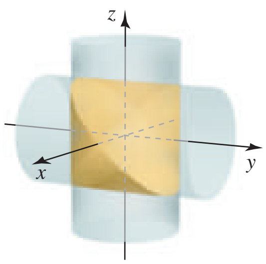
35. One of the wedges formed when the cylinder $x^2 + y^2 = 4$ is cut by the planes $z = 0$ and $y = z$



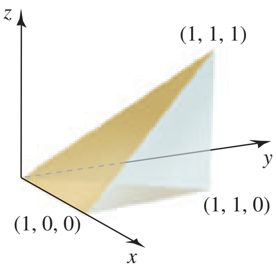
36. The solid bounded by the parabolic cylinders $z = y^2 + 1$ and $z = 2 - x^2$



37. The solid common to the two cylinders $x^2 + y^2 = 4$ and $x^2 + z^2 = 4$



38. The tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(1, 1, 0)$, and $(1, 1, 1)$



39. Single to double integral Evaluate $\int_0^{1/2}(\sin^{-1} 2x - \sin^{-1} x)\,dx$ by converting it to a double integral.

40. Tetrahedron limits Let D be the tetrahedron with vertices at $(0, 0, 0)$, $(1, 0, 0)$, $(0, 2, 0)$, and $(0, 0, 3)$. Suppose the volume of D is to be found using a triple integral. Give the limits of integration for the six possible orderings of the variables.

41. Polar to Cartesian Evaluate $\int_0^{\pi/4}\int_0^{\sec\theta} r^3\,dr\,d\theta$ using rectangular coordinates, where (r, θ) are polar coordinates.

42–43. Average value

42. Find the average of the *square* of the distance between the origin and the points in the solid paraboloid $D = \{(x, y, z)\colon 0 \le z \le 4 - x^2 - y^2\}$.

43. Find the average x-coordinate of the points in the prism $D = \{(x, y, z)\colon 0 \le x \le 1, 0 \le y \le 3 - 3x, 0 \le z \le 2\}$.

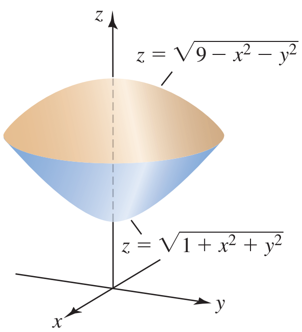
44–45. Integrals in cylindrical coordinates *Evaluate the following integrals in cylindrical coordinates.*

44. $$\int_0^3\int_0^{\sqrt{9-x^2}}\int_0^3 (x^2 + y^2)^{3/2}\,dz\,dy\,dx$$

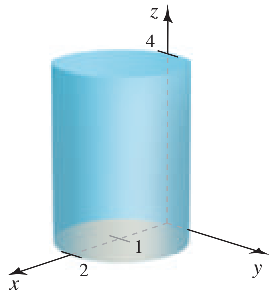
45. $$\int_{-1}^1\int_{-2}^2\int_0^{\sqrt{1-y^2}} \frac{1}{(1 + x^2 + y^2)^2}\,dx\,dz\,dy$$

46–47. Volumes in cylindrical coordinates *Use integration in cylindrical coordinates to find the volume of the following solids.*

46. The solid bounded by the hemisphere $z = \sqrt{9 - x^2 - y^2}$ and the hyperboloid $z = \sqrt{1 + x^2 + y^2}$.



47. The solid cylinder whose height is 4 and whose base is the disk $\{(r, \theta): 0 \le r \le 2\cos\theta\}$



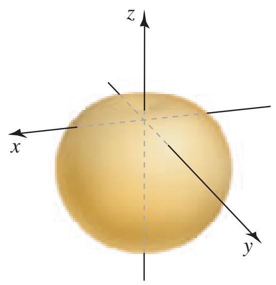
48–49. Integrals in spherical coordinates *Evaluate the following integrals in spherical coordinates.*

48. $\displaystyle\int_0^{2\pi}\int_0^{\pi/2}\int_0^{2\cos\varphi} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$

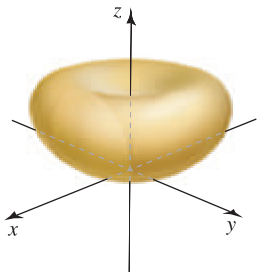
49. $\displaystyle\int_0^{\pi}\int_0^{\pi/4}\int_{2\sec\varphi}^{4\sec\varphi} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$

50–52. Volumes in spherical coordinates *Use integration in spherical coordinates to find the volume of the following solids.*

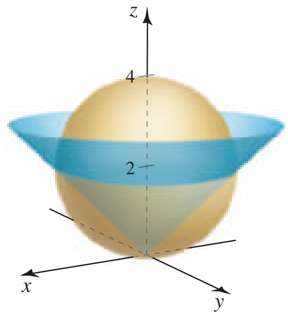
50. The solid cardioid of revolution $D = \{(\rho, \varphi, \theta): 0 \le \rho \le (1 - \cos\varphi)/2, 0 \le \varphi \le \pi, 0 \le \theta \le 2\pi\}$



51. The solid rose petal of revolution $D = \{(\rho, \varphi, \theta): 0 \le \rho \le 4\sin 2\varphi, 0 \le \varphi \le \pi/2, 0 \le \theta \le 2\pi\}$



52. The solid above the cone $\varphi = \pi/4$ and inside the sphere $\rho = 4\cos\varphi$



53–56. Center of mass of constant-density plates *Find the center of mass (centroid) of the following thin, constant-density plates. Sketch the region corresponding to the plate and indicate the location of the center of mass. Use symmetry whenever possible to simplify your work.*

53. The region bounded by $y = \sin x$ and $y = 0$ between $x = 0$ and $x = \pi$

54. The region bounded by $y = x^3$ and $y = x^2$ between $x = 0$ and $x = 1$

55. The half-annulus $\{(r, \theta): 2 \le r \le 4, 0 \le \theta \le \pi\}$

56. The region bounded by $y = x^2$ and $y = a^2 - x^2$, where $a > 0$

57–58. Center of mass of constant-density solids *Find the center of mass of the following solids, assuming a constant density. Use symmetry whenever possible and choose a convenient coordinate system.*

57. The paraboloid bowl bounded by $z = x^2 + y^2$ and $z = 36$

58. The tetrahedron bounded by $z = 4 - x - 2y$ and the coordinate planes

59–60. Variable-density solids *Find the coordinates of the center of mass of the following solids with the given density.*

59. The upper half of the ball $\left\{(\rho, \varphi, \theta): 0 \le \rho \le 16, 0 \le \varphi \le \dfrac{\pi}{2}, 0 \le \theta \le 2\pi\right\}$ with density $f(\rho, \varphi, \theta) = 1 + \rho/4$

60. The cube in the first octant bounded by the planes $x = 2$, $y = 2$, and $z = 2$, with $\rho(x, y, z) = 1 + x + y + z$

61–64. Center of mass for general objects *Consider the following two- and three-dimensional regions. Compute the center of mass, assuming constant density. All parameters are positive real numbers.*

61. A solid is bounded by a paraboloid with a circular base of radius R and height h. How far from the base is the center of mass?

62. Let R be the region enclosed by an equilateral triangle with sides of length s. What is the perpendicular distance between the center of mass of R and the edges of R?

63. A sector of a circle in the first quadrant is bounded between the x-axis, the line $y = x$, and the circle $x^2 + y^2 = a^2$. What are the coordinates of the center of mass?

64. An ice cream cone is bounded above by the sphere $x^2 + y^2 + z^2 = a^2$ and below by the upper half of the cone $z^2 = x^2 + y^2$. What are the coordinates of the center of mass?

65. Slicing a conical cake A cake is shaped like a solid cone with radius 4 and height 2, with its base on the xy-plane. A wedge of the cake is removed by making two slices from the axis of the cone outward, perpendicular to the xy-plane and separated by an angle of Q radians, where $0 < Q < 2\pi$.

a. Use a double integral to find the volume of the slice for $Q = \pi/4$. Use geometry to check your answer.

b. Use a double integral to find the volume of the slice for any $0 < Q < 2\pi$. Use geometry to check your answer.

66. Volume and weight of a fish tank A spherical fish tank with a radius of 1 ft is filled with water to a level 6 in below the top of the tank.

a. Determine the volume and weight of the water in the fish tank. (The weight density of water is about $62.5\ \text{lb/ft}^3$.)

b. How much additional water must be added to completely fill the tank?

67–70. Transforming a square *Let* $S = \{(u, v): 0 \le u \le 1, 0 \le v \le 1\}$ *be a unit square in the uv-plane. Find the image of S in the xy-plane under the following transformations.*

67. $T: x = v, y = u$

68. $T: x = -v, y = u$

69. $T: x = 3u + v, y = u + 3v$

70. $T: x = u, y = 2v + 2$

71–74. Computing Jacobians *Compute the Jacobian* $J(u, v)$ *of the following transformations.*

71. $T: x = 4u - v, y = -2u + 3v$

72. $T: x = u + v, y = u - v$

73. $T: x = 3u, y = 2v + 2$

74. $T: x = u^2 - v^2, y = 2uv$

75–78. Double integrals—transformation given *To evaluate the following integrals, carry out these steps.*

a. *Sketch the original region of integration R and the new region S using the given change of variables.*

b. *Find the limits of integration for the new integral with respect to u and v.*

c. *Compute the Jacobian.*

d. *Change variables and evaluate the new integral.*

75. $\iint_R xy^2\, dA$; $R = \{(x, y): y/3 \le x \le (y + 6)/3, 0 \le y \le 3\}$; use $x = u + v/3, y = v$.

76. $\iint_R 3xy^2\, dA$; $R = \{(x, y): 0 \le x \le 2, x \le y \le x + 4\}$; use $x = 2u, y = 4v + 2u$.

77. $\iint_R (x - y + 1)(x - y)^9\, dy\, dx$, $R = \{(x, y): 0 \le y \le x \le 1\}$; use $x = u + v, y = v - u$.

78. $\iint_R xy^2\, dA$; R is the region between the hyperbolas $xy = 1$ and $xy = 4$ and the lines $y = 1$ and $y = 4$; use $x = u/v, y = v$.

79–80. Double integrals *Evaluate the following integrals using a change of variables. Sketch the original and new regions of integration, R and S.*

79. $\iint_R y^4\, dA$; R is the region bounded by the hyperbolas $xy = 1$ and $xy = 4$ and the lines $y/x = 1$ and $y/x = 3$.

80. $\iint_R (y^2 + xy - 2x^2)\, dA$; R is the region bounded by the lines $y = x$, $y = x - 3$, $y = -2x + 3$, and $y = -2x - 3$.

81–82. Triple integrals *Use a change of variables to evaluate the following integrals.*

81. $\iiint_D yz\, dV$; D is bounded by the planes $x + 2y = 1$, $x + 2y = 2$, $x - z = 0$, $x - z = 2$, $2y - z = 0$, and $2y - z = 3$.

82. $\iiint_D x\, dV$; D is bounded by the planes $y - 2x = 0$, $y - 2x = 1$, $z - 3y = 0$, $z - 3y = 1$, $z - 4x = 0$, and $z - 4x = 3$.

Chapter 16 Guided Projects

Applications of the material in this chapter and related topics can be found in the following Guided Projects. For additional information, see the Preface.

- How big are n-balls?
- Electrical field integrals
- The tilted cylinder problem
- The exponential Eiffel Tower
- Moments of inertia
- Gravitational fields

17

Vector Calculus

Chapter Preview This culminating chapter of the text provides a beautiful, unifying conclusion to our study of calculus. Many ideas and themes that have appeared throughout the text come together in these final pages. First, we combine vector-valued functions (Chapter 14) and functions of several variables (Chapter 15) to form *vector fields*. Once vector fields have been introduced and illustrated through their many applications, we explore the calculus of vector fields. Concepts such as limits and continuity carry over directly. The extension of derivatives to vector fields leads to two new operations that underlie this chapter: the *curl* and the *divergence*. When integration is extended to vector fields, we discover new versions of the Fundamental Theorem of Calculus. The chapter ends with a final look at the Fundamental Theorem of Calculus and the several related forms in which it has appeared throughout the text.

17.1 Vector Fields

We live in a world filled with phenomena that can be represented by vector fields. Imagine sitting in a window seat looking out at the wing of an airliner. Although you can't see it, air is rushing over and under the wing. Focus on a point near the wing and visualize the motion of the air at that point at a single instant of time. The motion is described by a velocity vector with three components—for example, east-west, north-south, and up-down. At another point near the wing at the same time, the air is moving at a different speed and direction, and a different velocity vector is associated with that point. In general, at one instant in time, every point around the wing has a velocity vector associated with it (**Figure 17.1**). This collection of velocity vectors—a unique vector for each point in space—is a function called a *vector field*.

Other examples of vector fields include the wind patterns in a hurricane (**Figure 17.2a**) and the circulation of water in a heat exchanger (**Figure 17.2b**). Gravitational, magnetic, and electric force fields are also represented by vector fields (**Figure 17.2c**), as are the stresses and strains in buildings and bridges. Beyond physics and engineering, the transport of a chemical pollutant in a lake and human migration patterns can be modeled by vector fields.

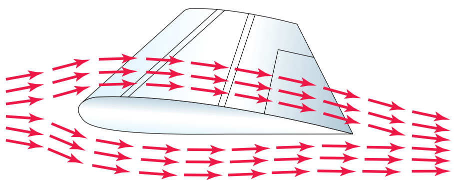
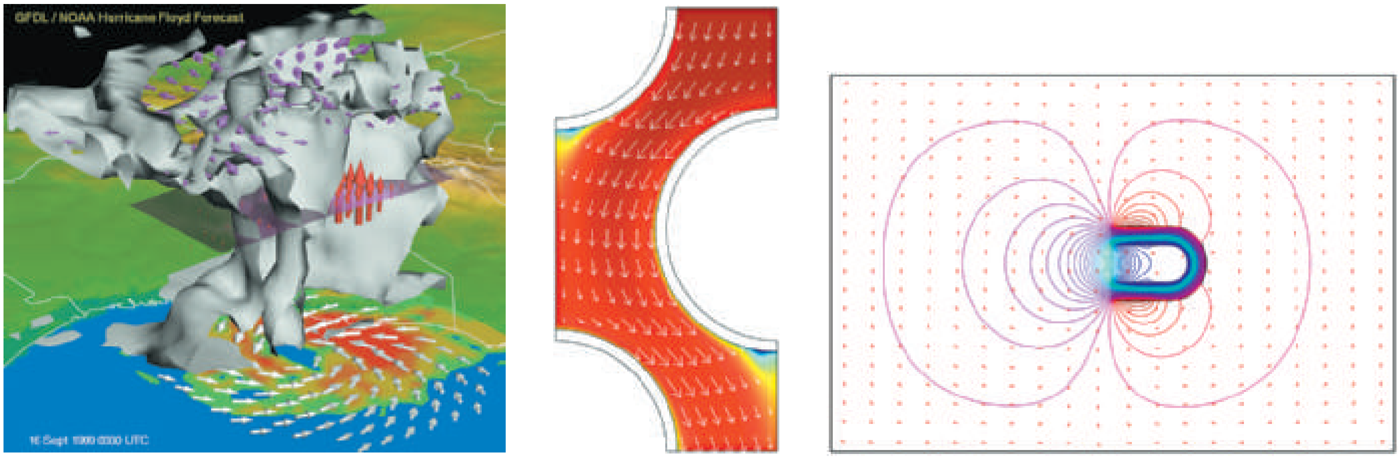


Figure 17.1



(a) (b) (c)

Figure 17.2

Vector Fields in Two Dimensions

To solidify the idea of a vector field, we begin by exploring vector fields in $\mathbb{R}^2$. From there, it is a short step to vector fields in $\mathbb{R}^3$.

➤ Notice that a vector field is both a vector-valued function (Chapter 14) and a function of several (Chapter 15).

> **DEFINITION Vector Fields in Two Dimensions**
>
> Let f and g be defined on a region R of $\mathbb{R}^2$. A **vector field** in $\mathbb{R}^2$ is a function $\mathbf{F}$ that assigns to each point in R a vector $\langle f(x, y), g(x, y) \rangle$. The vector field is written as
>
> $$\mathbf{F}(x, y) = \langle f(x, y), g(x, y) \rangle \quad \text{or}$$
> $$\mathbf{F}(x, y) = f(x, y)\,\mathbf{i} + g(x, y)\,\mathbf{j}.$$
>
> A vector field $\mathbf{F} = \langle f, g \rangle$ is continuous or differentiable on a region R of $\mathbb{R}^2$ if f and g are continuous or differentiable on R, respectively.

A vector field cannot be represented graphically in its entirety. Instead, we plot a representative sample of vectors that illustrates the general appearance of the vector field. Consider the vector field defined by

$$\mathbf{F}(x, y) = \langle 2x, 2y \rangle = 2x\,\mathbf{i} + 2y\,\mathbf{j}.$$

At selected points $P(x, y)$, we plot a vector with its tail at P equal to the value of $\mathbf{F}(x, y)$. For example, $\mathbf{F}(1, 1) = \langle 2, 2 \rangle$, so we draw a vector equal to $\langle 2, 2 \rangle$ with its tail at the point $(1, 1)$. Similarly, $\mathbf{F}(-2, -3) = \langle -4, -6 \rangle$, so at the point $(-2, -3)$, we draw a vector equal to $\langle -4, -6 \rangle$. We can make the following general observations about the vector field $\mathbf{F}(x, y) = \langle 2x, 2y \rangle$.

- For every (x, y) except $(0, 0)$, the vector $\mathbf{F}(x, y)$ points in the direction of $\langle 2x, 2y \rangle$, which is directly outward from the origin.
- The length of $\mathbf{F}(x, y)$ is $|\mathbf{F}| = |\langle 2x, 2y \rangle| = 2\sqrt{x^2 + y^2}$, which increases with distance from the origin.

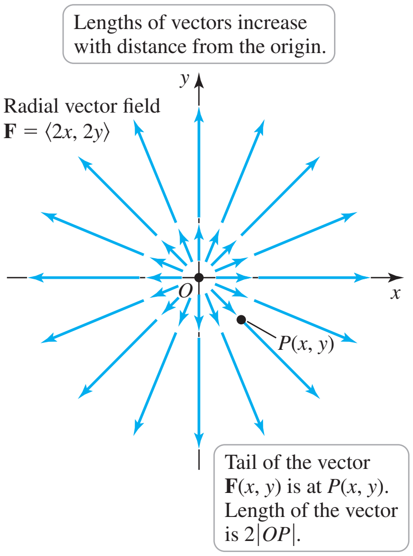


Figure 17.3

The vector field $\mathbf{F} = \langle 2x, 2y \rangle$ is an example of a *radial vector field* because its vectors point radially away from the origin (**Figure 17.3**). If $\mathbf{F}$ represents the velocity of a fluid moving in two dimensions, the graph of the vector field gives a vivid image of how a small object, such as a cork, moves through the fluid. In this case, at every point of the vector field, a particle moves in the direction of the arrow at that point with a speed equal to the length of the arrow. For this reason, vector fields are sometimes called *flows*. When sketching vector fields, it is often useful to draw continuous curves that are aligned with the vector field. Such curves are called *flow curves* or *streamlines*; we examine their properties in greater detail later in this section.

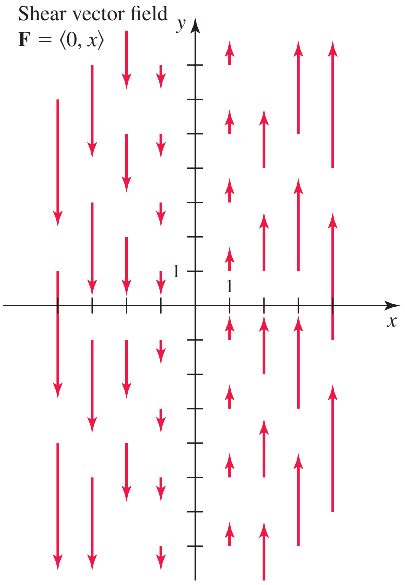


Figure 17.4

EXAMPLE 1 **Vector fields** Sketch representative vectors of the following vector fields.

a. $\mathbf{F}(x, y) = \langle 0, x \rangle = x\,\mathbf{j}$ (a shear field)

b. $\mathbf{F}(x, y) = \langle 1 - y^2, 0 \rangle = (1 - y^2)\,\mathbf{i}$, for $|y| \le 1$ (channel flow)

c. $\mathbf{F}(x, y) = \langle -y, x \rangle = -y\,\mathbf{i} + x\,\mathbf{j}$ (a rotation field)

SOLUTION

a. This vector field is independent of y. Furthermore, because the x-component of $\mathbf{F}$ is zero, all vectors in the field (for $x \neq 0$) point in the y-direction: upward for $x > 0$ and downward for $x < 0$. The magnitudes of the vectors in the field increase with distance from the y-axis (Figure 17.4). The flow curves for this field are vertical lines. If $\mathbf{F}$ represents a velocity field, a particle right of the y-axis moves upward, a particle left of the y-axis moves downward, and a particle on the y-axis is stationary.

b. In this case, the vector field is independent of x and the y-component of $\mathbf{F}$ is zero. Because $1 - y^2 > 0$ for $|y| < 1$, vectors in this region point in the positive x-direction. The x-component of the vector field is zero at the boundaries $y = \pm 1$ and increases to 1 along the center of the strip, $y = 0$. This vector field might model the flow of water in a straight shallow channel (Figure 17.5); its flow curves are horizontal lines, indicating motion in the direction of the positive x-axis.

c. It often helps to determine the vector field along the coordinate axes.

- When $y = 0$ (along the x-axis), we have $\mathbf{F}(x, 0) = \langle 0, x \rangle$. With $x > 0$, this vector field consists of vectors pointing upward, increasing in length as x increases. With $x < 0$, the vectors point downward, increasing in length as $|x|$ increases.
- When $x = 0$ (along the y-axis), we have $\mathbf{F}(0, y) = \langle -y, 0 \rangle$. If $y > 0$, the vectors point in the negative x-direction, increasing in length as y increases. If $y < 0$, the vectors point in the positive x-direction, increasing in length as $|y|$ increases.

A few more representative vectors show that this vector field has a counterclockwise rotation about the origin; the magnitudes of the vectors increase with distance from the origin (Figure 17.6).

➤ Drawing vectors with their actual length often leads to cluttered pictures of vector fields. For this reason, most of the vector fields in this chapter are illustrated with proportional scaling: All vectors are multiplied by a scalar chosen to make the vector field as understandable as possible.

➤ A useful observation for two-dimensional vector fields $\mathbf{F} = \langle f, g \rangle$ is that the slope of the vector at (x, y) is $g(x, y)/f(x, y)$. In Example 1a, the slopes are everywhere undefined; in part (b), the slopes are everywhere 0, and in part (c), the slopes are $-x/y$.

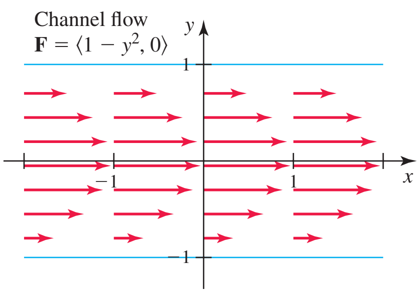


Figure 17.5

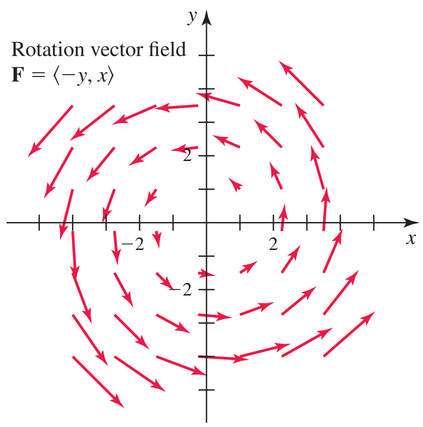


Figure 17.6

Related Exercises 10, 11, 13 ◄

QUICK CHECK 1 If the vector field in Example 1c describes the velocity of a fluid and you place a small cork in the plane at $(2, 0)$, what path will it follow? ◄

Radial Vector Fields in $\mathbb{R}^2$ Radial vector fields in $\mathbb{R}^2$ have the property that their vectors point directly toward or away from the origin at all points (except the origin), parallel to the position vectors $\mathbf{r} = \langle x, y \rangle$. We will work with radial vector fields of the form

$$\mathbf{F}(x, y) = \frac{\mathbf{r}}{|\mathbf{r}|^p} = \frac{\langle x, y \rangle}{|\mathbf{r}|^p} = \underbrace{\frac{\mathbf{r}}{|\mathbf{r}|}}_{\text{unit vector}} \underbrace{\frac{1}{|\mathbf{r}|^{p-1}}}_{\text{magnitude}},$$

where p is a real number. Figure 17.7 illustrates radial fields with $p = 1$ and $p = 3$. These vector fields (and their three-dimensional counterparts) play an important role in many applications. For example, central forces, such as gravitational or electrostatic forces between point masses or charges, are described by radial vector fields with $p = 3$. These forces obey an inverse square law in which the magnitude of the force is proportional to $1/|\mathbf{r}|^2$.

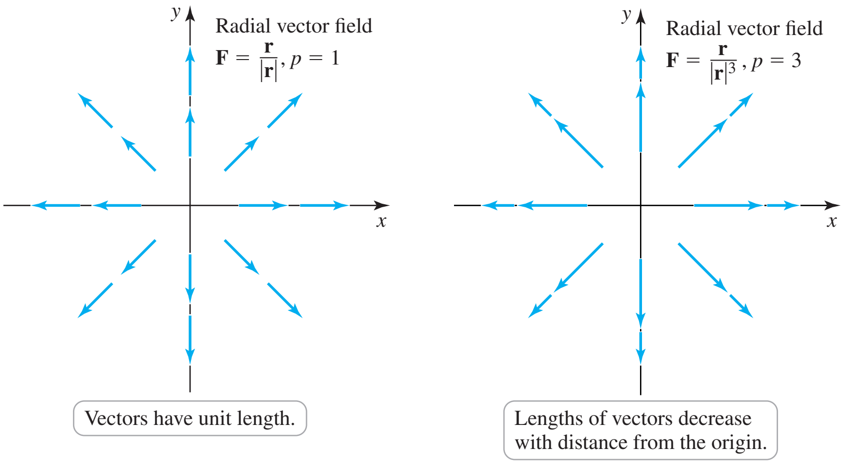


Figure 17.7

> **DEFINITION Radial Vector Fields in $\mathbb{R}^2$**
>
> Let $\mathbf{r} = \langle x, y \rangle$. A vector field of the form $\mathbf{F} = f(x, y)\,\mathbf{r}$, where f is a scalar-valued function, is a **radial vector field**. Of specific interest are the radial vector fields
>
> $$\mathbf{F}(x, y) = \frac{\mathbf{r}}{|\mathbf{r}|^p} = \frac{\langle x, y \rangle}{|\mathbf{r}|^p},$$
>
> where p is a real number. At every point (except the origin), the vectors of this field are directed outward from the origin with a magnitude of $|\mathbf{F}| = \dfrac{1}{|\mathbf{r}|^{p-1}}$.

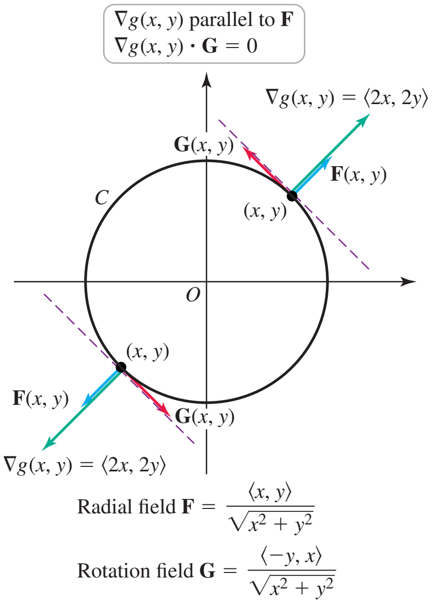


Figure 17.8

QUICK CHECK 2 In Example 2, verify that $\nabla g(x, y) \cdot \mathbf{G}(x, y) = 0$. In parts (a) and (b) of Example 2, verify that $|\mathbf{F}| = 1$ and $|\mathbf{G}| = 1$ at all points excluding the origin. ◄

EXAMPLE 2 Normal and tangent vectors Let C be the circle $x^2 + y^2 = a^2$, where $a > 0$.

a. Show that at each point of C, the radial vector field $\mathbf{F}(x, y) = \dfrac{\mathbf{r}}{|\mathbf{r}|} = \dfrac{\langle x, y \rangle}{\sqrt{x^2 + y^2}}$ is orthogonal to the line tangent to C at that point.

b. Show that at each point of C, the rotation vector field $\mathbf{G}(x, y) = \dfrac{\langle -y, x \rangle}{\sqrt{x^2 + y^2}}$ is parallel to the line tangent to C at that point.

SOLUTION Let $g(x, y) = x^2 + y^2$. The circle C described by the equation $g(x, y) = a^2$ may be viewed as a level curve of the surface $z = x^2 + y^2$. As shown in Theorem 15.12 (Section 15.5), the gradient $\nabla g(x, y) = \langle 2x, 2y \rangle$ is orthogonal to the line tangent to C at (x, y) (Figure 17.8).

a. Notice that $\nabla g(x, y)$ is parallel to $\mathbf{F} = \langle x, y \rangle / |\mathbf{r}|$ at the point (x, y). It follows that $\mathbf{F}$ is also orthogonal the line tangent to C at (x, y).

b. Notice that

$$\nabla g(x, y) \cdot \mathbf{G}(x, y) = \langle 2x, 2y \rangle \cdot \frac{\langle -y, x \rangle}{|\mathbf{r}|} = 0.$$

Therefore, $\nabla g(x, y)$ is orthogonal to the vector field $\mathbf{G}$ at (x, y), which implies that $\mathbf{G}$ is parallel to the tangent line at (x, y).

Related Exercises 27–28 ◄

Vector Fields in Three Dimensions

Vector fields in three dimensions are conceptually the same as vector fields in two dimensions. The vector $\mathbf{F}$ now has three components, each of which depends on three variables.

> **DEFINITION Vector Fields and Radial Vector Fields in $\mathbb{R}^3$**
>
> Let f, g, and h be defined on a region D of $\mathbb{R}^3$. A **vector field** in $\mathbb{R}^3$ is a function $\mathbf{F}$ that assigns to each point in D a vector $\langle f(x, y, z), g(x, y, z), h(x, y, z) \rangle$. The vector field is written as
>
> $$\mathbf{F}(x, y, z) = \langle f(x, y, z), g(x, y, z), h(x, y, z) \rangle \quad \text{or}$$
> $$\mathbf{F}(x, y, z) = f(x, y, z)\,\mathbf{i} + g(x, y, z)\,\mathbf{j} + h(x, y, z)\,\mathbf{k}.$$
>
> A vector field $\mathbf{F} = \langle f, g, h \rangle$ is continuous or differentiable on a region D of $\mathbb{R}^3$ if f, g, and h are continuous or differentiable on D, respectively. Of particular importance are the **radial vector fields**
>
> $$\mathbf{F}(x, y, z) = \frac{\mathbf{r}}{|\mathbf{r}|^p} = \frac{\langle x, y, z \rangle}{|\mathbf{r}|^p},$$
>
> where p is a real number.

EXAMPLE 3 **Vector fields in $\mathbb{R}^3$** Sketch and discuss the following vector fields.

a. $\mathbf{F}(x, y, z) = \langle x, y, e^{-z} \rangle$, for $z \geq 0$

b. $\mathbf{F}(x, y, z) = \langle 0, 0, 1 - x^2 - y^2 \rangle$, for $x^2 + y^2 \leq 1$

SOLUTION

a. First consider the x- and y-components of $\mathbf{F}$ in the xy-plane $(z = 0)$, where $\mathbf{F} = \langle x, y, 1 \rangle$. This vector field looks like a radial field in the first two components, increasing in magnitude with distance from the z-axis. However, each vector also has a constant vertical component of 1. In horizontal planes $z = z_0 > 0$, the radial pattern remains the same, but the vertical component decreases as z increases. As $z \to \infty$, $e^{-z} \to 0$ and the vector field approaches a horizontal radial field (Figure 17.9).

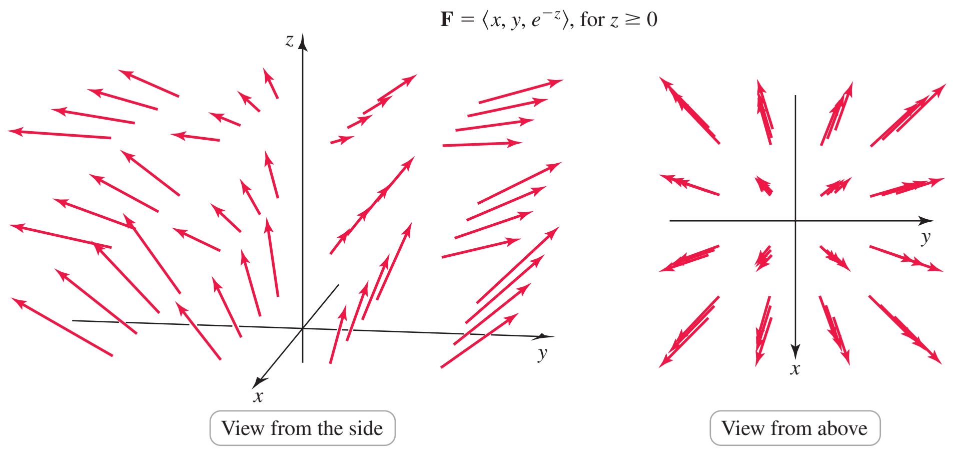


Figure 17.9

b. Regarding $\mathbf{F}$ as a velocity field for points in and on the cylinder $x^2 + y^2 = 1$, there is no motion in the x- or y-direction. The z-component of the vector field may be written $1 - r^2$, where $r^2 = x^2 + y^2$ is the square of the distance from the z-axis. We see that the z-component increases from 0 on the boundary of the cylinder $(r = 1)$ to a maximum value of 1 along the centerline of the cylinder $(r = 0)$ (Figure 17.10). This vector field models the flow of a fluid inside a tube (such as a blood vessel).

Related Exercises 20, 22 ◀

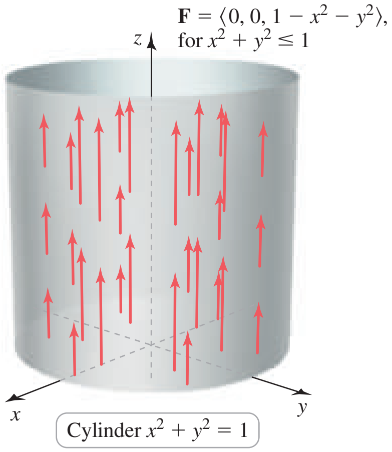


Figure 17.10

Physicists often use the convention that a gradient field and its potential function are related by $\mathbf{F} = -\nabla\varphi$ (with a negative sign).

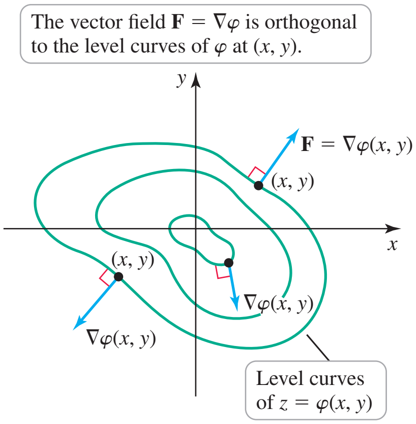


Figure 17.11

Gradient Fields and Potential Functions One way to generate a vector field is to start with a differentiable scalar-valued function φ, take its gradient, and let $\mathbf{F} = \nabla\varphi$. A vector field defined as the gradient of a scalar-valued function φ is called a *gradient field*, and φ is called a *potential function*.

Suppose φ is a differentiable function on a region R of $\mathbb{R}^2$ and consider the surface $z = \varphi(x, y)$. Recall from Chapter 15 that this function may also be represented by level curves in the xy-plane. At each point (a, b) on a level curve, the gradient $\nabla\varphi(a, b) = \langle \varphi_x(a, b), \varphi_y(a, b) \rangle$ is orthogonal to the level curve at (a, b) (Figure 17.11). Therefore, the vectors of $\mathbf{F} = \nabla\varphi$ point in a direction orthogonal to the level curves of φ.

The idea extends to gradients of functions of three variables. If φ is differentiable on a region D of $\mathbb{R}^3$, then $\mathbf{F} = \nabla\varphi = \langle \varphi_x, \varphi_y, \varphi_z \rangle$ is a vector field that points in a direction orthogonal to the level *surfaces* of φ.

Gradient fields are useful because of the physical meaning of the gradient. For example, if φ represents the temperature in a conducting material, then the gradient field $\mathbf{F} = \nabla\varphi$ evaluated at a point indicates the direction in which the temperature increases most rapidly at that point. According to a basic physical law, heat diffuses in the direction of the vector field $-\mathbf{F} = -\nabla\varphi$, the direction in which the temperature *decreases* most rapidly; that is, heat flows "down the gradient" from relatively hot regions to cooler regions. Similarly, water on a smooth surface tends to flow down the elevation gradient.

QUICK CHECK 3 Find the gradient field associated with the function $\varphi(x, y, z) = xyz$. ◄

A potential function plays the role of an antiderivative of a vector field: Derivatives of the potential function produce the vector field. If φ is a potential function for a gradient field, then $\varphi + C$ is also a potential function for that gradient field, for any constant C.

DEFINITION Gradient Fields and Potential Functions

Let φ be differentiable on a region of $\mathbb{R}^2$ or $\mathbb{R}^3$. The vector field $\mathbf{F} = \nabla\varphi$ is a **gradient field** and the function φ is a **potential function** for $\mathbf{F}$.

EXAMPLE 4 Gradient fields

a. Sketch and interpret the gradient field associated with the temperature function $T = 200 - x^2 - y^2$ on the circular plate $R = \{(x, y): x^2 + y^2 \le 25\}$.

b. Sketch and interpret the gradient field associated with the velocity potential $\varphi = \tan^{-1} xy$.

SOLUTION

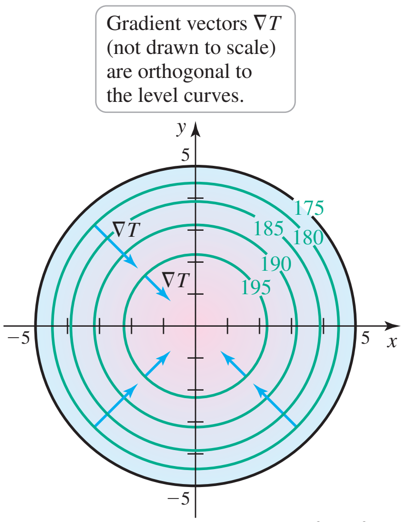
a. The gradient field associated with T is

$$\mathbf{F} = \nabla T = \langle -2x, -2y \rangle = -2\langle x, y \rangle.$$

This vector field points inward toward the origin at all points of R except $(0, 0)$. The magnitudes of the vectors,

$$|\mathbf{F}| = \sqrt{(-2x)^2 + (-2y)^2} = 2\sqrt{x^2 + y^2},$$

are greatest on the edge of the disk R, where $x^2 + y^2 = 25$ and $|\mathbf{F}| = 10$. The magnitudes of the vectors in the field decrease toward the center of the plate with $|\mathbf{F}(0, 0)| = 0$. Figure 17.12 shows the level curves of the temperature function with several gradient vectors, all orthogonal to the level curves. Note that the plate is hottest at the center and coolest on the edge, so heat diffuses *outward*, in the direction opposite that of the gradient.



Level curves of $T(x, y) = 200 - x^2 - y^2$

Figure 17.12

b. The gradient of a velocity potential gives the velocity components of a two-dimensional flow; that is, $\mathbf{F} = \langle u, v \rangle = \nabla\varphi$, where u and v are the velocities in the x- and y-directions, respectively. Computing the gradient, we find that

$$\mathbf{F} = \langle \varphi_x, \varphi_y \rangle = \left\langle \frac{1}{1 + (xy)^2} \cdot y, \frac{1}{1 + (xy)^2} \cdot x \right\rangle = \left\langle \frac{y}{1 + x^2y^2}, \frac{x}{1 + x^2y^2} \right\rangle.$$

Notice that the level curves of φ are the hyperbolas $xy = C$ or $y = C/x$. At all points, the vector field is orthogonal to the level curves (Figure 17.13).

Related Exercises 38, 47 ◄

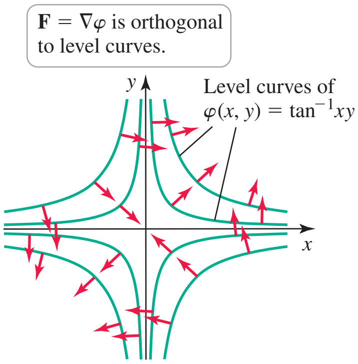


Figure 17.13

Equipotential Curves and Surfaces The preceding example illustrates a beautiful geometric connection between a gradient field and its associated potential function. Let φ be a potential function for the vector field $\mathbf{F}$ in $\mathbb{R}^2$; that is, $\mathbf{F} = \nabla\varphi$. The level curves of a potential function are called **equipotential curves** (curves on which the potential function is constant).

Because the equipotential curves are level curves of φ, the vector field $\mathbf{F} = \nabla\varphi$ is everywhere orthogonal to the equipotential curves (Figure 17.14). The vector field may be visualized by drawing continuous **flow curves** or **streamlines** that are everywhere orthogonal to the equipotential curves. These ideas also apply to vector fields in $\mathbb{R}^3$, in which case the vector field is orthogonal to the **equipotential surfaces**.

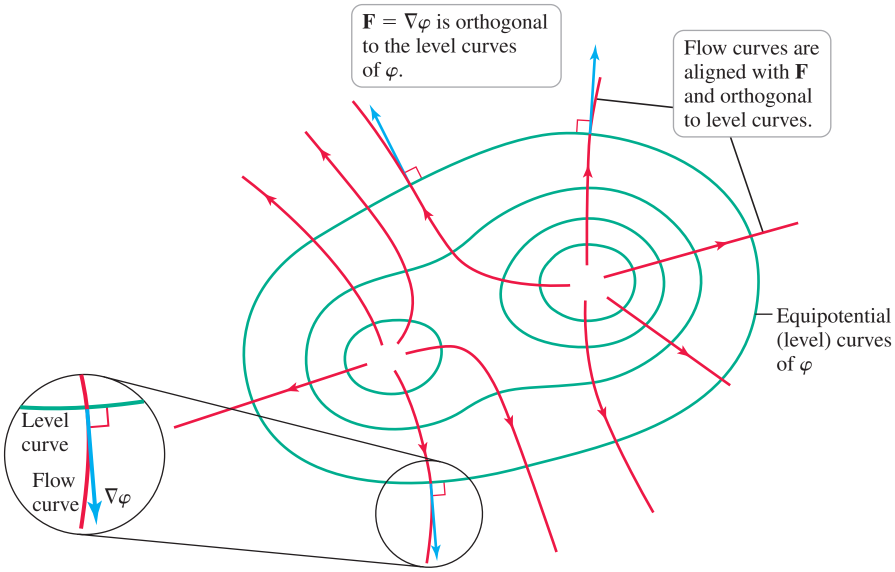


Figure 17.14

EXAMPLE 5 **Equipotential curves** The equipotential curves for the potential function $\varphi(x, y) = (x^2 - y^2)/2$ are shown in green in Figure 17.15.

a. Find the gradient field associated with φ and verify that the gradient field is orthogonal to the equipotential curve at $(2, 1)$.

b. Verify that the vector field $\mathbf{F} = \nabla\varphi$ is orthogonal to the equipotential curves at all points (x, y).

SOLUTION

a. The level (or equipotential) curves are the hyperbolas $(x^2 - y^2)/2 = C$, where C is a constant. The slope at any point on a level curve $\varphi(x, y) = C$ (Section 15.4) is

$$\frac{dy}{dx} = -\frac{\varphi_x}{\varphi_y} = \frac{x}{y}.$$

At the point $(2, 1)$, the slope of the level curve is $dy/dx = 2$, so the vector tangent to the curve points in the direction $\langle 1, 2 \rangle$. The gradient field is given by $\mathbf{F} = \nabla\varphi = \langle x, -y \rangle$, so $\mathbf{F}(2, 1) = \nabla\varphi(2, 1) = \langle 2, -1 \rangle$. The dot product of the tangent vector $\langle 1, 2 \rangle$ and the gradient is $\langle 1, 2 \rangle \cdot \langle 2, -1 \rangle = 0$; therefore, the two vectors are orthogonal.

b. In general, the line tangent to the equipotential curve at (x, y) is parallel to the vector $\langle y, x \rangle$, and the vector field at that point is $\mathbf{F} = \langle x, -y \rangle$. The vector field and the tangent vectors are orthogonal because $\langle y, x \rangle \cdot \langle x, -y \rangle = 0$.

Related Exercise 52 ◄

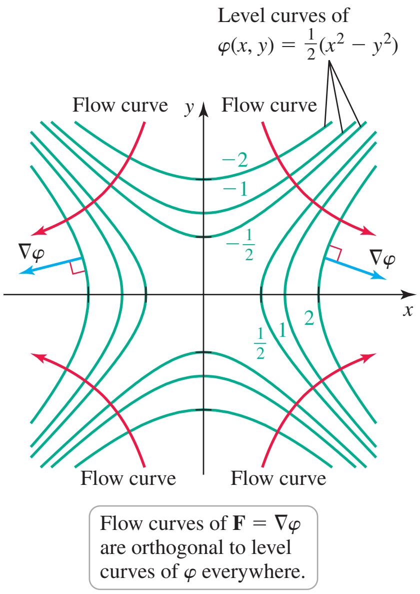


Figure 17.15

➤ We use the fact that a line with slope a/b points in the direction of the vectors $\langle 1, a/b \rangle$ or $\langle b, a \rangle$.

SECTION 17.1 EXERCISES

Getting Started

1. How is a vector field $\mathbf{F} = \langle f, g, h \rangle$ used to describe the motion of air at one instant in time?

2. Sketch the vector field $\mathbf{F} = \langle x, y \rangle$.

3. How do you graph the vector field $\mathbf{F} = \langle f(x, y), g(x, y) \rangle$?

4. Given a differentiable, scalar-valued function φ, why is the gradient of φ a vector field?

5. Interpret the gradient field of the temperature function $T = f(x, y)$.

6. Show that all the vectors in vector field $\mathbf{F} = \dfrac{\sqrt{2}\langle x, y \rangle}{\sqrt{x^2 + y^2}}$ have the same length, and state the length of the vectors.

7. Sketch a few representative vectors of vector field $\mathbf{F} = \langle 0, 1 \rangle$ along the line $y = 2$.

Practice Exercises

8–23. Sketching vector fields *Sketch the following vector fields.*

8. $\mathbf{F} = \langle 1, 0 \rangle$

9. $\mathbf{F} = \langle -1, 1 \rangle$

10. $\mathbf{F} = \langle 1, y \rangle$

11. $\mathbf{F} = \langle x, 0 \rangle$

12. $\mathbf{F} = \langle -x, -y \rangle$

13. $\mathbf{F} = \langle x, -y \rangle$

14. $\mathbf{F} = \langle 2x, 3y \rangle$

15. $\mathbf{F} = \langle y, -x \rangle$

16. $\mathbf{F} = \langle x + y, y \rangle$

17. $\mathbf{F} = \langle x, y - x \rangle$

18. $\mathbf{F} = \left\langle \dfrac{x}{\sqrt{x^2 + y^2}}, \dfrac{y}{\sqrt{x^2 + y^2}} \right\rangle$

19. $\mathbf{F} = \langle e^{-x}, 0 \rangle$

20. $\mathbf{F} = \langle 0, 0, 1 \rangle$

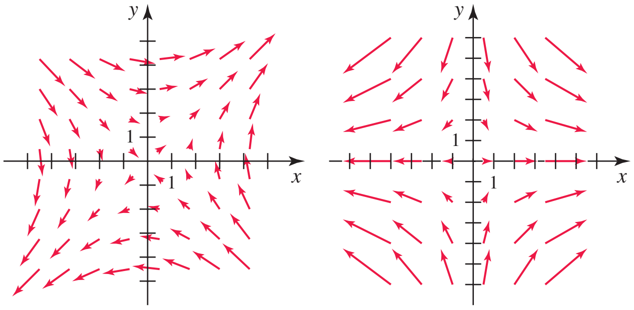
T **21.** $\mathbf{F} = \langle 1, 0, z \rangle$

T **22.** $\mathbf{F} = \dfrac{\langle x, y, z \rangle}{\sqrt{x^2 + y^2 + z^2}}$

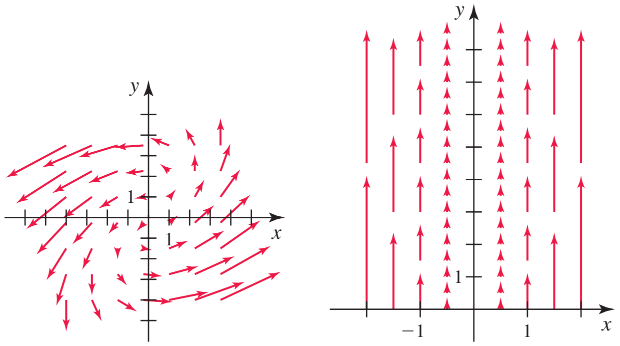
T **23.** $\mathbf{F} = \langle y, -x, 0 \rangle$

24. Matching vector fields with graphs Match vector fields a–d with graphs A–D.

a. $\mathbf{F} = \langle 0, x^2 \rangle$ **b.** $\mathbf{F} = \langle x - y, x \rangle$
c. $\mathbf{F} = \langle 2x, -y \rangle$ **d.** $\mathbf{F} = \langle y, x \rangle$



(A) (B)



(C) (D)

25–30. Normal and tangential components *For the vector field* $\mathbf{F}$ *and curve C, complete the following:*

a. *Determine the points (if any) along the curve C at which the vector field* $\mathbf{F}$ *is tangent to C.*

b. *Determine the points (if any) along the curve C at which the vector field* $\mathbf{F}$ *is normal to C.*

c. *Sketch C and a few representative vectors of* $\mathbf{F}$ *on C.*

25. $\mathbf{F} = \left\langle \dfrac{1}{2}, 0 \right\rangle$; $C = \{(x, y): y - x^2 = 1\}$

26. $\mathbf{F} = \left\langle \dfrac{y}{2}, -\dfrac{x}{2} \right\rangle$; $C = \{(x, y): y - x^2 = 1\}$

27. $\mathbf{F} = \langle x, y \rangle$; $C = \{(x, y): x^2 + y^2 = 4\}$

28. $\mathbf{F} = \langle y, -x \rangle$; $C = \{(x, y): x^2 + y^2 = 1\}$

29. $\mathbf{F} = \langle x, y \rangle$; $C = \{(x, y): x = 1\}$

30. $\mathbf{F} = \langle y, x \rangle$; $C = \{(x, y): x^2 + y^2 = 1\}$

31–34. Design your own vector field *Specify the component functions of a vector field* $\mathbf{F}$ *in* $\mathbb{R}^2$ *with the following properties. Solutions are not unique.*

31. $\mathbf{F}$ is everywhere normal to the line $y = x$.

32. $\mathbf{F}$ is everywhere normal to the line $x = 2$.

33. At all points except $(0, 0)$, $\mathbf{F}$ has unit magnitude and points away from the origin along radial lines.

34. The flow of $\mathbf{F}$ is counterclockwise around the origin, increasing in magnitude with distance from the origin.

35–42. Gradient fields *Find the gradient field* $\mathbf{F} = \nabla\varphi$ *for the following potential functions* φ.

35. $\varphi(x, y) = x^2y - y^2x$

36. $\varphi(x, y) = \sqrt{xy}$

37. $\varphi(x, y) = x/y$

38. $\varphi(x, y) = \tan^{-1}(x/y)$

39. $\varphi(x, y, z) = \dfrac{x^2 + y^2 + z^2}{2}$

40. $\varphi(x, y, z) = \ln(1 + x^2 + y^2 + z^2)$

41. $\varphi(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$

42. $\varphi(x, y, z) = e^{-z}\sin(x + y)$

43–46. Gradient fields on curves *For the potential function φ and points A, B, C, and D on the level curve $\varphi(x, y) = 0$, complete the following steps.*

- ***a.*** *Find the gradient field $\mathbf{F} = \nabla\varphi$.*
- ***b.*** *Evaluate $\mathbf{F}$ at the points A, B, C, and D.*
- ***c.*** *Plot the level curve $\varphi(x, y) = 0$ and the vectors $\mathbf{F}$ at the points A, B, C, and D.*

43. $\varphi(x, y) = y - 2x$; $A(-1, -2)$, $B(0, 0)$, $C(1, 2)$, and $D(2, 4)$

44. $\varphi(x, y) = \frac{1}{2}x^2 - y$; $A(-2, 2)$, $B(-1, 1/2)$, $C(1, 1/2)$, and $D(2, 2)$

45. $\varphi(x, y) = -y + \sin x$; $A(\pi/2, 1)$, $B(\pi, 0)$, $C(3\pi/2, -1)$, and $D(2\pi, 0)$

T 46. $\varphi(x, y) = \dfrac{32 - x^4 - y^4}{32}$; $A(2, 2)$, $B(-2, 2)$, $C(-2, -2)$, and $D(2, -2)$

T 47–48. Gradient fields *Find the gradient field $\mathbf{F} = \nabla\varphi$ for the potential function φ. Sketch a few level curves of φ and a few vectors of $\mathbf{F}$.*

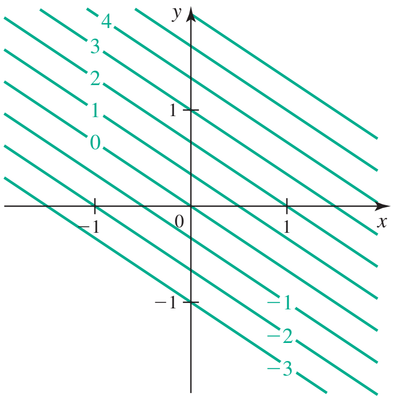
47. $\varphi(x, y) = x^2 + y^2$, for $x^2 + y^2 \le 16$

48. $\varphi(x, y) = x + y$, for $|x| \le 2$, $|y| \le 2$

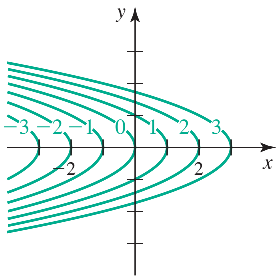
49–52. Equipotential curves *Consider the following potential functions and the graphs of their equipotential curves.*

- ***a.*** *Find the associated gradient field $\mathbf{F} = \nabla\varphi$.*
- ***b.*** *Show that the vector field is orthogonal to the equipotential curve at the point $(1, 1)$. Illustrate this result on the figure.*
- ***c.*** *Show that the vector field is orthogonal to the equipotential curve at all points (x, y).*
- ***d.*** *Sketch two flow curves representing $\mathbf{F}$ that are everywhere orthogonal to the equipotential curves.*

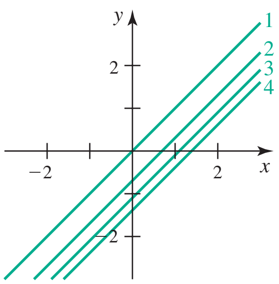
49. $\varphi(x, y) = 2x + 3y$



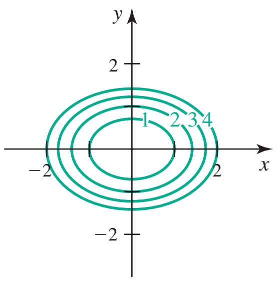
50. $\varphi(x, y) = x + y^2$



51. $\varphi(x, y) = e^{x-y}$



52. $\varphi(x, y) = x^2 + 2y^2$



53. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

- **a.** The vector field $\mathbf{F} = \langle 3x^2, 1 \rangle$ is a gradient field for both $\varphi_1(x, y) = x^3 + y$ and $\varphi_2(x, y) = y + x^3 + 100$.
- **b.** The vector field $\mathbf{F} = \dfrac{\langle y, x \rangle}{\sqrt{x^2 + y^2}}$ is constant in direction and magnitude on the unit circle.
- **c.** The vector field $\mathbf{F} = \dfrac{\langle y, x \rangle}{\sqrt{x^2 + y^2}}$ is neither a radial field nor a rotation field.

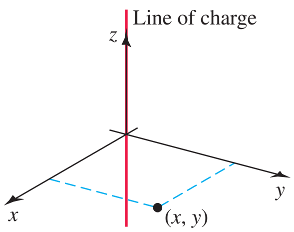
Explorations and Challenges

54. Electric field due to a point charge The electric field in the xy-plane due to a point charge at $(0, 0)$ is a gradient field with a potential function $V(x, y) = \dfrac{k}{\sqrt{x^2 + y^2}}$, where $k > 0$ is a physical constant.

- **a.** Find the components of the electric field in the x- and y-directions, where $\mathbf{E}(x, y) = -\nabla V(x, y)$.
- **b.** Show that the vectors of the electric field point in the radial direction (outward from the origin) and the radial component of $\mathbf{E}$ can be expressed as $E_r = \dfrac{k}{r^2}$, where $r = \sqrt{x^2 + y^2}$.
- **c.** Show that the vector field is orthogonal to the equipotential curves at all points in the domain of V.

55. Electric field due to a line of charge The electric field in the xy-plane due to an infinite line of charge along the z-axis is a gradient field with a potential function $V(x, y) = c \ln\left(\dfrac{r_0}{\sqrt{x^2 + y^2}}\right)$, where $c > 0$ is a constant and r_0 is a reference distance at which the potential is assumed to be 0 (see figure).

- **a.** Find the components of the electric field in the x- and y-directions, where $\mathbf{E}(x, y) = -\nabla V(x, y)$.
- **b.** Show that the electric field at a point in the xy-plane is directed outward from the origin and has magnitude $|\mathbf{E}| = c/r$, where $r = \sqrt{x^2 + y^2}$.
- **c.** Show that the vector field is orthogonal to the equipotential curves at all points in the domain of V.



56. Gravitational force due to a mass The gravitational force on a point mass m due to a point mass M is a gradient field with potential $U(r) = GMm/r$, where G is the gravitational constant and $r = \sqrt{x^2 + y^2 + z^2}$ is the distance between the masses.

- **a.** Find the components of the gravitational force in the x-, y-, and z-directions, where $\mathbf{F}(x, y, z) = -\nabla U(x, y, z)$.
- **b.** Show that the gravitational force points in the radial direction (outward from point mass M) and the radial component is $F(r) = GMm/r^2$.
- **c.** Show that the vector field is orthogonal to the equipotential surfaces at all points in the domain of U.

57–61. Flow curves in the plane *Let* $\mathbf{F}(x, y) = \langle f(x, y), g(x, y) \rangle$ *be defined on* $\mathbb{R}^2$.

57. Explain why the flow curves or streamlines of **F** satisfy $y' = \dfrac{g(x, y)}{f(x, y)}$ and are everywhere tangent to the vector field.

T 58. Find and graph the flow curves for the vector field $\mathbf{F} = \langle 1, x \rangle$.

T 59. Find and graph the flow curves for the vector field $\mathbf{F} = \langle x, x \rangle$.

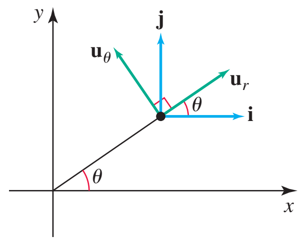
T 60. Find and graph the flow curves for the vector field $\mathbf{F} = \langle y, x \rangle$. Note that $\dfrac{d}{dx}(y^2) = 2yy'(x)$.

T 61. Find and graph the flow curves for the vector field $\mathbf{F} = \langle -y, x \rangle$.

62–63. Unit vectors in polar coordinates

62. Vectors in $\mathbb{R}^2$ may also be expressed in terms of polar coordinates. The standard coordinate unit vectors in polar coordinates are denoted $\mathbf{u}_r$ and $\mathbf{u}_\theta$ (see figure). Unlike the coordinate unit vectors in Cartesian coordinates, $\mathbf{u}_r$ and $\mathbf{u}_\theta$ change their direction depending on the point (r, θ). Use the figure to show that for $r > 0$, the following relationships among the unit vectors in Cartesian and polar coordinates hold:

$$\mathbf{u}_r = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j} \qquad \mathbf{i} = \mathbf{u}_r\cos\theta - \mathbf{u}_\theta\sin\theta$$
$$\mathbf{u}_\theta = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j} \qquad \mathbf{j} = \mathbf{u}_r\sin\theta + \mathbf{u}_\theta\cos\theta$$



63. Verify that the relationships in Exercise 62 are consistent when $\theta = 0, \pi/2, \pi$, and $3\pi/2$.

64–66. Vector fields in polar coordinates *A vector field in polar coordinates has the form* $\mathbf{F}(r, \theta) = f(r, \theta)\mathbf{u}_r + g(r, \theta)\mathbf{u}_\theta$, *where the unit vectors are defined in Exercise 62. Sketch the following vector fields and express them in Cartesian coordinates.*

64. $\mathbf{F} = \mathbf{u}_r$ **65.** $\mathbf{F} = \mathbf{u}_\theta$ **66.** $\mathbf{F} = r\,\mathbf{u}_\theta$

67. Cartesian vector field to polar vector field Write the vector field $\mathbf{F} = \langle -y, x \rangle$ in polar coordinates and sketch the field.

QUICK CHECK ANSWERS

1. The particle follows a circular path around the origin.
3. $\nabla\varphi = \langle yz, xz, xy \rangle$ ◄

17.2 Line Integrals

With integrals of a single variable, we integrate over intervals in $\mathbb{R}$ (the real line). With double and triple integrals, we integrate over regions in $\mathbb{R}^2$ or $\mathbb{R}^3$. *Line integrals* (which really should be called *curve integrals*) are another class of integrals that play an important role in vector calculus. They are used to integrate either scalar-valued functions or vector fields along curves.

Suppose a thin, circular plate has a known temperature distribution and you must compute the average temperature along the edge of the plate. The required calculation involves integrating the temperature function over the *curved* boundary of the plate. Similarly, to calculate the amount of work needed to put a satellite into orbit, we integrate the gravitational force (a vector field) along the curved path of the satellite. Both these calculations require line integrals. As you will see, line integrals take several different forms. It is the goal of this section to distinguish among these various forms and show how and when each form should be used.

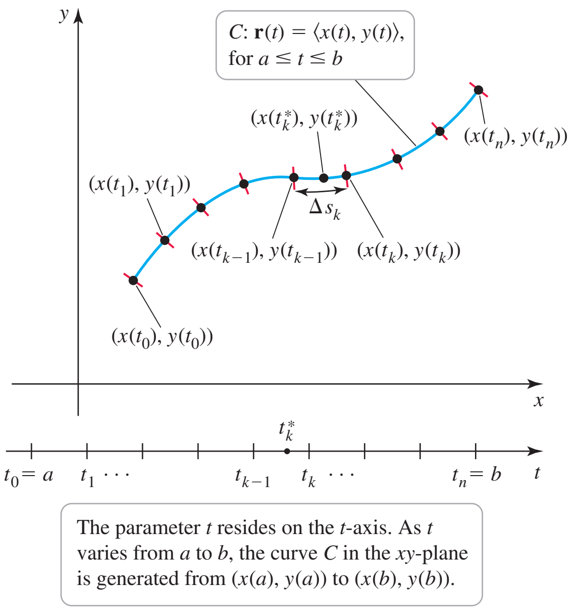


Figure 17.16

Scalar Line Integrals in the Plane

We focus first on line integrals of scalar-valued functions over curves in the *xy*-plane. Assume C is a smooth curve of finite length given by $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, for $a \le t \le b$. We divide $[a, b]$ into n subintervals using the grid points

$$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b.$$

This partition of $[a, b]$ divides C into n subarcs (**Figure 17.16**), where the arc length of the kth subarc is denoted Δs_k. Let t_k^* be a point in the kth subinterval $[t_{k-1}, t_k]$, which corresponds to a point $(x(t_k^*), y(t_k^*))$ on the kth subarc of C, for $k = 1, 2, \ldots, n$.

Now consider a scalar-valued function $z = f(x, y)$ defined on a region containing C. Evaluating f at $(x(t_k^*), y(t_k^*))$ and multiplying this value by Δs_k, we form the sum

$$S_n = \sum_{k=1}^{n} f(x(t_k^*), y(t_k^*))\Delta s_k,$$

which is similar to a Riemann sum. We now let Δ be the maximum value of $\{\Delta s_1, \dots, \Delta s_n\}$. If the limit of the sum as $n \to \infty$ and $\Delta \to 0$ exists over all partitions, then the limit is called *the line integral of f over C*.

DEFINITION Scalar Line Integral in the Plane

Suppose the scalar-valued function f is defined on a region containing the smooth curve C given by $\mathbf{r}(t) = \langle x(t), y(t)\rangle$, for $a \le t \le b$. The **line integral of f over C** is

$$\int_C f(x(t), y(t))\,ds = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x(t_k^*), y(t_k^*))\Delta s_k,$$

provided this limit exists over all partitions of $[a, b]$. When the limit exists, f is said to be **integrable** on C.

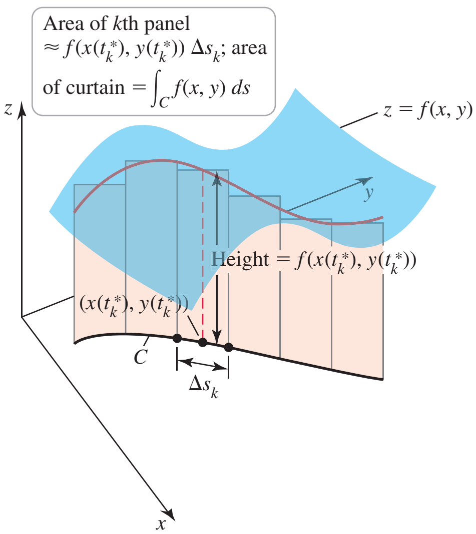


Figure 17.17

The more compact notations $\int_C f(\mathbf{r}(t))\,ds$, $\int_C f(x, y)\,ds$, and $\int_C f\,ds$ are also used for the line integral of f over C. It can be shown that if f is continuous on a region containing C, then f is integrable over C.

There are several useful interpretations of the line integral of a scalar function. If $f(x, y) = 1$, the line integral $\int_C ds$ gives the length of the curve C, just as the ordinary integral $\int_a^b dx$ gives the length of the interval $[a, b]$, which is $b - a$. If $f(x, y) \ge 0$ on C, then $\int_C f(x, y)\,ds$ can be viewed as the area of one side of the vertical, curtain-like surface that lies between the graphs of f and C (Figure 17.17). This interpretation results from regarding the product $f(x(t_k^*), y(t_k^*))\Delta s_k$ as an approximation to the area of the kth panel of the curtain. Similarly, if f is a density function for a thin wire represented by the curve C, then $\int_C f(x, y)\,ds$ gives the mass of the wire—the product $f(x(t_k^*), y(t_k^*))\Delta s_k$ is an approximation to the mass of the kth piece of the wire (Exercises 35–36).

Evaluating Line Integrals

The line integral of f over C given in the definition is not an ordinary Riemann integral, because the integrand is expressed as a function of t while the variable of integration is the arc length parameter s. We need a practical way to evaluate such integrals; the key is to use a change of variables to convert a line integral into an ordinary integral. Let C be given by $\mathbf{r}(t) = \langle x(t), y(t)\rangle$, for $a \le t \le b$. Recall from Section 14.4 that the length of C over the interval $[a, t]$ is

$$s(t) = \int_a^t |\mathbf{r}'(u)|\,du.$$

Differentiating both sides of this equation and using the Fundamental Theorem of Calculus yields $s'(t) = |\mathbf{r}'(t)|$. We now make a standard change of variables using the relationship

$$ds = s'(t)\,dt = |\mathbf{r}'(t)|\,dt.$$

QUICK CHECK 1 Explain mathematically why differentiating the arc length integral leads to $s'(t) = |\mathbf{r}'(t)|$. ◄

➤ If t represents time, then the relationship $ds = |\mathbf{r}'(t)|dt$ is a generalization of the familiar formula

distance $=$ speed $\cdot$ time.

Relying on a result from advanced calculus, the original line integral with respect to s can be converted into an ordinary integral with respect to t:

$$\int_C f\,ds = \int_a^b f(x(t), y(t))\underbrace{|\mathbf{r}'(t)|\,dt}_{ds}.$$

➤ The value of a line integral of a scalar-valued function is independent of the parameterization of C and independent of the direction in which C is traversed (Exercises 64–65).

THEOREM 17.1 Evaluating Scalar Line Integrals in $\mathbb{R}^2$
Let f be continuous on a region containing a smooth curve C: $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, for $a \leq t \leq b$. Then

$$\int_C f\,ds = \int_a^b f(x(t), y(t))\,|\mathbf{r}'(t)|\,dt$$
$$= \int_a^b f(x(t), y(t))\sqrt{x'(t)^2 + y'(t)^2}\,dt.$$

If t represents time and C is the path of a moving object, then $|\mathbf{r}'(t)|$ is the speed of the object. The *speed factor* $|\mathbf{r}'(t)|$ that appears in the integral relates distance traveled along the curve as measured by s to the elapsed time as measured by the parameter t.

Notice that if $f(x, y) = 1$, then the line integral is $\int_a^b \sqrt{x'(t)^2 + y'(t)^2}\,dt$, which is the arc length formula for C. Theorem 17.1 leads to the following procedure for evaluating line integrals.

PROCEDURE Evaluating the Line Integral $\int_C f\,ds$

1. Find a parametric description of C in the form $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, for $a \leq t \leq b$.
2. Compute $|\mathbf{r}'(t)| = \sqrt{x'(t)^2 + y'(t)^2}$.
3. Make substitutions for x and y in the integrand and evaluate an ordinary integral:

$$\int_C f\,ds = \int_a^b f(x(t), y(t))\,|\mathbf{r}'(t)|\,dt.$$

➤ When we compute the average value by an ordinary integral, we divide by the length of the interval of integration. Analogously, when we compute the average value by a line integral, we divide by the length of the curve L:

$$\bar{f} = \frac{1}{L}\int_C f\,ds.$$

EXAMPLE 1 Average temperature on a circle The temperature of the circular plate $R = \{(x, y)\colon x^2 + y^2 \leq 1\}$ is $T(x, y) = 100(x^2 + 2y^2)$. Find the average temperature along the edge of the plate.

SOLUTION Calculating the average value requires integrating the temperature function over the boundary circle $C = \{(x, y)\colon x^2 + y^2 = 1\}$ and dividing by the length (circumference) of C. The first step is to find a parametric description for C. We use the standard parameterization for a unit circle centered at the origin, $\mathbf{r} = \langle x, y \rangle = \langle \cos t, \sin t \rangle$, for $0 \leq t \leq 2\pi$. Next, we compute the speed factor

$$|\mathbf{r}'(t)| = \sqrt{x'(t)^2 + y'(t)^2} = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1.$$

We substitute $x = \cos t$ and $y = \sin t$ into the temperature function and express the line integral as an ordinary integral with respect to t:

$$\int_C T(x, y)\,ds = \int_0^{2\pi} \underbrace{100(x(t)^2 + 2y(t)^2)}_{T(t)}\,\underbrace{|\mathbf{r}'(t)|}_{1}\,dt \qquad \text{Write the line integral as an ordinary integral with respect to } t;\ ds = |\mathbf{r}'(t)|\,dt.$$

$$= 100\int_0^{2\pi}(\cos^2 t + 2\sin^2 t)\,dt \qquad \text{Substitute for } x \text{ and } y.$$

$$= 100\underbrace{\int_0^{2\pi}(1 + \sin^2 t)\,dt}_{3\pi} \qquad \cos^2 t + \sin^2 t = 1$$

$$= 300\pi. \qquad \text{Use } \sin^2 t = \frac{1 - \cos 2t}{2} \text{ and integrate.}$$

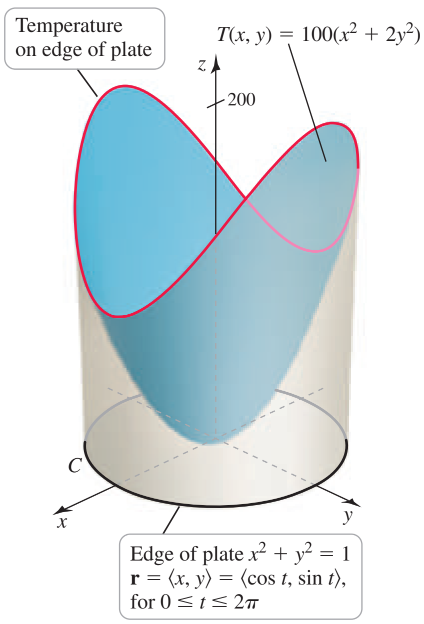


Figure 17.18

➤ The line integral in Example 1 also gives the area of the vertical cylindrical curtain that hangs between the surface and C in Figure 17.18.

The geometry of this line integral is shown in **Figure 17.18**. The temperature function on the boundary of C is a function of t. The line integral is an ordinary integral with respect to t over the interval $[0, 2\pi]$. To find the average value, we divide the line integral of the temperature by the length of the curve, which is 2π. Therefore, the average temperature on the boundary of the plate is $300\pi/(2\pi) = 150$.

Related Exercise 17 ◄

QUICK CHECK 2 Suppose $\mathbf{r}(t) = \langle t, 0 \rangle$, for $a \le t \le b$, is a parametric description of C; note that C is the interval $[a, b]$ on the x-axis. Show that $\int_C f(x, y)\, ds = \int_a^b f(t, 0)\, dt$, which is an ordinary, single-variable integral introduced in Chapter 5. ◄

➤ If $f(x, y, z) = 1$, then the line integral gives the length of C.

➤ Recall that a parametric equation of a line is

$$\mathbf{r}(t) = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle,$$

where $\langle x_0, y_0, z_0 \rangle$ is a position vector associated with a fixed point on the line and $\langle a, b, c \rangle$ is a vector parallel to the line.

Line Integrals in $\mathbb{R}^3$

The argument that leads to line integrals on plane curves extends immediately to three or more dimensions. Here is the corresponding evaluation theorem for line integrals in $\mathbb{R}^3$.

THEOREM 17.2 Evaluating Scalar Line Integrals in $\mathbb{R}^3$
Let f be continuous on a region containing a smooth curve C: $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, for $a \le t \le b$. Then

$$\begin{aligned}\int_C f\, ds &= \int_a^b f(x(t), y(t), z(t))\,|\mathbf{r}'(t)|\, dt \\ &= \int_a^b f(x(t), y(t), z(t))\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt.\end{aligned}$$

EXAMPLE 2 Line integrals in $\mathbb{R}^3$ Evaluate $\int_C (xy + 2z)\, ds$ on the following line segments.

a. The line segment from $P(1, 0, 0)$ to $Q(0, 1, 1)$

b. The line segment from $Q(0, 1, 1)$ to $P(1, 0, 0)$

SOLUTION

a. A parametric description of the line segment from $P(1, 0, 0)$ to $Q(0, 1, 1)$ is

$$\mathbf{r}(t) = \langle 1, 0, 0 \rangle + t\langle -1, 1, 1 \rangle = \langle 1 - t, t, t \rangle, \text{ for } 0 \le t \le 1.$$

The speed factor is

$$|\mathbf{r}'(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} = \sqrt{(-1)^2 + 1^2 + 1^2} = \sqrt{3}.$$

Substituting $x = 1 - t$, $y = t$, and $z = t$, the value of the line integral is

$$\begin{aligned}\int_C (xy + 2z)\, ds &= \int_0^1 \big(\underbrace{(1 - t)}_{x}\,\underbrace{t}_{y} + \underbrace{2t}_{2z}\big)\sqrt{3}\, dt && \text{Substitute for } x, y, \text{ and } z. \\ &= \sqrt{3}\int_0^1 (3t - t^2)\, dt && \text{Simplify.} \\ &= \sqrt{3}\left(\frac{3t^2}{2} - \frac{t^3}{3}\right)\Bigg|_0^1 && \text{Integrate.} \\ &= \frac{7\sqrt{3}}{6}. && \text{Evaluate.}\end{aligned}$$

b. The line segment from $Q(0, 1, 1)$ to $P(1, 0, 0)$ may be described parametrically by

$$\mathbf{r}(t) = \langle 0, 1, 1 \rangle + t\langle 1, -1, -1 \rangle = \langle t, 1 - t, 1 - t \rangle, \text{ for } 0 \le t \le 1.$$

The speed factor is

$$|\mathbf{r}'(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} = \sqrt{1^2 + (-1)^2 + (-1)^2} = \sqrt{3}.$$

We substitute $x = t$, $y = 1 - t$, and $z = 1 - t$ and do a calculation similar to that in part (a). The value of the line integral is again $\dfrac{7\sqrt{3}}{6}$, emphasizing the fact that a scalar line integral is independent of the orientation and parameterization of the curve.

Related Exercises 32–33 ◄

EXAMPLE 3 Flight of an eagle An eagle soars on the ascending spiral path

$$C: \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle = \left\langle 2400 \cos \frac{t}{2}, 2400 \sin \frac{t}{2}, 500t \right\rangle,$$

where x, y, and z are measured in feet and t is measured in minutes. How far does the eagle fly over the time interval $0 \le t \le 10$?

➤ Because we are finding the length of a curve, the integrand in this line integral is $f(x, y, z) = 1$.

SOLUTION The distance traveled is found by integrating the element of arc length ds along C, that is, $L = \int_C ds$. We now make a change of variables to the parameter t using

$$\begin{aligned} |\mathbf{r}'(t)| &= \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} \\ &= \sqrt{\left(-1200 \sin \frac{t}{2}\right)^2 + \left(1200 \cos \frac{t}{2}\right)^2 + 500^2} && \text{Substitute derivatives.} \\ &= \sqrt{1200^2 + 500^2} = 1300. && \sin^2 \frac{t}{2} + \cos^2 \frac{t}{2} = 1 \end{aligned}$$

It follows that the distance traveled is

$$L = \int_C ds = \int_0^{10} |\mathbf{r}'(t)|\, dt = \int_0^{10} 1300\, dt = 13{,}000 \text{ ft.}$$

Related Exercise 39 ◄

QUICK CHECK 3 What is the speed of the eagle in Example 3? ◄

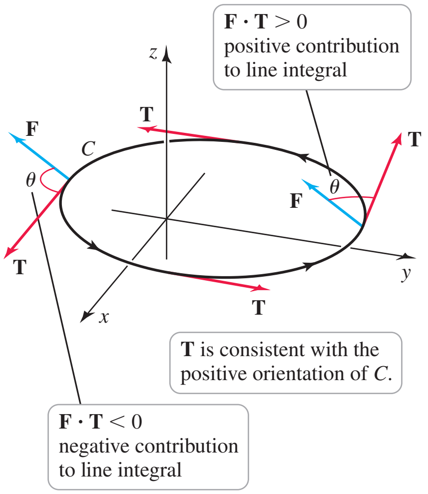


Figure 17.19

Line Integrals of Vector Fields

Line integrals along curves in $\mathbb{R}^2$ or $\mathbb{R}^3$ may also have integrands that involve vector fields. Such line integrals are different from scalar line integrals in two respects.

- Recall that an *oriented curve* is a parameterized curve for which a direction is specified. The *positive* orientation is the direction in which the curve is generated as the parameter increases. For example, the positive orientation of the circle $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$, for $0 \le t \le 2\pi$, is counterclockwise. As we will see, vector line integrals must be evaluated on oriented curves, and the value of a line integral depends on the orientation.
- The line integral of a vector field $\mathbf{F}$ along an oriented curve involves a specific component of $\mathbf{F}$ relative to the curve. We begin by defining vector line integrals for the *tangential* component of $\mathbf{F}$, a situation that has many physical applications.

Let C: $\mathbf{r}(s) = \langle x(s), y(s), z(s) \rangle$ be a smooth oriented curve in $\mathbb{R}^3$ parameterized by arc length and let $\mathbf{F}$ be a vector field that is continuous on a region containing C. At each point of C, the unit tangent vector $\mathbf{T}$ points in the positive direction on C (Figure 17.19). The component of $\mathbf{F}$ in the direction of $\mathbf{T}$ at a point of C is $|\mathbf{F}| \cos \theta$, where θ is the angle between $\mathbf{F}$ and $\mathbf{T}$. Because $\mathbf{T}$ is a unit vector,

$$|\mathbf{F}| \cos \theta = |\mathbf{F}||\mathbf{T}| \cos \theta = \mathbf{F} \cdot \mathbf{T}.$$

The first line integral of a vector field $\mathbf{F}$ that we introduce is the line integral of the scalar $\mathbf{F} \cdot \mathbf{T}$ along the curve C. When we integrate $\mathbf{F} \cdot \mathbf{T}$ along C, the effect is to add up the components of $\mathbf{F}$ in the direction of C at each point of C.

➤ The component of $\mathbf{F}$ in the direction of $\mathbf{T}$ is the scalar component of $\mathbf{F}$ in the direction of $\mathbf{T}$, $\text{scal}_{\mathbf{T}}\, \mathbf{F}$, as defined in Section 13.3. Note that $|\mathbf{T}| = 1$.

> **DEFINITION Line Integral of a Vector Field**
>
> Let $\mathbf{F}$ be a vector field that is continuous on a region containing a smooth oriented curve C parameterized by arc length. Let $\mathbf{T}$ be the unit tangent vector at each point of C consistent with the orientation. The line integral of $\mathbf{F}$ over C is $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$.

➤ Some texts let $d\mathbf{s}$ stand for $\mathbf{T}\, ds$. Then the line integral $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$ is written $\int_C \mathbf{F} \cdot d\mathbf{s}$.

Just as we did for line integrals of scalar-valued functions, we need a method for evaluating vector line integrals when the parameter is not the arc length. Suppose C has a parameterization $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, for $a \le t \le b$. Recall from Section 14.2 that the unit tangent vector at a point on the curve is $\mathbf{T} = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$. Using the fact that $ds = |\mathbf{r}'(t)|\, dt$, the line integral becomes

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_a^b \mathbf{F} \cdot \underbrace{\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}}_{T} \underbrace{|\mathbf{r}'(t)|\, dt}_{ds} = \int_a^b \mathbf{F} \cdot \mathbf{r}'(t)\, dt.$$

> Keep in mind that $f(t)$ stands for $f(x(t), y(t), z(t))$ with analogous expressions for $g(t)$ and $h(t)$.

This integral may be written in several equivalent forms. If $\mathbf{F} = \langle f, g, h \rangle$, then the line integral is expressed in component form as

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_a^b \mathbf{F} \cdot \mathbf{r}'(t)\, dt = \int_a^b (f(t)x'(t) + g(t)y'(t) + h(t)z'(t))\, dt.$$

Another useful form is obtained by noting that

$$dx = x'(t)\, dt, \qquad dy = y'(t)\, dt, \qquad dz = z'(t)\, dt.$$

Making these replacements in the previous integral results in the form

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_C f\, dx + g\, dy + h\, dz.$$

Finally, if we let $d\mathbf{r} = \langle dx, dy, dz \rangle$, then $f\, dx + g\, dy + h\, dz = \mathbf{F} \cdot d\mathbf{r}$, and we have

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_C \mathbf{F} \cdot d\mathbf{r}.$$

It is helpful to become familiar with these various forms of the line integral.

Different Forms of Line Integrals of Vector Fields

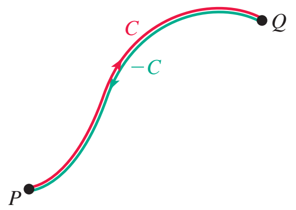
The line integral $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$ may be expressed in the following forms, where $\mathbf{F} = \langle f, g, h \rangle$ and C has a parameterization $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, for $a \le t \le b$:

$$\begin{aligned}\int_a^b \mathbf{F} \cdot \mathbf{r}'(t)\, dt &= \int_a^b (f(t)x'(t) + g(t)y'(t) + h(t)z'(t))\, dt \\ &= \int_C f\, dx + g\, dy + h\, dz \\ &= \int_C \mathbf{F} \cdot d\mathbf{r}.\end{aligned}$$

For line integrals in the plane, we let $\mathbf{F} = \langle f, g \rangle$ and assume C is parameterized in the form $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, for $a \le t \le b$. Then

$$\int_a^b \mathbf{F} \cdot \mathbf{r}'(t)\, dt = \int_a^b (f(t)x'(t) + g(t)y'(t))\, dt = \int_C f\, dx + g\, dy = \int_C \mathbf{F} \cdot d\mathbf{r}.$$

> We use the convention that $-C$ is the curve C with the opposite orientation.



EXAMPLE 4 **Different paths** Evaluate $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$ with $\mathbf{F} = \langle y - x, x \rangle$ on the following oriented paths in $\mathbb{R}^2$ (Figure 17.20).

a. The quarter-circle C_1 from $P(0, 1)$ to $Q(1, 0)$

b. The quarter-circle $-C_1$ from $Q(1, 0)$ to $P(0, 1)$

c. The path C_2 from $P(0, 1)$ to $Q(1, 0)$ via two line segments through $O(0, 0)$

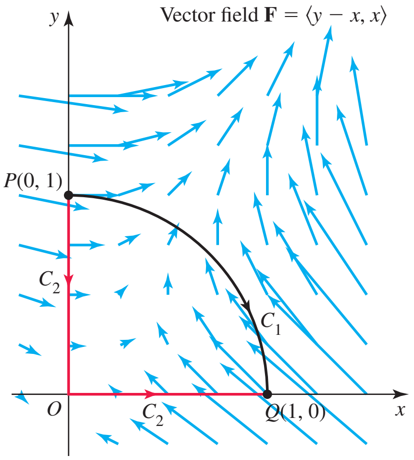


Figure 17.20

SOLUTION

a. Working in $\mathbb{R}^2$, a parametric description of the curve C_1 with the required (clockwise) orientation is $\mathbf{r}(t) = \langle \sin t, \cos t \rangle$, for $0 \le t \le \pi/2$. Along C_1, the vector field is

$$\mathbf{F} = \langle y - x, x \rangle = \langle \cos t - \sin t, \sin t \rangle.$$

The velocity vector is $\mathbf{r}'(t) = \langle \cos t, -\sin t \rangle$, so the integrand of the line integral is

$$\mathbf{F} \cdot \mathbf{r}'(t) = \langle \cos t - \sin t, \sin t \rangle \cdot \langle \cos t, -\sin t \rangle = \underbrace{\cos^2 t - \sin^2 t}_{\cos 2t} - \underbrace{\sin t \cos t}_{\frac{1}{2}\sin 2t}.$$

The value of the line integral of $\mathbf{F}$ over C_1 is

$$\begin{aligned}\int_0^{\pi/2} \mathbf{F}\cdot\mathbf{r}'(t)\,dt &= \int_0^{\pi/2}\left(\cos 2t - \frac{1}{2}\sin 2t\right)dt && \text{Substitute for } \mathbf{F}\cdot\mathbf{r}'(t).\\ &= \left(\frac{1}{2}\sin 2t + \frac{1}{4}\cos 2t\right)\Big|_0^{\pi/2} && \text{Evaluate integral.}\\ &= -\frac{1}{2}. && \text{Simplify.}\end{aligned}$$

b. A parameterization of the curve $-C_1$ from Q to P is $\mathbf{r}(t) = \langle \cos t, \sin t\rangle$, for $0 \le t \le \pi/2$. The vector field along the curve is

$$\mathbf{F} = \langle y - x, x\rangle = \langle \sin t - \cos t, \cos t\rangle,$$

and the velocity vector is $\mathbf{r}'(t) = \langle -\sin t, \cos t\rangle$. A calculation similar to that in part (a) results in

$$\int_{-C_1} \mathbf{F}\cdot\mathbf{T}\,ds = \int_0^{\pi/2}\mathbf{F}\cdot\mathbf{r}'(t)\,dt = \frac{1}{2}.$$

Comparing the results of parts (a) and (b), we see that reversing the orientation of C_1 reverses the sign of the line integral of the vector field.

c. The path C_2 consists of two line segments.

- The segment from P to O is parameterized by $\mathbf{r}(t) = \langle 0, 1 - t\rangle$, for $0 \le t \le 1$. Therefore, $\mathbf{r}'(t) = \langle 0, -1\rangle$ and $\mathbf{F} = \langle y - x, x\rangle = \langle 1 - t, 0\rangle$. On this segment, $\mathbf{T} = \langle 0, -1\rangle$.
- The line segment from O to Q is parameterized by $\mathbf{r}(t) = \langle t, 0\rangle$, for $0 \le t \le 1$. Therefore, $\mathbf{r}'(t) = \langle 1, 0\rangle$ and $\mathbf{F} = \langle y - x, x\rangle = \langle -t, t\rangle$. On this segment, $\mathbf{T} = \langle 1, 0\rangle$.

➤ Line integrals of vector fields satisfy properties similar to those of ordinary integrals. Suppose C is a smooth curve from A to B, C_1 is the curve from A to P, and C_2 is the curve from P to B, where P is a point on C between A and B. Then

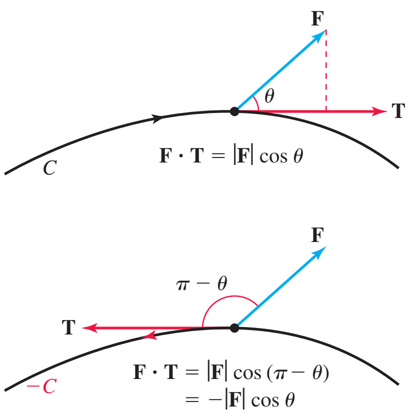
$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{C_1}\mathbf{F}\cdot d\mathbf{r} + \int_{C_2}\mathbf{F}\cdot d\mathbf{r}.$$

The line integral is split into two parts and evaluated as follows:

$$\begin{aligned}\int_{C_2}\mathbf{F}\cdot\mathbf{T}\,ds &= \int_{PO}\mathbf{F}\cdot\mathbf{T}\,ds + \int_{OQ}\mathbf{F}\cdot\mathbf{T}\,ds\\ &= \int_0^1\langle 1 - t, 0\rangle\cdot\langle 0, -1\rangle\,dt + \int_0^1\langle -t, t\rangle\cdot\langle 1, 0\rangle\,dt && \text{Substitute for } x, y, \mathbf{r}'.\\ &= \int_0^1 0\,dt + \int_0^1(-t)\,dt && \text{Simplify.}\\ &= -\frac{1}{2}. && \text{Evaluate integrals.}\end{aligned}$$

The line integrals in parts (a) and (c) have the same value and run from P to Q, but along different paths. We might ask: For what vector fields are the values of a line integral independent of path? We return to this question in Section 17.3.

Related Exercises 42–43 ◄



Reversing the orientation of C changes the sign of $\mathbf{F}\cdot\mathbf{T}$ at each point on C.

Figure 17.21

The solutions to parts (a) and (b) of Example 4 illustrate a general result that applies to line integrals of vector fields:

$$\int_{-C}\mathbf{F}\cdot\mathbf{T}\,ds = -\int_C \mathbf{F}\cdot\mathbf{T}\,ds.$$

Figure 17.21 provides the justification of this fact: Reversing the orientation of C changes the sign of $\mathbf{F}\cdot\mathbf{T}$ at each point of C, which changes the sign of the line integral.

Work Integrals A common application of line integrals of vector fields is computing the work done in moving an object in a force field (for example, a gravitational or electric field). First recall (Section 6.7) that if $\mathbf{F}$ is a *constant* force field, the work done in moving an object a distance d along the x-axis is $W = F_x d$, where $F_x = |\mathbf{F}|\cos\theta$ is the component

➤ Remember that the value of $\int_C f\,ds$ (the line integral of a *scalar* function) does not depend on the orientation of C.

QUICK CHECK 4 Suppose a two-dimensional force field is everywhere directed outward from the origin, and C is a circle centered at the origin. What is the angle between the field and the unit vectors tangent to C? ◄

of the force along the x-axis (Figure 17.22a). Only the component of $\mathbf{F}$ in the direction of motion contributes to the work. More generally, if $\mathbf{F}$ is a *variable* force field, the work done in moving an object from $x = a$ to $x = b$ is $W = \int_a^b F_x(x)\,dx$, where again F_x is the component of the force $\mathbf{F}$ in the direction of motion (parallel to the x-axis, Figure 17.22b).

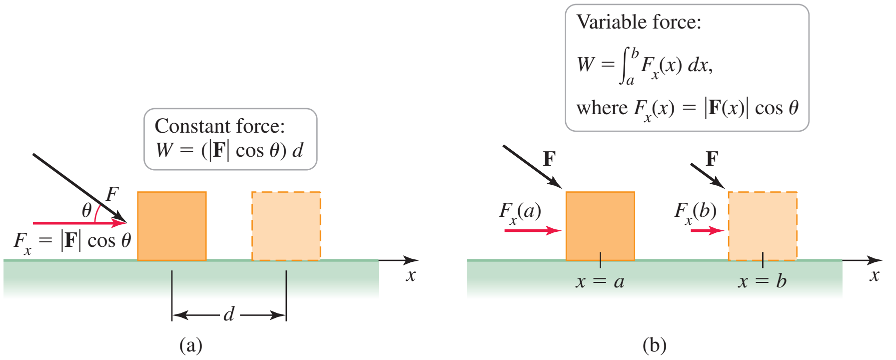


Figure 17.22

We now take this progression one step further. Let $\mathbf{F}$ be a variable force field defined in a region D of $\mathbb{R}^3$ and suppose C is a smooth oriented curve in D along which an object moves. The direction of motion at each point of C is given by the unit tangent vector $\mathbf{T}$. Therefore, the component of $\mathbf{F}$ in the direction of motion is $\mathbf{F}\cdot\mathbf{T}$, which is the tangential component of $\mathbf{F}$ along C. Summing the contributions to the work at each point of C, the work done in moving an object along C in the presence of the force is the line integral of $\mathbf{F}\cdot\mathbf{T}$ (Figure 17.23).

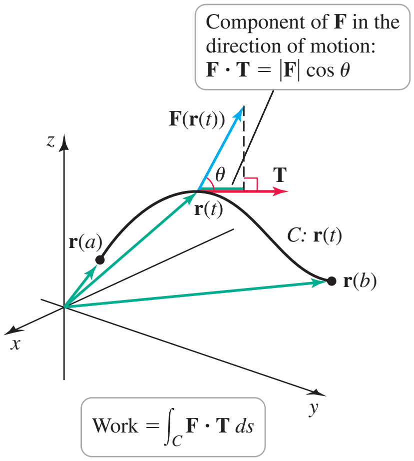


Figure 17.23

➤ Just to be clear, a work integral is nothing more than a line integral of the tangential component of a force field.

DEFINITION Work Done in a Force Field

Let $\mathbf{F}$ be a continuous force field in a region D of $\mathbb{R}^3$. Let

$$C: \mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle, \text{ for } a \le t \le b,$$

be a smooth curve in D with a unit tangent vector $\mathbf{T}$ consistent with the orientation. The work done in moving an object along C in the positive direction is

$$W = \int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_a^b \mathbf{F}\cdot\mathbf{r}'(t)\,dt.$$

EXAMPLE 5 An inverse square force Gravitational and electrical forces between point masses and point charges obey inverse square laws: They act along the line joining the centers and they vary as $1/r^2$, where r is the distance between the centers. The force of attraction (or repulsion) of an inverse square force field is given by the vector field $\mathbf{F} = \dfrac{k\langle x, y, z\rangle}{(x^2+y^2+z^2)^{3/2}}$, where k is a physical constant. Because $\mathbf{r} = \langle x, y, z\rangle$, this force may also be written $\mathbf{F} = \dfrac{k\mathbf{r}}{|\mathbf{r}|^3}$. Find the work done in moving an object along the following paths.

a. C_1 is the line segment from $(1, 1, 1)$ to (a, a, a), where $a > 1$.

b. C_2 is the extension of C_1 produced by letting $a \to \infty$.

SOLUTION

a. A parametric description of C_1 consistent with the orientation is $\mathbf{r}(t) = \langle t, t, t\rangle$, for $1 \le t \le a$, with $\mathbf{r}'(t) = \langle 1, 1, 1\rangle$. In terms of the parameter t, the force field is

$$\mathbf{F} = \frac{k\langle x, y, z\rangle}{(x^2+y^2+z^2)^{3/2}} = \frac{k\langle t, t, t\rangle}{(3t^2)^{3/2}}.$$

The dot product that appears in the work integral is

$$\mathbf{F}\cdot\mathbf{r}'(t) = \frac{k\langle t, t, t\rangle}{(3t^2)^{3/2}}\cdot\langle 1, 1, 1\rangle = \frac{3kt}{3\sqrt{3}\,t^3} = \frac{k}{\sqrt{3}\,t^2}.$$

Therefore, the work done is

$$W = \int_1^a \mathbf{F} \cdot \mathbf{r}'(t)\, dt = \frac{k}{\sqrt{3}} \int_1^a t^{-2}\, dt = \frac{k}{\sqrt{3}}\left(1 - \frac{1}{a}\right).$$

b. The path C_2 is obtained by letting $a \to \infty$ in part (a). The required work is

$$W = \lim_{a \to \infty} \frac{k}{\sqrt{3}}\left(1 - \frac{1}{a}\right) = \frac{k}{\sqrt{3}}.$$

If **F** is a gravitational field, this result implies that the work required to escape Earth's gravitational field is finite (which makes space flight possible).

Related Exercise 55 ◄

Circulation and Flux of a Vector Field

Line integrals are useful for investigating two important properties of vector fields: *circulation* and *flux*. These properties apply to any vector field, but they are particularly relevant and easy to visualize if you think of **F** as the velocity field for a moving fluid.

➤ In the definition of circulation, a *closed curve* is a curve whose initial and terminal points are the same, as defined formally in Section 17.3.

➤ Although we define circulation integrals for smooth curves, these integrals may be computed on piecewise-smooth curves. We adopt the convention that *piecewise* refers to a curve with finitely many pieces.

Circulation We assume $\mathbf{F} = \langle f, g, h \rangle$ is a continuous vector field on a region D of $\mathbb{R}^3$, and we take C to be a *closed* smooth oriented curve in D. The *circulation* of **F** along C is a measure of how much of the vector field points in the direction of C. More simply, as you travel along C in the positive direction, how much of the vector field is at your back and how much of it is in your face? To determine the circulation, we simply "add up" the components of **F** in the direction of the unit tangent vector **T** at each point. Therefore, circulation integrals are another example of line integrals of vector fields.

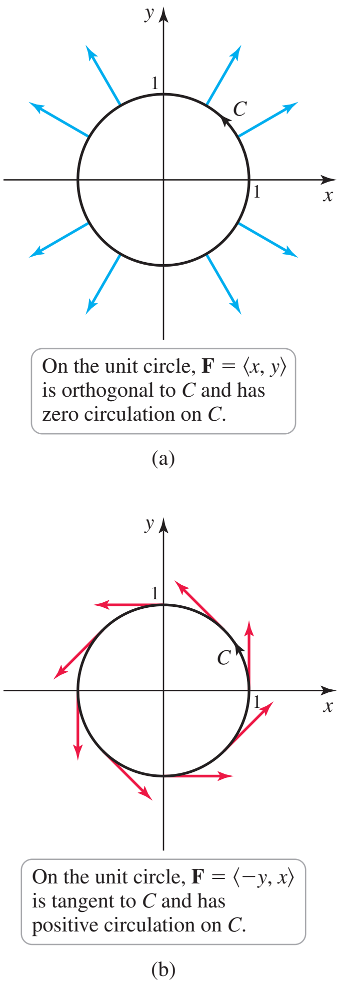


Figure 17.24

> **DEFINITION Circulation**
>
> Let **F** be a continuous vector field on a region D of $\mathbb{R}^3$, and let C be a closed smooth oriented curve in D. The **circulation** of **F** on C is $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$, where **T** is the unit vector tangent to C consistent with the orientation.

EXAMPLE 6 Circulation of two-dimensional flows Let C be the unit circle with counterclockwise orientation. Find the circulation on C of the following vector fields.

a. The radial vector field $\mathbf{F} = \langle x, y \rangle$

b. The rotation vector field $\mathbf{F} = \langle -y, x \rangle$

SOLUTION

a. The unit circle with the specified orientation is described parametrically by $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$, for $0 \le t \le 2\pi$. Therefore, $\mathbf{r}'(t) = \langle -\sin t, \cos t \rangle$ and the circulation of the radial field $\mathbf{F} = \langle x, y \rangle$ is

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_0^{2\pi} \mathbf{F} \cdot \mathbf{r}'(t)\, dt \qquad \text{Evaluation of a line integral}$$

$$= \int_0^{2\pi} \underbrace{\langle \cos t, \sin t \rangle}_{\mathbf{F} = \langle x, y \rangle} \cdot \underbrace{\langle -\sin t, \cos t \rangle}_{\mathbf{r}'(t)}\, dt \qquad \text{Substitute for } \mathbf{F} \text{ and } \mathbf{r}'.$$

$$= \int_0^{2\pi} 0\, dt = 0. \qquad \text{Simplify.}$$

The tangential component of the radial field is zero everywhere on C, so the circulation is zero (**Figure 17.24a**).

b. The circulation for the rotation field $\mathbf{F} = \langle -y, x \rangle$ is

$$\begin{aligned}\int_C \mathbf{F} \cdot \mathbf{T}\, ds &= \int_0^{2\pi} \mathbf{F} \cdot \mathbf{r}'(t)\, dt && \text{Evaluation of a line integral}\\ &= \int_0^{2\pi} \underbrace{\langle -\sin t, \cos t \rangle}_{\mathbf{F} = \langle -y, x\rangle} \cdot \underbrace{\langle -\sin t, \cos t \rangle}_{\mathbf{r}'(t)}\, dt && \text{Substitute for } \mathbf{F} \text{ and } \mathbf{r}'.\\ &= \int_0^{2\pi} \underbrace{(\sin^2 t + \cos^2 t)}_{1}\, dt && \text{Simplify.}\\ &= 2\pi.\end{aligned}$$

In this case, at every point of C, the rotation field is in the direction of the tangent vector; the result is a positive circulation (**Figure 17.24b**).

Related Exercise 57 ◀

EXAMPLE 7 **Circulation of a three-dimensional flow** Find the circulation of the vector field $\mathbf{F} = \langle z, x, -y \rangle$ on the tilted ellipse C: $\mathbf{r}(t) = \langle \cos t, \sin t, \cos t \rangle$, for $0 \le t \le 2\pi$ (**Figure 17.25a**).

SOLUTION We first determine that

$$\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle = \langle -\sin t, \cos t, -\sin t \rangle.$$

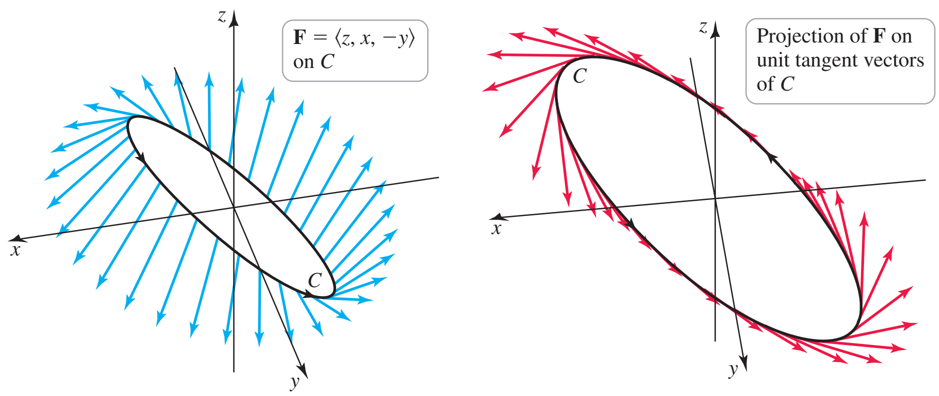


Figure 17.25

Substituting $x = \cos t$, $y = \sin t$, and $z = \cos t$ into $\mathbf{F} = \langle z, x, -y \rangle$, the circulation is

$$\begin{aligned}\int_C \mathbf{F} \cdot \mathbf{T}\, ds &= \int_0^{2\pi} \mathbf{F} \cdot \mathbf{r}'(t)\, dt && \text{Evaluation of a line integral}\\ &= \int_0^{2\pi} \langle \cos t, \cos t, -\sin t \rangle \cdot \langle -\sin t, \cos t, -\sin t \rangle\, dt && \text{Substitute for } \mathbf{F} \text{ and } \mathbf{r}'.\\ &= \int_0^{2\pi} (-\sin t \cos t + 1)\, dt && \text{Simplify; } \sin^2 t + \cos^2 t = 1.\\ &= 2\pi. && \text{Evaluate integral.}\end{aligned}$$

Figure 17.25b shows the projection of the vector field on the unit tangent vectors at various points on C. The circulation is the "sum" of the scalar components associated with these projections, which, in this case, is positive.

Related Exercise 53 ◀

➤ In the definition of flux, the non-self-intersecting property of C means that C is a *simple* curve, as defined formally in Section 17.3.

Flux of Two-Dimensional Vector Fields Assume $\mathbf{F} = \langle f, g \rangle$ is a continuous vector field on a region R of $\mathbb{R}^2$. We let C be a smooth oriented curve in R that does not intersect itself; C may or may not be closed. To compute the *flux* of the vector field across C, we "add up" the components of $\mathbf{F}$ *orthogonal* or *normal* to C at each point of C. Notice that every point on C has two unit vectors normal to C. Therefore, we let $\mathbf{n}$ denote the unit vector in the xy-plane normal to C in a direction to be defined momentarily. Once the direction of $\mathbf{n}$ is defined, the component of $\mathbf{F}$ normal to C is $\mathbf{F} \cdot \mathbf{n}$, and the flux is the line integral of $\mathbf{F} \cdot \mathbf{n}$ along C, which we denote $\int_C \mathbf{F} \cdot \mathbf{n}\, ds$.

➤ Recall that $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

The first step is to define the unit normal vector at a point P of C. Because C lies in the xy-plane, the unit vector $\mathbf{T}$ tangent at P also lies in the xy-plane. Therefore, its z-component is 0, and we let $\mathbf{T} = \langle T_x, T_y, 0 \rangle$. As always, $\mathbf{k} = \langle 0, 0, 1 \rangle$ is the unit vector in the z-direction. Because a unit vector $\mathbf{n}$ in the xy-plane normal to C is orthogonal to both $\mathbf{T}$ and $\mathbf{k}$, we determine the direction of $\mathbf{n}$ by letting $\mathbf{n} = \mathbf{T} \times \mathbf{k}$. This choice has two implications.

- If C is a closed curve oriented counterclockwise (when viewed from above), the unit normal vector points *outward* along the curve (Figure 17.26a). When $\mathbf{F}$ also points outward at a point on C, the angle θ between $\mathbf{F}$ and $\mathbf{n}$ satisfies $0 \le \theta < \frac{\pi}{2}$ (Figure 17.26b). At all such points, $\mathbf{F} \cdot \mathbf{n} > 0$ and there is a positive contribution to the flux across C. When $\mathbf{F}$ points inward at a point on C, $\frac{\pi}{2} < \theta \le \pi$ and $\mathbf{F} \cdot \mathbf{n} < 0$, which means there is a negative contribution to the flux at that point.
- If C is not a closed curve, the unit normal vector points to the right (when viewed from above) as the curve is traversed in the positive direction.

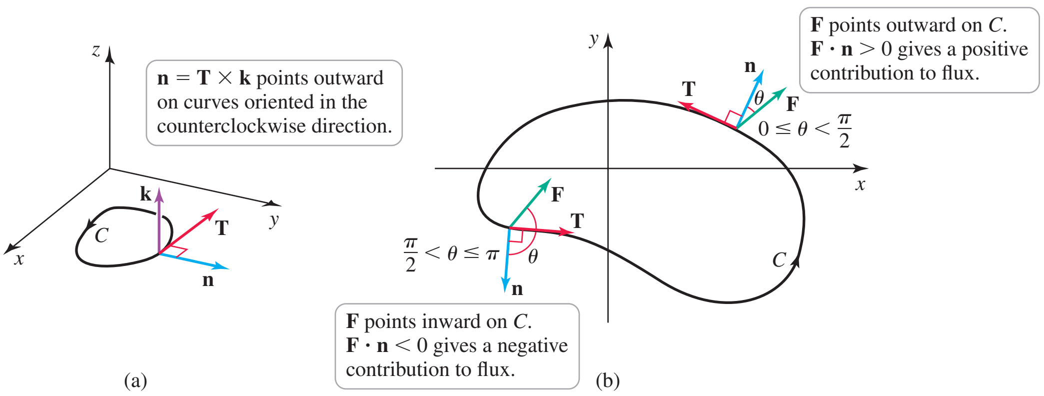


Figure 17.26

Calculating the cross product that defines the unit normal vector $\mathbf{n}$, we find that

$$\mathbf{n} = \mathbf{T} \times \mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ T_x & T_y & 0 \\ 0 & 0 & 1 \end{vmatrix} = T_y \mathbf{i} - T_x \mathbf{j}.$$

QUICK CHECK 5 Sketch a closed curve on a sheet of paper and draw a unit tangent vector $\mathbf{T}$ on the curve pointing in the counterclockwise direction. Explain why $\mathbf{n} = \mathbf{T} \times \mathbf{k}$ is an *outward* unit normal vector. ◄

Because $\mathbf{T} = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$, the components of $\mathbf{T}$ are

$$\mathbf{T} = \langle T_x, T_y, 0 \rangle = \frac{\langle x'(t), y'(t), 0 \rangle}{|\mathbf{r}'(t)|}.$$

We now have an expression for the unit normal vector:

$$\mathbf{n} = T_y \mathbf{i} - T_x \mathbf{j} = \frac{y'(t)}{|\mathbf{r}'(t)|}\mathbf{i} - \frac{x'(t)}{|\mathbf{r}'(t)|}\mathbf{j} = \frac{\langle y'(t), -x'(t) \rangle}{|\mathbf{r}'(t)|}.$$

To evaluate the flux integral $\int_C \mathbf{F} \cdot \mathbf{n}\, ds$, we make a familiar change of variables by letting $ds = |\mathbf{r}'(t)|\, dt$. The flux of $\mathbf{F} = \langle f, g \rangle$ across C is then

$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_a^b \mathbf{F} \cdot \underbrace{\frac{\langle y'(t), -x'(t) \rangle}{|\mathbf{r}'(t)|}}_{\mathbf{n}} \underbrace{|\mathbf{r}'(t)|\, dt}_{ds} = \int_a^b (f(t)y'(t) - g(t)x'(t))\, dt.$$

This is one useful form of the flux integral. Alternatively, we can note that $dx = x'(t)\,dt$ and $dy = y'(t)\,dt$ and write

$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_C f\, dy - g\, dx.$$

➤ Like circulation integrals, flux integrals may be computed on piecewise-smooth curves by finding the flux on each piece and adding the results.

DEFINITION Flux

Let $\mathbf{F} = \langle f, g \rangle$ be a continuous vector field on a region R of $\mathbb{R}^2$. Let C: $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, for $a \le t \le b$, be a smooth oriented curve in R that does not intersect itself. The **flux** of the vector field $\mathbf{F}$ across C is

$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_a^b (f(t)y'(t) - g(t)x'(t))\, dt,$$

where $\mathbf{n} = \mathbf{T} \times \mathbf{k}$ is the unit normal vector and $\mathbf{T}$ is the unit tangent vector consistent with the orientation. If C is a closed curve with counterclockwise orientation, $\mathbf{n}$ is the outward normal vector, and the flux integral gives the **outward flux** across C.

The concepts of circulation and flux can be visualized in terms of headwinds and crosswinds. Suppose the wind patterns in your neighborhood can be modeled with a vector field $\mathbf{F}$ (that doesn't change with time). Now imagine taking a walk around the block in a counterclockwise direction along a closed path. At different points along your walk, you encounter winds from various directions and with various speeds. The circulation of the wind field $\mathbf{F}$ along your path is the net amount of headwind (negative contribution) and tailwind (positive contribution) that you encounter during your walk. The flux of $\mathbf{F}$ across your path is the net amount of crosswind (positive from your left and negative from your right) encountered on your walk.

EXAMPLE 8 Flux of two-dimensional flows Find the outward flux across the unit circle with counterclockwise orientation for the following vector fields.

a. The radial vector field $\mathbf{F} = \langle x, y \rangle$

b. The rotation vector field $\mathbf{F} = \langle -y, x \rangle$

SOLUTION

a. The unit circle with counterclockwise orientation has a description $\mathbf{r}(t) = \langle x(t), y(t) \rangle = \langle \cos t, \sin t \rangle$, for $0 \le t \le 2\pi$. Therefore, $x'(t) = -\sin t$ and $y'(t) = \cos t$. The components of $\mathbf{F}$ are $f = x(t) = \cos t$ and $g = y(t) = \sin t$. It follows that the outward flux is

$$\int_a^b (f(t)y'(t) - g(t)x'(t))\, dt = \int_0^{2\pi} (\underbrace{\cos t}_{f(t)}\, \underbrace{\cos t}_{y'(t)} - \underbrace{\sin t}_{g(t)}\, \underbrace{(-\sin t)}_{x'(t)})\, dt$$

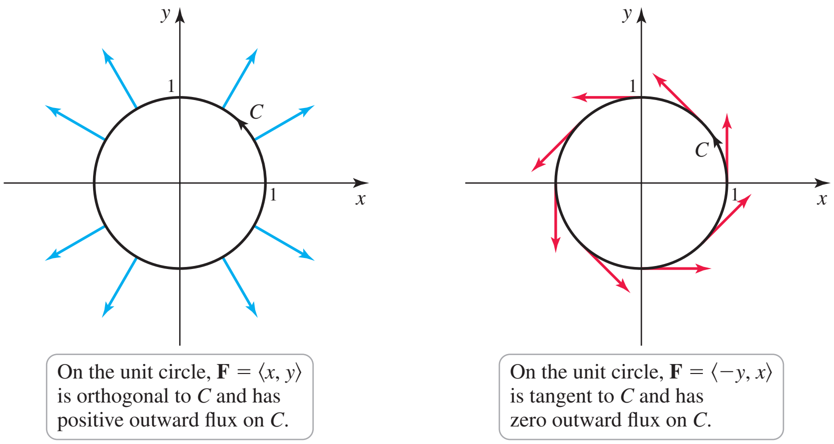
$$= \int_0^{2\pi} 1\, dt = 2\pi. \qquad \cos^2 t + \sin^2 t = 1$$

Because the radial field points outward and is aligned with the unit normal vectors on C, the outward flux is positive (Figure 17.27a).

b. For the rotation field, $f = -y(t) = -\sin t$ and $g = x(t) = \cos t$. The outward flux is

$$\int_a^b (f(t)y'(t) - g(t)x'(t))\, dt = \int_0^{2\pi} (\underbrace{-\sin t}_{f(t)}\, \underbrace{\cos t}_{y'(t)} - \underbrace{\cos t}_{g(t)}\, \underbrace{(-\sin t)}_{x'(t)})\, dt$$

$$= \int_0^{2\pi} 0\, dt = 0.$$



(a) (b)

Figure 17.27

Because the rotation field is orthogonal to **n** at all points of C, the outward flux across C is zero (Figure 17.27b). The results of Examples 6 and 8 are worth remembering: On a unit circle centered at the origin, the *radial* vector field $\langle x, y \rangle$ has outward flux 2π and zero circulation. The *rotation* vector field $\langle -y, x \rangle$ has zero outward flux and circulation 2π.

Related Exercises 59–60 ◀

SECTION 17.2 EXERCISES

Getting Started

1. How does a line integral differ from the single-variable integral $\int_a^b f(x)\,dx$?

2. If a curve C is given by $\mathbf{r}(t) = \langle t, t^2 \rangle$, what is $|\mathbf{r}'(t)|$?

3. Given that C is the curve $\mathbf{r}(t) = \langle \cos t, t \rangle$, for $\pi/2 \le t \le \pi$, convert the line integral $\displaystyle\int_C \frac{x}{y}\,ds$ to an ordinary integral. Do not evaluate the integral.

4–7. *Find a parametric description* $\mathbf{r}(t)$ *for the following curves.*

4. The segment of the curve $x = \sin \pi y$ from $(0, 0)$ to $(0, 3)$

5. The line segment from $(1, 2, 3)$ to $(5, 4, 0)$

6. The quarter-circle from $(1, 0)$ to $(0, 1)$ with its center at the origin

7. The segment of the parabola $x = y^2 + 1$ from $(5, 2)$ to $(17, 4)$

8. Find an expression for the vector field $\mathbf{F} = \langle x - y, y - x \rangle$ (in terms of t) along the unit circle $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$.

9. Suppose C is the curve $\mathbf{r}(t) = \langle t, t^3 \rangle$, for $0 \le t \le 2$, and $\mathbf{F} = \langle x, 2y \rangle$. Evaluate $\int_C \mathbf{F} \cdot \mathbf{T}\,ds$ using the following steps.
   a. Convert the line integral $\int_C \mathbf{F} \cdot \mathbf{T}\,ds$ to an ordinary integral.
   b. Evaluate the integral in part (a).

10. Suppose C is the circle $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$, for $0 \le t \le 2\pi$, and $\mathbf{F} = \langle 1, x \rangle$. Evaluate $\int_C \mathbf{F} \cdot \mathbf{n}\,ds$ using the following steps.
    a. Convert the line integral $\int_C \mathbf{F} \cdot \mathbf{n}\,ds$ to an ordinary integral.
    b. Evaluate the integral in part (a).

11. State two other forms for the line integral $\int_C \mathbf{F} \cdot \mathbf{T}\,ds$ given that $\mathbf{F} = \langle f, g, h \rangle$.

12–13. *Assume f is continuous on a region containing the smooth curve C from point A to point B and suppose* $\int_C f\,ds = 10$.

12. Explain the meaning of the curve $-C$ and state the value of $\int_{-C} f\,ds$.

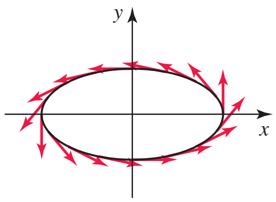
13. Suppose P is a point on the curve C between A and B, where C_1 is the part of the curve from A to P, and C_2 is the part of the curve from P to B. Assuming $\int_{C_1} f\,ds = 3$, find the value of $\int_{C_2} f\,ds$.

14. Consider the graph of an ellipse C, oriented counterclockwise. Graphs of the vector fields $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, and $\mathbf{F}_4$ along the curve C are given (see figures). Along C, $\mathbf{F}_1$ and $\mathbf{F}_4$ are tangent to C, and $\mathbf{F}_2$ and $\mathbf{F}_3$ are normal to C. Determine whether the following integrals are positive, negative, or equal to 0.

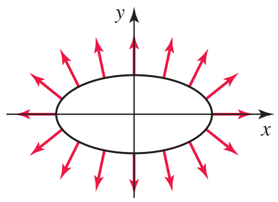
    a. $\displaystyle\int_C \mathbf{F}_1 \cdot \mathbf{T}\,ds$ **b.** $\displaystyle\int_C \mathbf{F}_2 \cdot \mathbf{T}\,ds$ **c.** $\displaystyle\int_C \mathbf{F}_3 \cdot \mathbf{T}\,ds$

    d. $\displaystyle\int_C \mathbf{F}_4 \cdot \mathbf{T}\,ds$ **e.** $\displaystyle\int_C \mathbf{F}_1 \cdot \mathbf{n}\,ds$ **f.** $\displaystyle\int_C \mathbf{F}_2 \cdot \mathbf{n}\,ds$

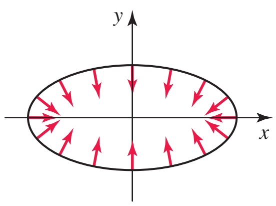
    g. $\displaystyle\int_C \mathbf{F}_3 \cdot \mathbf{n}\,ds$ **h.** $\displaystyle\int_C \mathbf{F}_4 \cdot \mathbf{n}\,ds$



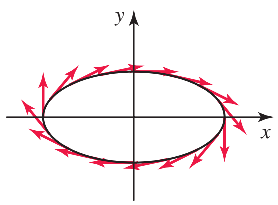
Vectors of $\mathbf{F}_1$ along C



Vectors of $\mathbf{F}_2$ along C



Vectors of $\mathbf{F}_3$ along C



Vectors of $\mathbf{F}_4$ along C

15. How is the circulation of a vector field on a closed smooth oriented curve calculated?

16. Given a two-dimensional vector field $\mathbf{F}$ and a smooth oriented curve C, what is the meaning of the flux of $\mathbf{F}$ across C?

Practice Exercises

17–34. Scalar line integrals *Evaluate the following line integrals along the curve C.*

17. $\int_C xy\,ds$; C is the unit circle $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$, for $0 \le t \le 2\pi$.

18. $\int_C (x^2 - 2y^2)\,ds$; C is the line segment $\mathbf{r}(t) = \left\langle \frac{t}{\sqrt{2}}, \frac{t}{\sqrt{2}} \right\rangle$, for $0 \le t \le 4$.

19. $\int_C (2x + y)\,ds$; C is the line segment $\mathbf{r}(t) = \langle 3t, 4t \rangle$, for $0 \le t \le 2$.

20. $\int_C x\,ds$; C is the curve $\mathbf{r}(t) = \langle t^3, 4t \rangle$, for $0 \le t \le 1$.

21. $\int_C xy^3\,ds$; C is the quarter-circle $\mathbf{r}(t) = \langle 2\cos t, 2\sin t \rangle$, for $0 \le t \le \pi/2$.

22. $\int_C 3x \cos y\,ds$; C is the curve $\mathbf{r}(t) = \langle \sin t, t \rangle$, for $0 \le t \le \pi/2$.

23. $\int_C (y - z)\,ds$; C is the helix $\mathbf{r}(t) = \langle 3\cos t, 3\sin t, 4t \rangle$, for $0 \le t \le 2\pi$.

24. $\int_C (x - y + 2z)\,ds$; C is the circle $\mathbf{r}(t) = \langle 1, 3\cos t, 3\sin t \rangle$, for $0 \le t \le 2\pi$.

25. $\int_C (x^2 + y^2)\,ds$; C is the circle of radius 4 centered at $(0, 0)$.

26. $\int_C (x^2 + y^2)\,ds$; C is the line segment from $(0, 0)$ to $(5, 5)$.

27. $\int_C \frac{x}{x^2 + y^2}\,ds$; C is the line segment from $(1, 1)$ to $(10, 10)$.

28. $\int_C (xy)^{1/3}\,ds$; C is the curve $y = x^2$, for $0 \le x \le 1$.

29. $\int_C xy\,ds$; C is a portion of the ellipse $\frac{x^2}{4} + \frac{y^2}{16} = 1$ in the first quadrant, oriented counterclockwise.

30. $\int_C (2x - 3y)\,ds$; C is the line segment from $(-1, 0)$ to $(0, 1)$ followed by the line segment from $(0, 1)$ to $(1, 0)$.

31. $\int_C (x + y + z)\,ds$; C is the semicircle $\mathbf{r}(t) = \langle 2\cos t, 0, 2\sin t \rangle$, for $0 \le t \le \pi$.

32. $\int_C \frac{xy}{z}\,ds$; C is the line segment from $(1, 4, 1)$ to $(3, 6, 3)$.

33. $\int_C xz\,ds$; C is the line segment from $(0, 0, 0)$ to $(3, 2, 6)$ followed by the line segment from $(3, 2, 6)$ to $(7, 9, 10)$.

34. $\int_C xe^{yz}\,ds$; C is $\mathbf{r}(t) = \langle t, 2t, -2t \rangle$, for $0 \le t \le 2$.

35–36. Mass and density *A thin wire represented by the smooth curve C with a density ρ (mass per unit length) has a mass* $M = \int_C \rho\,ds$. *Find the mass of the following wires with the given density.*

T 35. C: $\{(x, y): y = 2x^2, 0 \le x \le 3\}$; $\rho(x, y) = 1 + xy$

36. C: $\mathbf{r}(\theta) = \langle \cos\theta, \sin\theta \rangle$, for $0 \le \theta \le \pi$; $\rho(\theta) = 2\theta/\pi + 1$

37–38. Average values *Find the average value of the following functions on the given curves.*

37. $f(x, y) = x + 2y$ on the line segment from $(1, 1)$ to $(2, 5)$

38. $f(x, y) = xe^y$ on the unit circle centered at the origin

39–40. Length of curves *Use a scalar line integral to find the length of the following curves.*

39. $\mathbf{r}(t) = \left\langle 20\sin\frac{t}{4}, 20\cos\frac{t}{4}, \frac{t}{2} \right\rangle$, for $0 \le t \le 2$

40. $\mathbf{r}(t) = \langle 30\sin t, 40\sin t, 50\cos t \rangle$, for $0 \le t \le 2\pi$

41–46. Line integrals of vector fields in the plane *Given the following vector fields and oriented curves C, evaluate* $\int_C \mathbf{F} \cdot \mathbf{T}\,ds$.

41. $\mathbf{F} = \langle x, y \rangle$ on the parabola $\mathbf{r}(t) = \langle 4t, t^2 \rangle$, for $0 \le t \le 1$

42. $\mathbf{F} = \langle -y, x \rangle$ on the semicircle $\mathbf{r}(t) = \langle 4\cos t, 4\sin t \rangle$, for $0 \le t \le \pi$

43. $\mathbf{F} = \langle y, x \rangle$ on the line segment from $(1, 1)$ to $(5, 10)$

44. $\mathbf{F} = \langle -y, x \rangle$ on the parabola $y = x^2$ from $(0, 0)$ to $(1, 1)$

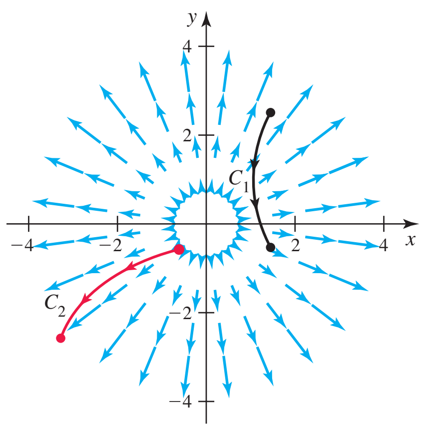
45. $\mathbf{F} = \frac{\langle x, y \rangle}{(x^2 + y^2)^{3/2}}$ on the curve $\mathbf{r}(t) = \langle t^2, 3t^2 \rangle$, for $1 \le t \le 2$

46. $\mathbf{F} = \frac{\langle x, y \rangle}{x^2 + y^2}$ on the line segment $\mathbf{r}(t) = \langle t, 4t \rangle$, for $1 \le t \le 10$

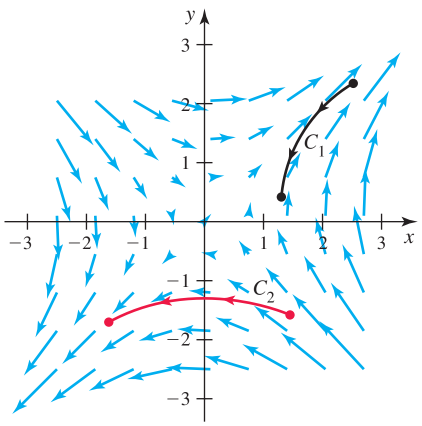
47–48. Line integrals from graphs *Determine whether* $\int_C \mathbf{F} \cdot d\mathbf{r}$ *along the paths* C_1 *and* C_2 *shown in the following vector fields is positive or negative. Explain your reasoning.*

a. $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$ ***b.*** $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$

47.



48.



49–56. Work integrals *Given the force field* **F**, *find the work required to move an object on the given oriented curve.*

49. $\mathbf{F} = \langle y, -x \rangle$ on the line segment from $(1, 2)$ to $(0, 0)$ followed by the line segment from $(0, 0)$ to $(0, 4)$

50. $\mathbf{F} = \langle x, y \rangle$ on the line segment from $(-1, 0)$ to $(0, 8)$ followed by the line segment from $(0, 8)$ to $(2, 8)$

51. $\mathbf{F} = \langle y, x \rangle$ on the parabola $y = 2x^2$ from $(0, 0)$ to $(2, 8)$

52. $\mathbf{F} = \langle y, -x \rangle$ on the line segment $y = 10 - 2x$ from $(1, 8)$ to $(3, 4)$

53. $\mathbf{F} = \langle x, y, z \rangle$ on the tilted ellipse $\mathbf{r}(t) = \langle 4 \cos t, 4 \sin t, 4 \cos t \rangle$, for $0 \le t \le 2\pi$

54. $\mathbf{F} = \langle -y, x, z \rangle$ on the helix $\mathbf{r}(\mathbf{t}) = \left\langle 2 \cos t, 2 \sin t, \dfrac{t}{2\pi} \right\rangle$, for $0 \le t \le 2\pi$

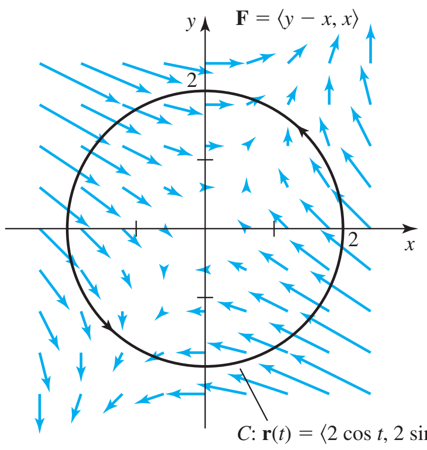
55. $\mathbf{F} = \dfrac{\langle x, y, z \rangle}{(x^2 + y^2 + z^2)^{3/2}}$ on the line segment from $(1, 1, 1)$ to $(10, 10, 10)$

T 56. $\mathbf{F} = \dfrac{\langle x, y, z \rangle}{x^2 + y^2 + z^2}$ on the line segment from $(1, 1, 1)$ to $(8, 4, 2)$

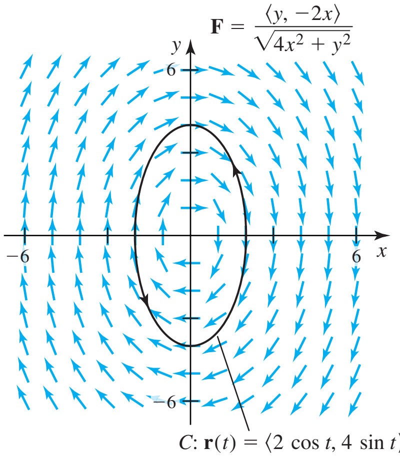
57–58. Circulation *Consider the following vector fields* **F** *and closed oriented curves C in the plane (see figures).*

a. Based on the picture, make a conjecture about whether the circulation of **F** *on C is positive, negative, or zero.*
b. Compute the circulation and interpret the result.

57. $\mathbf{F} = \langle y - x, x \rangle$; C: $\mathbf{r}(t) = \langle 2 \cos t, 2 \sin t \rangle$, for $0 \le t \le 2\pi$



58. $\mathbf{F} = \dfrac{\langle y, -2x \rangle}{\sqrt{4x^2 + y^2}}$; C: $\mathbf{r}(t) = \langle 2 \cos t, 4 \sin t \rangle$, for $0 \le t \le 2\pi$



59–60. Flux *Consider the vector fields and curves in Exercises 57–58.*

a. Based on the picture, make a conjecture about whether the outward flux of **F** *across C is positive, negative, or zero.*
b. Compute the flux for the vector fields and curves.

59. **F** and C given in Exercise 57

60. **F** and C given in Exercise 58

61. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. If a curve has a parametric description $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, where t is the arc length, then $|\mathbf{r}'(t)| = 1$.
b. The vector field $\mathbf{F} = \langle y, x \rangle$ has both zero circulation along and zero flux across the unit circle centered at the origin.
c. If at all points of a path a force acts in a direction orthogonal to the path, then no work is done in moving an object along the path.
d. The flux of a vector field across a curve in $\mathbb{R}^2$ can be computed using a line integral.

62. Flying into a headwind An airplane flies in the xz-plane, where x increases in the eastward direction and $z \ge 0$ represents vertical distance above the ground. A wind blows horizontally out of the west, producing a force $\mathbf{F} = \langle 150, 0 \rangle$. On which path between the points $(100, 50)$ and $(-100, 50)$ is more work done overcoming the wind?

a. The line segment $\mathbf{r}(t) = \langle x(t), z(t) \rangle = \langle -t, 50 \rangle$, for $-100 \le t \le 100$
b. The arc of the circle $\mathbf{r}(t) = \langle 100 \cos t, 50 + 100 \sin t \rangle$, for $0 \le t \le \pi$

63. Flying into a headwind

a. How does the result of Exercise 62 change if the force due to the wind is $\mathbf{F} = \langle 141, 50 \rangle$ (approximately the same magnitude, but a different direction)?
b. How does the result of Exercise 62 change if the force due to the wind is $\mathbf{F} = \langle 141, -50 \rangle$ (approximately the same magnitude, but a different direction)?

64. Changing orientation Let $f(x, y) = x + 2y$ and let C be the unit circle.

a. Find a parameterization of C with a counterclockwise orientation and evaluate $\int_C f\, ds$.

b. Find a parameterization of C with a clockwise orientation and evaluate $\int_C f\, ds$.

c. Compare the results of parts (a) and (b).

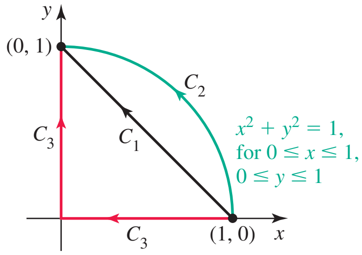
65. Changing orientation Let $f(x, y) = x$ and let C be the segment of the parabola $y = x^2$ joining $O(0, 0)$ and $P(1, 1)$.

a. Find a parameterization of C in the direction from O to P. Evaluate $\int_C f\, ds$.

b. Find a parameterization of C in the direction from P to O. Evaluate $\int_C f\, ds$.

c. Compare the results of parts (a) and (b).

66. Work in a rotation field Consider the rotation field $\mathbf{F} = \langle -y, x \rangle$ and the three paths shown in the figure. Compute the work done on each of the three paths. Does it appear that the line integral $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$ is independent of the path, where C is any path from $(1, 0)$ to $(0, 1)$?



67. Work in a hyperbolic field Consider the hyperbolic force field $\mathbf{F} = \langle y, x \rangle$ (the streamlines are hyperbolas) and the three paths shown in the figure for Exercise 66. Compute the work done in the presence of $\mathbf{F}$ on each of the three paths. Does it appear that the line integral $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$ is independent of the path, where C is any path from $(1, 0)$ to $(0, 1)$?

68–72. Assorted line integrals *Evaluate each line integral using the given curve C.*

68. $\displaystyle\int_C x^2\, dx + dy + y\, dz$; C is the curve $\mathbf{r}(t) = \langle t, 2t, t^2 \rangle$, for $0 \le t \le 3$.

T 69. $\displaystyle\int_C x^3 y\, dx + xz\, dy + (x + y)^2\, dz$; C is the helix $\mathbf{r}(t) = \langle 2t, \sin t, \cos t \rangle$, for $0 \le t \le 4\pi$.

T 70. $\displaystyle\int_C \frac{x^2}{y^4}\, ds$; C is the segment of the parabola $x = 3y^2$ from $(3, 1)$ to $(27, 3)$.

71. $\displaystyle\int_C \frac{y}{\sqrt{x^2 + y^2}}\, dx - \frac{x}{\sqrt{x^2 + y^2}}\, dy$; C is a quarter-circle from $(0, 4)$ to $(4, 0)$.

72. $\displaystyle\int_C (x + y)\, dx + (x - y)\, dy + x\, dz$; C is the line segment from $(1, 2, 4)$ to $(3, 8, 13)$.

73. Flux across curves in a vector field Consider the vector field $\mathbf{F} = \langle y, x \rangle$ shown in the figure.

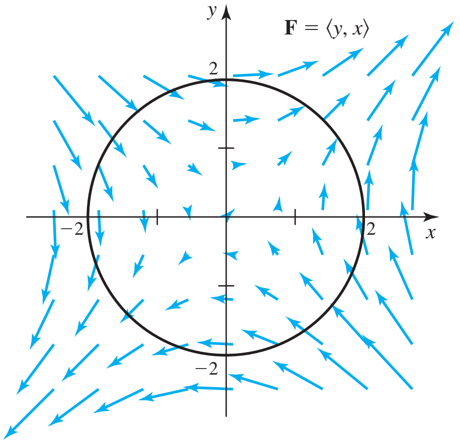
a. Compute the outward flux across the quarter-circle C: $\mathbf{r}(t) = \langle 2\cos t, 2\sin t \rangle$, for $0 \le t \le \pi/2$.

b. Compute the outward flux across the quarter-circle C: $\mathbf{r}(t) = \langle 2\cos t, 2\sin t \rangle$, for $\pi/2 \le t \le \pi$.

c. Explain why the flux across the quarter-circle in the third quadrant equals the flux computed in part (a).

d. Explain why the flux across the quarter-circle in the fourth quadrant equals the flux computed in part (b).

e. What is the outward flux across the full circle?



Explorations and Challenges

74–75. Zero circulation fields

74. For what values of b and c does the vector field $\mathbf{F} = \langle by, cx \rangle$ have zero circulation on the unit circle centered at the origin and oriented counterclockwise?

75. Consider the vector field $\mathbf{F} = \langle ax + by, cx + dy \rangle$. Show that $\mathbf{F}$ has zero circulation on any oriented circle centered at the origin, for any a, b, c, and d, provided $b = c$.

76–77. Zero flux fields

76. For what values of a and d does the vector field $\mathbf{F} = \langle ax, dy \rangle$ have zero flux across the unit circle centered at the origin and oriented counterclockwise?

77. Consider the vector field $\mathbf{F} = \langle ax + by, cx + dy \rangle$. Show that $\mathbf{F}$ has zero flux across any oriented circle centered at the origin, for any a, b, c, and d, provided $a = -d$.

78. Heat flux in a plate A square plate $R = \{(x, y): 0 \le x \le 1, 0 \le y \le 1\}$ has a temperature distribution $T(x, y) = 100 - 50x - 25y$.

a. Sketch two level curves of the temperature in the plate.

b. Find the gradient of the temperature $\nabla T(x, y)$.

c. Assume the flow of heat is given by the vector field $\mathbf{F} = -\nabla T(x, y)$. Compute $\mathbf{F}$.

d. Find the outward heat flux across the boundary $\{(x, y): x = 1, 0 \le y \le 1\}$.

e. Find the outward heat flux across the boundary $\{(x, y): 0 \le x \le 1, y = 1\}$.

79. Inverse force fields Consider the radial field $\mathbf{F} = \dfrac{\mathbf{r}}{|\mathbf{r}|^p} = \dfrac{\langle x, y, z \rangle}{|\mathbf{r}|^p}$, where $p > 1$ (the inverse square law corresponds to $p = 3$). Let C be the line segment from $(1, 1, 1)$ to (a, a, a), where $a > 1$, given by $\mathbf{r}(t) = \langle t, t, t \rangle$, for $1 \le t \le a$.

a. Find the work done in moving an object along C with $p = 2$.

b. If $a \to \infty$ in part (a), is the work finite?

c. Find the work done in moving an object along C with $p = 4$.

d. If $a \to \infty$ in part (c), is the work finite?

e. Find the work done in moving an object along C for any $p > 1$.

f. If $a \to \infty$ in part (e), for what values of p is the work finite?

80. Line integrals with respect to *dx* and *dy* Given a vector field $\mathbf{F} = \langle f, 0 \rangle$ and curve C with parameterization $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, for $a \le t \le b$, we see that the line integral $\int_C f\,dx + g\,dy$ simplifies to $\int_C f\,dx$.

a. Show that $\int_C f\,dx = \int_a^b f(t)x'(t)\,dt$.

b. Use the vector field $\mathbf{F} = \langle 0, g \rangle$ to show that $\int_C g\,dy = \int_a^b g(t)y'(t)\,dt$.

c. Evaluate $\int_C xy\,dx$, where C is the line segment from $(0, 0)$ to $(5, 12)$.

d. Evaluate $\int_C xy\,dy$, where C is a segment of the parabola $x = y^2$ from $(1, -1)$ to $(1, 1)$.

81–82. Looking ahead: Area from line integrals *The area of a region R in the plane, whose boundary is the curve C, may be computed using line integrals with the formula*

$$\text{area of } R = \int_C x\,dy = -\int_C y\,dx.$$

81. Let R be the rectangle with vertices $(0, 0)$, $(a, 0)$, $(0, b)$, and (a, b), and let C be the boundary of R oriented counterclockwise. Use the formula $A = \int_C x\,dy$ to verify that the area of the rectangle is ab.

82. Let $R = \{(r, \theta): 0 \le r \le a, 0 \le \theta \le 2\pi\}$ be the disk of radius a centered at the origin, and let C be the boundary of R oriented counterclockwise. Use the formula $A = -\int_C y\,dx$ to verify that the area of the disk is πa^2.

QUICK CHECK ANSWERS

1. The Fundamental Theorem of Calculus says that $\dfrac{d}{dt}\displaystyle\int_a^t f(u)\,du = f(t)$, which applies to differentiating the arc length integral. **2.** Note that $x = t$, $y = 0$, and $|\mathbf{r}'(t)| = \sqrt{1^2 + 0^2} = 1$. Therefore, $\displaystyle\int_C f(x, y)\,ds = \int_a^b f(t, 0)\,dt$. **3.** 1300 ft/min **4.** $\pi/2$ **5.** $\mathbf{T}$ and $\mathbf{k}$ are unit vectors, so $\mathbf{n}$ is a unit vector. By the right-hand rule for cross products, $\mathbf{n}$ points outward from the curve. ◄

17.3 Conservative Vector Fields

This is an action-packed section in which several fundamental ideas come together. At the heart of the matter are two questions:

- When can a vector field be expressed as the gradient of a potential function? A vector field with this property will be defined as a *conservative* vector field.
- What special properties do conservative vector fields have?

After some preliminary definitions, we present a test to determine whether a vector field in $\mathbb{R}^2$ or $\mathbb{R}^3$ is conservative. This test is followed by a procedure to find a potential function for a conservative field. We then develop several equivalent properties shared by all conservative vector fields.

Types of Curves and Regions

Many of the results in the remainder of this text rely on special properties of regions and curves. It's best to collect these definitions in one place for easy reference.

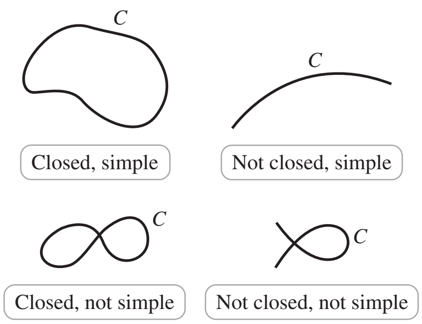


Figure 17.28

➤ Recall that all points of an open set are interior points. An open set does not contain its boundary points.

➤ Roughly speaking, connected means that R is all in one piece and simply connected in $\mathbb{R}^2$ means that R has no holes. $\mathbb{R}^2$ and $\mathbb{R}^3$ are themselves connected and simply connected.

DEFINITION Simple and Closed Curves

Suppose a curve C (in $\mathbb{R}^2$ or $\mathbb{R}^3$) is described parametrically by $\mathbf{r}(t)$, where $a \le t \le b$. Then C is a **simple curve** if $\mathbf{r}(t_1) \ne \mathbf{r}(t_2)$ for all t_1 and t_2, with $a < t_1 < t_2 < b$; that is, C never intersects itself between its endpoints. The curve C is **closed** if $\mathbf{r}(a) = \mathbf{r}(b)$; that is, the initial and terminal points of C are the same (Figure 17.28).

In all that follows, we generally assume that R in $\mathbb{R}^2$ (or D in $\mathbb{R}^3$) is an open region. Open regions are further classified according to whether they are *connected* and whether they are *simply connected*.

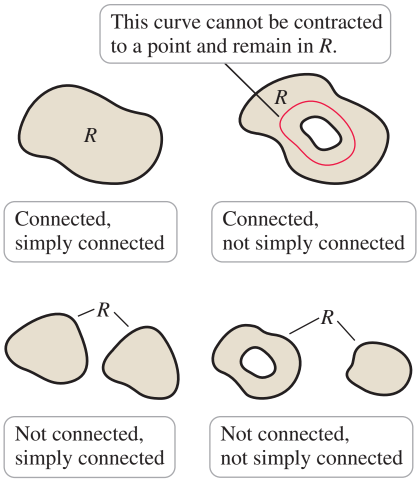


Figure 17.29

QUICK CHECK 1 Is a figure-8 curve simple? Closed? Is a torus connected? Simply connected? ◄

➤ The term *conservative* refers to conservation of energy. See Exercise 66 for an example of conservation of energy in a conservative force field.

➤ Depending on the context and the interpretation of the vector field, the potential function φ may be defined such that $\mathbf{F} = -\nabla\varphi$ (with a negative sign).

DEFINITION Connected and Simply Connected Regions

An open region R in $\mathbb{R}^2$ (or D in $\mathbb{R}^3$) is **connected** if it is possible to connect any two points of R by a continuous curve lying in R. An open region R is **simply connected** if every closed simple curve in R can be deformed and contracted to a point in R (Figure 17.29).

Test for Conservative Vector Fields

We begin with the central definition of this section.

DEFINITION Conservative Vector Field

A vector field $\mathbf{F}$ is said to be **conservative** on a region (in $\mathbb{R}^2$ or $\mathbb{R}^3$) if there exists a scalar function φ such that $\mathbf{F} = \nabla\varphi$ on that region.

Suppose the components of $\mathbf{F} = \langle f, g, h \rangle$ have continuous first partial derivatives on a region D in $\mathbb{R}^3$. Also assume $\mathbf{F}$ is conservative, which means by definition that there is a potential function φ such that $\mathbf{F} = \nabla\varphi$. Matching the components of $\mathbf{F}$ and $\nabla\varphi$, we see that $f = \varphi_x$, $g = \varphi_y$, and $h = \varphi_z$. Recall from Theorem 15.4 that if a function has continuous second partial derivatives, the order of differentiation in the second partial derivatives does not matter. Under these conditions on φ, we conclude the following:

- $\varphi_{xy} = \varphi_{yx}$, which implies that $f_y = g_x$,
- $\varphi_{xz} = \varphi_{zx}$, which implies that $f_z = h_x$, and
- $\varphi_{yz} = \varphi_{zy}$, which implies that $g_z = h_y$.

These observations constitute half of the proof of the following theorem. The remainder of the proof is given in Section 17.4.

THEOREM 17.3 Test for Conservative Vector Fields

Let $\mathbf{F} = \langle f, g, h \rangle$ be a vector field defined on a connected and simply connected region D of $\mathbb{R}^3$, where f, g, and h have continuous first partial derivatives on D. Then $\mathbf{F}$ is a conservative vector field on D (there is a potential function φ such that $\mathbf{F} = \nabla\varphi$) if and only if

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}, \qquad \frac{\partial f}{\partial z} = \frac{\partial h}{\partial x}, \qquad \text{and} \qquad \frac{\partial g}{\partial z} = \frac{\partial h}{\partial y}.$$

For vector fields in $\mathbb{R}^2$, we have the single condition $\dfrac{\partial f}{\partial y} = \dfrac{\partial g}{\partial x}$.

EXAMPLE 1 Testing for conservative fields Determine whether the following vector fields are conservative on $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively.

a. $\mathbf{F} = \langle e^x \cos y, -e^x \sin y \rangle$ **b.** $\mathbf{F} = \langle 2xy - z^2, x^2 + 2z, 2y - 2xz \rangle$

SOLUTION

a. Letting $f(x, y) = e^x \cos y$ and $g(x, y) = -e^x \sin y$, we see that

$$\frac{\partial f}{\partial y} = -e^x \sin y = \frac{\partial g}{\partial x}.$$

The conditions of Theorem 17.3 are met and $\mathbf{F}$ is conservative.

b. Letting $f(x, y, z) = 2xy - z^2$, $g(x, y, z) = x^2 + 2z$, and $h(x, y, z) = 2y - 2xz$, we have

$$\frac{\partial f}{\partial y} = 2x = \frac{\partial g}{\partial x}, \qquad \frac{\partial f}{\partial z} = -2z = \frac{\partial h}{\partial x}, \qquad \frac{\partial g}{\partial z} = 2 = \frac{\partial h}{\partial y}.$$

By Theorem 17.3, $\mathbf{F}$ is conservative. *Related Exercises 13–14* ◄

Finding Potential Functions

Like antiderivatives, potential functions are determined up to an arbitrary additive constant. Unless an additive constant in a potential function has some physical meaning, it is usually omitted. Given a conservative vector field, there are several methods for finding a potential function. One method is shown in the following example. Another approach is illustrated in Exercise 71.

QUICK CHECK 2 Explain why a potential function for a conservative vector field is determined up to an additive constant. ◄

EXAMPLE 2 **Finding potential functions** Find a potential function for the conservative vector fields in Example 1.

a. $\mathbf{F} = \langle e^x \cos y, -e^x \sin y \rangle$

b. $\mathbf{F} = \langle 2xy - z^2, x^2 + 2z, 2y - 2xz \rangle$

SOLUTION

a. A potential function φ for $\mathbf{F} = \langle f, g \rangle$ has the property that $\mathbf{F} = \nabla\varphi$ and satisfies the conditions

$$\varphi_x = f(x, y) = e^x \cos y \quad \text{and} \quad \varphi_y = g(x, y) = -e^x \sin y.$$

The first equation is integrated with respect to x (holding y fixed) to obtain

$$\int \varphi_x \, dx = \int e^x \cos y \, dx,$$

which implies that

$$\varphi(x, y) = e^x \cos y + c(y).$$

In this case, the "constant of integration" $c(y)$ is an arbitrary function of y. You can check the preceding calculation by noting that

$$\frac{\partial\varphi}{\partial x} = \frac{\partial}{\partial x}\left(e^x \cos y + c(y)\right) = e^x \cos y = f(x, y).$$

➤ This procedure may begin with either of the two conditions, $\varphi_x = f$ or $\varphi_y = g$.

To find the arbitrary function $c(y)$, we differentiate $\varphi(x, y) = e^x \cos y + c(y)$ with respect to y and equate the result to g (recall that $\varphi_y = g$):

$$\varphi_y = -e^x \sin y + c'(y) \quad \text{and} \quad g = -e^x \sin y.$$

We conclude that $c'(y) = 0$, which implies that $c(y)$ is any real number, which we typically take to be zero. So a potential function is $\varphi(x, y) = e^x \cos y$, a result that may be checked by differentiation.

b. The method of part (a) is more elaborate with three variables. A potential function φ must now satisfy these conditions:

➤ This procedure may begin with any of the three conditions.

$$\varphi_x = f = 2xy - z^2, \qquad \varphi_y = g = x^2 + 2z, \quad \text{and} \quad \varphi_z = h = 2y - 2xz.$$

Integrating the first condition with respect to x (holding y and z fixed), we have

$$\varphi = \int (2xy - z^2)\, dx = x^2 y - xz^2 + c(y, z).$$

Because the integration is with respect to x, the arbitrary "constant" is a function of y and z. To find $c(y, z)$, we differentiate φ with respect to y, which results in

$$\varphi_y = x^2 + c_y(y, z).$$

Equating φ_y and $g = x^2 + 2z$, we see that $c_y(y, z) = 2z$. To obtain $c(y, z)$, we integrate $c_y(y, z) = 2z$ with respect to y (holding z fixed), which results in $c(y, z) = 2yz + d(z)$. The "constant" of integration is now a function of z, which we call $d(z)$. At this point, a potential function looks like

$$\varphi(x, y, z) = x^2 y - xz^2 + 2yz + d(z).$$

To determine $d(z)$, we differentiate φ with respect to z:

$$\varphi_z = -2xz + 2y + d'(z).$$

QUICK CHECK 3 Verify by differentiation that the potential functions found in Example 2 produce the corresponding vector fields. ◄

Equating φ_z and $h = 2y - 2xz$, we see that $d'(z) = 0$, or $d(z)$ is a real number, which we generally take to be zero. Putting it all together, a potential function is

$$\varphi = x^2y - xz^2 + 2yz.$$

Related Exercises 19, 24 ◄

> **PROCEDURE** **Finding Potential Functions in $\mathbb{R}^3$**
>
> Suppose $\mathbf{F} = \langle f, g, h \rangle$ is a conservative vector field. To find φ such that $\mathbf{F} = \nabla\varphi$, use the following steps:
>
> 1. Integrate $\varphi_x = f$ with respect to x to obtain φ, which includes an arbitrary function $c(y, z)$.
> 2. Compute φ_y and equate it to g to obtain an expression for $c_y(y, z)$.
> 3. Integrate $c_y(y, z)$ with respect to y to obtain $c(y, z)$, including an arbitrary function $d(z)$.
> 4. Compute φ_z and equate it to h to get $d(z)$.
>
> A similar procedure beginning with $\varphi_y = g$ or $\varphi_z = h$ may be easier in some cases.

Fundamental Theorem for Line Integrals and Path Independence

➤ Compare the two versions of the Fundamental Theorem.

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

$$\int_C \nabla\varphi \cdot d\mathbf{r} = \varphi(B) - \varphi(A)$$

Knowing how to find potential functions, we now investigate their properties. The first property is one of several beautiful parallels to the Fundamental Theorem of Calculus.

> **THEOREM 17.4** **Fundamental Theorem for Line Integrals**
>
> Let R be a region in $\mathbb{R}^2$ or $\mathbb{R}^3$ and let φ be a differentiable potential function defined on R. If $\mathbf{F} = \nabla\varphi$ (which means that $\mathbf{F}$ is conservative), then
>
> $$\int_C \mathbf{F} \cdot \mathbf{T}\,ds = \int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(B) - \varphi(A),$$
>
> for all points A and B in R and all piecewise-smooth oriented curves C in R from A to B.

Proof: Let the curve C in $\mathbb{R}^3$ be given by $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, for $a \le t \le b$, where $\mathbf{r}(a)$ and $\mathbf{r}(b)$ are the position vectors for the points A and B, respectively. By the Chain Rule, the rate of change of φ with respect to t along C is

$$\begin{aligned}
\frac{d\varphi}{dt} &= \frac{\partial\varphi}{\partial x}\frac{dx}{dt} + \frac{\partial\varphi}{\partial y}\frac{dy}{dt} + \frac{\partial\varphi}{\partial z}\frac{dz}{dt} && \text{Chain Rule} \\
&= \left\langle \frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle && \text{Identify the dot product.} \\
&= \nabla\varphi \cdot \mathbf{r}'(t) && \mathbf{r} = \langle x, y, z \rangle \\
&= \mathbf{F} \cdot \mathbf{r}'(t). && \mathbf{F} = \nabla\varphi
\end{aligned}$$

Evaluating the line integral and using the Fundamental Theorem of Calculus, it follows that

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_a^b \mathbf{F} \cdot \mathbf{r}'(t)\,dt \\
&= \int_a^b \frac{d\varphi}{dt}\,dt && \mathbf{F} \cdot \mathbf{r}'(t) = \frac{d\varphi}{dt} \\
&= \varphi(B) - \varphi(A). && \text{Fundamental Theorem of Calculus; } t = b \text{ corresponds to } B \text{ and } t = a \text{ corresponds to } A.
\end{aligned}$$

◄

Here is the meaning of Theorem 17.4: If $\mathbf{F}$ is a conservative vector field, then the value of a line integral of $\mathbf{F}$ depends only on the endpoints of the path. For this reason, we say the line integral is *independent of path*, which means that to evaluate line integrals of conservative vector fields, we do not need a parameterization of the path.

If we think of φ as an antiderivative of the vector field $\mathbf{F}$, then the parallel to the Fundamental Theorem of Calculus is clear. The line integral of $\mathbf{F}$ is the difference of the values of φ evaluated at the endpoints. Theorem 17.4 motivates the following definition.

DEFINITION Independence of Path

Let $\mathbf{F}$ be a continuous vector field with domain R. If $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ for all piecewise-smooth curves C_1 and C_2 in R with the same initial and terminal points, then the line integral is **independent of path**.

An important question concerns the converse of Theorem 17.4. With additional conditions on the domain R, the converse turns out to be true.

THEOREM 17.5

Let $\mathbf{F}$ be a continuous vector field on an open connected region R in $\mathbb{R}^2$. If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path, then $\mathbf{F}$ is conservative; that is, there exists a potential function φ such that $\mathbf{F} = \nabla \varphi$ on R.

➤ We state and prove Theorem 17.5 in two variables. It is easily extended to three or more variables.

Proof: Let $P(a, b)$ and $Q(x, y)$ be interior points of R and define $\varphi(x, y) = \int_C \mathbf{F} \cdot d\mathbf{r}$, where C is a piecewise-smooth path from P to Q, and $\mathbf{F} = \langle f, g \rangle$. Because the integral defining φ is independent of path, any piecewise-smooth path in R from P to Q can be used. The goal is to compute the directional derivative $D_{\mathbf{u}}\varphi(x, y)$, where $\mathbf{u} = \langle u_1, u_2 \rangle$ is an arbitrary unit vector, and then show that $\mathbf{F} = \nabla \varphi$. We let $S(x + tu_1, y + tu_2)$ be a point in R near Q and then apply the definition of the directional derivative at Q:

$$\begin{aligned} D_{\mathbf{u}}\varphi(x, y) &= \lim_{t \to 0} \frac{\varphi(x + tu_1, y + tu_2) - \varphi(x, y)}{t} \\ &= \lim_{t \to 0} \frac{1}{t}\left(\int_P^S \mathbf{F} \cdot d\mathbf{r} - \int_P^Q \mathbf{F} \cdot d\mathbf{r} \right) \\ &= \lim_{t \to 0} \frac{1}{t} \int_Q^S \mathbf{F} \cdot d\mathbf{r}. \end{aligned}$$

Using path independence, we choose the path from Q to S to be a line parameterized by $\mathbf{r}(s) = \langle x + su_1, y + su_2 \rangle$, for $0 \le s \le t$. Noting that $\mathbf{r}'(s) = \mathbf{u}$, the directional derivative is

$$\begin{aligned} D_{\mathbf{u}}\varphi(x, y) &= \lim_{t \to 0} \frac{1}{t} \int_Q^S \mathbf{F} \cdot d\mathbf{r} \\ &= \lim_{t \to 0} \frac{1}{t} \int_0^t \mathbf{F}(x + su_1, y + su_2) \cdot \mathbf{r}'(s)\, ds && \text{Change line integral to ordinary integral.} \\ &= \lim_{t \to 0} \frac{\int_0^t \mathbf{F}(x + su_1, y + su_2) \cdot \mathbf{r}'(s)\, ds - \int_0^0 \mathbf{F}(x + su_1, y + su_2) \cdot \mathbf{r}'(s)\, ds}{t} && \text{Second integral equals 0.} \\ &= \frac{d}{dt} \int_0^t \mathbf{F}(x + su_1, y + su_2) \cdot \mathbf{u}\, ds \bigg|_{t=0} && \text{Identify difference quotient; } \mathbf{r}'(s) = \mathbf{u} \\ &= \mathbf{F}(x, y) \cdot \mathbf{u}. && \text{Fundamental Theorem of Calculus} \end{aligned}$$

Choosing $\mathbf{u} = \mathbf{i} = \langle 1, 0 \rangle$, we see that $D_{\mathbf{i}}\varphi(x, y) = \varphi_x(x, y) = \mathbf{F}(x, y) \cdot \mathbf{i} = f(x, y)$. Similarly, choosing $\mathbf{u} = \mathbf{j} = \langle 0, 1 \rangle$, we have $D_{\mathbf{j}}\varphi(x, y) = \varphi_y(x, y) = \mathbf{F}(x, y) \cdot \mathbf{j} = g(x, y)$. Therefore, $\mathbf{F} = \langle f, g \rangle = \langle \varphi_x, \varphi_y \rangle = \nabla\varphi$, and $\mathbf{F}$ is conservative. ➤

EXAMPLE 3 **Verifying path independence** Consider the potential function $\varphi(x, y) = (x^2 - y^2)/2$ and its gradient field $\mathbf{F} = \langle x, -y \rangle$.

- Let C_1 be the quarter-circle $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$, for $0 \le t \le \pi/2$, from $A(1, 0)$ to $B(0, 1)$.
- Let C_2 be the line $\mathbf{r}(t) = \langle 1 - t, t \rangle$, for $0 \le t \le 1$, also from A to B.

Evaluate the line integrals of $\mathbf{F}$ on C_1 and C_2, and show that both are equal to $\varphi(B) - \varphi(A)$.

SOLUTION On C_1, we have $\mathbf{r}'(t) = \langle -\sin t, \cos t \rangle$ and $\mathbf{F} = \langle x, -y \rangle = \langle \cos t, -\sin t \rangle$. The line integral on C_1 is

$$\begin{aligned}
\int_{C_1} \mathbf{F} \cdot d\mathbf{r} &= \int_a^b \mathbf{F} \cdot \mathbf{r}'(t)\, dt \\
&= \int_0^{\pi/2} \underbrace{\langle \cos t, -\sin t \rangle}_{\mathbf{F}} \cdot \underbrace{\langle -\sin t, \cos t \rangle\, dt}_{\mathbf{r}'(t)\, dt} && \text{Substitute for } \mathbf{F} \text{ and } \mathbf{r}'. \\
&= \int_0^{\pi/2} (-\sin 2t)\, dt && 2 \sin t \cos t = \sin 2t \\
&= \left(\frac{1}{2} \cos 2t\right)\Big|_0^{\pi/2} = -1. && \text{Evaluate the integral.}
\end{aligned}$$

On C_2, we have $\mathbf{r}'(t) = \langle -1, 1 \rangle$ and $\mathbf{F} = \langle x, -y \rangle = \langle 1 - t, -t \rangle$; therefore,

$$\begin{aligned}
\int_{C_2} \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \underbrace{\langle 1 - t, -t \rangle}_{\mathbf{F}} \cdot \underbrace{\langle -1, 1 \rangle\, dt}_{d\mathbf{r}} && \text{Substitute for } \mathbf{F} \text{ and } d\mathbf{r}. \\
&= \int_0^1 (-1)\, dt = -1. && \text{Simplify.}
\end{aligned}$$

The two line integrals have the same value, which is

$$\varphi(B) - \varphi(A) = \varphi(0, 1) - \varphi(1, 0) = -\frac{1}{2} - \frac{1}{2} = -1.$$

Related Exercises 31–32 ➤

EXAMPLE 4 **Line integral of a conservative vector field** Evaluate

$$\int_C ((2xy - z^2)\,\mathbf{i} + (x^2 + 2z)\,\mathbf{j} + (2y - 2xz)\,\mathbf{k}) \cdot d\mathbf{r},$$

where C is a simple curve from $A(-3, -2, -1)$ to $B(1, 2, 3)$.

SOLUTION This vector field is conservative and has a potential function $\varphi = x^2y - xz^2 + 2yz$ (Example 2). By the Fundamental Theorem for line integrals,

$$\begin{aligned}
&\int_C ((2xy - z^2)\,\mathbf{i} + (x^2 + 2z)\,\mathbf{j} + (2y - 2xz)\,\mathbf{k}) \cdot d\mathbf{r} \\
&\quad = \int_C \nabla \underbrace{(x^2y - xz^2 + 2yz)}_{\varphi} \cdot d\mathbf{r} \\
&\quad = \varphi(1, 2, 3) - \varphi(-3, -2, -1) = 16.
\end{aligned}$$

Related Exercise 34 ➤

QUICK CHECK 4 Explain why the vector field $\nabla(xy + xz - yz)$ is conservative. ➤

Line Integrals on Closed Curves

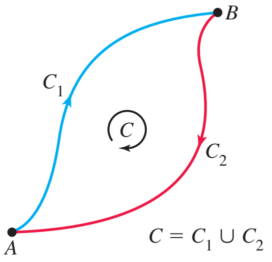
It is a short step to another characterization of conservative vector fields. Suppose C is a simple *closed* piecewise-smooth oriented curve in $\mathbb{R}^2$ or $\mathbb{R}^3$. To distinguish line integrals on closed curves, we adopt the notation $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where the small circle on the integral sign indicates that C is a closed curve. Let A be any point on C and think of A as both the initial point and the final point of C. Assuming $\mathbf{F}$ is a conservative vector field on an open connected region R containing C, it follows by Theorem 17.4 that

➤ Notice the analogy to $\int_a^a f(x)\,dx = 0$, which is true of all integrable functions.

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \varphi(A) - \varphi(A) = 0.$$

Because A is an arbitrary point on C, we see that the line integral of a conservative vector field on a closed curve is zero.

An argument can be made in the opposite direction as well: Suppose $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ on all simple closed piecewise-smooth oriented curves in a region R, and let A and B be distinct points in R. Let C_1 denote any curve from A to B, let C_2 be any curve from B to A (distinct from and not intersecting C_1), and let C be the closed curve consisting of C_1 followed by C_2 (**Figure 17.30**). Then



$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$

Figure 17.30

$$0 = \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}.$$

Therefore, $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = -\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{-C_2} \mathbf{F} \cdot d\mathbf{r}$, where $-C_2$ is the curve C_2 traversed in the opposite direction (from A to B). We see that the line integral has the same value on two arbitrary paths between A and B. It follows that the value of the line integral is independent of path, and by Theorem 17.5, $\mathbf{F}$ is conservative. This argument is a proof of the following theorem.

> **THEOREM 17.6 Line Integrals on Closed Curves**
> Let R be an open connected region in $\mathbb{R}^2$ or $\mathbb{R}^3$. Then $\mathbf{F}$ is a conservative vector field on R if and only if $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ on all simple closed piecewise-smooth oriented curves C in R.

EXAMPLE 5 A closed curve line integral in $\mathbb{R}^3$ Evaluate $\int_C \nabla(-xy + xz + yz) \cdot d\mathbf{r}$ on the curve C: $\mathbf{r}(t) = \langle \sin t, \cos t, \sin t \rangle$, for $0 \le t \le 2\pi$, without using Theorem 17.4 or Theorem 17.6.

SOLUTION The components of the vector field are

$$\mathbf{F} = \nabla(-xy + xz + yz) = \langle -y + z, -x + z, x + y \rangle.$$

Note that $\mathbf{r}'(t) = \langle \cos t, -\sin t, \cos t \rangle$ and $d\mathbf{r} = \mathbf{r}'(t)\,dt$. Substituting values of x, y, and z, the value of the line integral is

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C \langle -y + z, -x + z, x + y \rangle \cdot d\mathbf{r} \quad \text{Substitute for } \mathbf{F}.$$

$$= \int_0^{2\pi} \sin 2t\,dt \quad \text{Substitute for } x, y, z, d\mathbf{r}.$$

$$= -\frac{1}{2}\cos 2t\Big|_0^{2\pi} = 0. \quad \text{Evaluate integral.}$$

The line integral of this conservative vector field on the closed curve C is zero. In fact, by Theorem 17.6, the line integral vanishes on any simple closed piecewise-smooth oriented curve.

Related Exercise 50 ◄

Summary of the Properties of Conservative Vector Fields

We have established three equivalent properties of conservative vector fields **F** defined on an open connected region R in $\mathbb{R}^2$ (or D in $\mathbb{R}^3$).

- There exists a potential function φ such that $\mathbf{F} = \nabla\varphi$ (definition).
- $\int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(B) - \varphi(A)$ for all points A and B in R and all piecewise-smooth oriented curves C in R from A to B (path independence).
- $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ on all simple piecewise-smooth closed oriented curves C in R.

The connections between these properties were established by Theorems 17.4, 17.5, and 17.6 in the following way:

$$\text{Path independence} \underbrace{\Leftrightarrow}^{\text{Theorems 17.4 and 17.5}} \mathbf{F} \text{ is conservative } (\nabla\varphi = \mathbf{F}) \underbrace{\Leftrightarrow}^{\text{Theorem 17.6}} \oint_C \mathbf{F} \cdot d\mathbf{r} = \mathbf{0}.$$

SECTION 17.3 EXERCISES

Getting Started

1. Explain with pictures what is meant by a simple curve and a closed curve.
2. Explain with pictures what is meant by a connected region and a simply connected region.
3. How do you determine whether a vector field in $\mathbb{R}^2$ is conservative (has a potential function φ such that $\mathbf{F} = \nabla\varphi$)?
4. How do you determine whether a vector field in $\mathbb{R}^3$ is conservative?
5. Briefly describe how to find a potential function φ for a conservative vector field $\mathbf{F} = \langle f, g \rangle$.
6. If **F** is a conservative vector field on a region R, how do you evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is a path between two points A and B in R?
7. If **F** is a conservative vector field on a region R, what is the value of $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where C is a simple closed piecewise-smooth oriented curve in R?
8. Give three equivalent properties of conservative vector fields.

Practice Exercises

9–16. Testing for conservative vector fields *Determine whether the following vector fields are conservative (in $\mathbb{R}^2$ or $\mathbb{R}^3$).*

9. $\mathbf{F} = \langle 1, 1 \rangle$
10. $\mathbf{F} = \langle x, y \rangle$
11. $\mathbf{F} = \langle -y, x \rangle$
12. $\mathbf{F} = \langle -y, x + y \rangle$
13. $\mathbf{F} = \langle e^{-x} \cos y, e^{-x} \sin y \rangle$
14. $\mathbf{F} = \langle 2x^3 + xy^2, 2y^3 - x^2 y \rangle$
15. $\mathbf{F} = \langle yz \cos xz, \sin xz, xy \cos xz \rangle$
16. $\mathbf{F} = \langle ye^{x-z}, e^{x-z}, ye^{x-z} \rangle$

17–30. Finding potential functions *Determine whether the following vector fields are conservative on the specified region. If so, determine a potential function. Let R^* and D^* be open regions of $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively, that do not include the origin.*

17. $\mathbf{F} = \langle x, y \rangle$ on $\mathbb{R}^2$
18. $\mathbf{F} = \langle -y, -x \rangle$ on $\mathbb{R}^2$
19. $\mathbf{F} = \left\langle x^3 - xy, \frac{x^2}{2} + y \right\rangle$ on $\mathbb{R}^2$
20. $\mathbf{F} = \frac{\langle x, y \rangle}{x^2 + y^2}$ on R^*
21. $\mathbf{F} = \frac{\langle x, y \rangle}{\sqrt{x^2 + y^2}}$ on R^*
22. $\mathbf{F} = \langle y, x, x - y \rangle$ on $\mathbb{R}^3$
23. $\mathbf{F} = \langle z, 1, x \rangle$ on $\mathbb{R}^3$
24. $\mathbf{F} = \langle yz, xz, xy \rangle$ on $\mathbb{R}^3$
25. $\mathbf{F} = \langle e^z, e^z, e^z(x - y) \rangle$ on $\mathbb{R}^3$
26. $\mathbf{F} = \langle 1, -z, y \rangle$ on $\mathbb{R}^3$
27. $\mathbf{F} = \langle y + z, x + z, x + y \rangle$ on $\mathbb{R}^3$
28. $\mathbf{F} = \frac{\langle x, y, z \rangle}{x^2 + y^2 + z^2}$ on D^*
29. $\mathbf{F} = \frac{\langle x, y, z \rangle}{\sqrt{x^2 + y^2 + z^2}}$ on D^*
30. $\mathbf{F} = \langle x^3, 2y, -z^3 \rangle$ on $\mathbb{R}^3$

31–34. Evaluating line integrals *Use the given potential function φ of the gradient field* **F** *and the curve C to evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ in two ways.*

a. *Use a parametric description of C and evaluate the integral directly.*
b. *Use the Fundamental Theorem for line integrals.*

31. $\varphi(x, y) = xy$; C: $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$, for $0 \le t \le \pi$
32. $\varphi(x, y) = x + 3y$; C: $\mathbf{r}(t) = \langle 2 - t, t \rangle$, for $0 \le t \le 2$
33. $\varphi(x, y, z) = (x^2 + y^2 + z^2)/2$; C: $\mathbf{r}(t) = \langle \cos t, \sin t, t/\pi \rangle$, for $0 \le t \le 2\pi$
34. $\varphi(x, y, z) = xy + xz + yz$; C: $\mathbf{r}(t) = \langle t, 2t, 3t \rangle$, for $0 \le t \le 4$

35–38. Applying the Fundamental Theorem of Line Integrals *Suppose the vector field* **F** *is continuous on $\mathbb{R}^2$, $\mathbf{F} = \langle f, g \rangle = \nabla\varphi$, $\varphi(1, 2) = 7$, $\varphi(3, 6) = 10$, and $\varphi(6, 4) = 20$. Evaluate the following integrals for the given curve C, if possible.*

35. $\int_C \mathbf{F} \cdot d\mathbf{r}$; C: $\mathbf{r}(t) = \langle 2t - 1, t^2 + t \rangle$, for $1 \le t \le 2$
36. $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$; C is a smooth curve from $(1, 2)$ to $(6, 4)$.
37. $\int_C f\, dx + g\, dy$; C is the path consisting of the line segment from $A(6, 4)$ to $B(1, 2)$ followed by the line segment from $B(1, 2)$ to $C(3, 6)$.
38. $\oint_C \mathbf{F} \cdot d\mathbf{r}$; C is a circle, oriented clockwise, starting and ending at the point $A(6, 4)$.

39–44. Using the Fundamental Theorem for line integrals *Verify that the Fundamental Theorem for line integrals can be used to evaluate the given integral, and then evaluate the integral.*

39. $\int_C \langle 2x, 2y \rangle \cdot d\mathbf{r}$, where C is a smooth curve from $(0, 1)$ to $(3, 4)$

40. $\int_C \langle 1, 1, 1 \rangle \cdot d\mathbf{r}$, where C is a smooth curve from $(1, -1, 2)$ to $(3, 0, 7)$

41. $\int_C \nabla(e^{-x} \cos y) \cdot d\mathbf{r}$, where C is the line segment from $(0, 0)$ to $(\ln 2, 2\pi)$

42. $\int_C \nabla(1 + x^2yz) \cdot d\mathbf{r}$, where C is the helix $\mathbf{r}(t) = \langle \cos 2t, \sin 2t, t \rangle$, for $0 \leq t \leq 4\pi$

43. $\int_C \cos(2y - z)\, dx - 2x \sin(2y - z)\, dy + x \sin(2y - z)\, dz$, where C is the curve $\mathbf{r}(t) = \langle t^2, t, t \rangle$, for $0 \leq t \leq \pi$

44. $\int_C e^x y\, dx + e^x\, dy$, where C is the parabola $\mathbf{r}(t) = \langle t + 1, t^2 \rangle$, for $-1 \leq t \leq 3$

45–50. Line integrals of vector fields on closed curves *Evaluate* $\oint_C \mathbf{F} \cdot d\mathbf{r}$ *for the following vector fields and closed oriented curves C by parameterizing C. If the integral is not zero, give an explanation.*

45. $\mathbf{F} = \langle x, y \rangle$; C is the circle of radius 4 centered at the origin oriented counterclockwise.

46. $\mathbf{F} = \langle y, x \rangle$; C is the circle of radius 8 centered at the origin oriented counterclockwise.

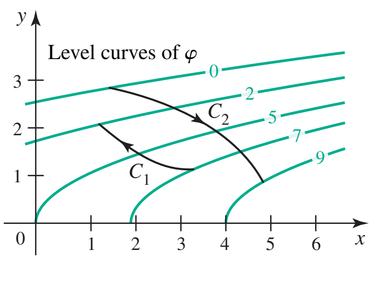
47. $\mathbf{F} = \langle x, y \rangle$; C is the triangle with vertices $(0, \pm 1)$ and $(1, 0)$ oriented counterclockwise.

48. $\mathbf{F} = \langle y, -x \rangle$; C is the circle of radius 3 centered at the origin oriented counterclockwise.

49. $\mathbf{F} = \langle x, y, z \rangle$; C: $\mathbf{r}(t) = \langle \cos t, \sin t, 2 \rangle$, for $0 \leq t \leq 2\pi$

50. $\mathbf{F} = \langle y - z, z - x, x - y \rangle$; C: $\mathbf{r}(t) = \langle \cos t, \sin t, \cos t \rangle$, for $0 \leq t \leq 2\pi$

51–52. Evaluating line integrals using level curves *Suppose the vector field* $\mathbf{F}$*, whose potential function is* φ*, is continuous on* $\mathbb{R}^2$*. Use the curves* C_1 *and* C_2 *and level curves of* φ *(see figure) to evaluate the following line integrals.*



51. $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$

52. $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$

53–56. Line integrals *Evaluate the following line integrals using a method of your choice.*

53. $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = \langle 2xy + z^2, x^2, 2xz \rangle$ and C is the circle $\mathbf{r}(t) = \langle 3 \cos t, 4 \cos t, 5 \sin t \rangle$, for $0 \leq t \leq 2\pi$

54. $\oint_C e^{-x}(\cos y\, dx + \sin y\, dy)$, where C is the square with vertices $(\pm 1, \pm 1)$ oriented counterclockwise

55. $\int_C \nabla(\sin xy) \cdot d\mathbf{r}$, where C is the line segment from $(0, 0)$ to $(2, \pi/4)$

56. $\int_C x^3\, dx + y^3\, dy$, where C is the curve $\mathbf{r}(t) = \langle 1 + \sin t, \cos^2 t \rangle$, for $0 \leq t \leq \pi/2$

57. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. If $\mathbf{F} = \langle -y, x \rangle$ and C is the circle of radius 4 centered at $(1, 0)$ oriented counterclockwise, then $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$.
b. If $\mathbf{F} = \langle x, -y \rangle$ and C is the circle of radius 4 centered at $(1, 0)$ oriented counterclockwise, then $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$.
c. A constant vector field is conservative on $\mathbb{R}^2$.
d. The vector field $\mathbf{F} = \langle f(x), g(y) \rangle$ is conservative on $\mathbb{R}^2$ (assume f and g are defined for all real numbers).
e. Gradient fields are conservative.

58. Closed-curve integrals Evaluate $\oint_C ds$, $\oint_C dx$, and $\oint_C dy$, where C is the unit circle oriented counterclockwise.

59–62. Work in force fields *Find the work required to move an object in the following force fields along a line segment between the given points. Check to see whether the force is conservative.*

59. $\mathbf{F} = \langle x, 2 \rangle$ from $A(0, 0)$ to $B(2, 4)$

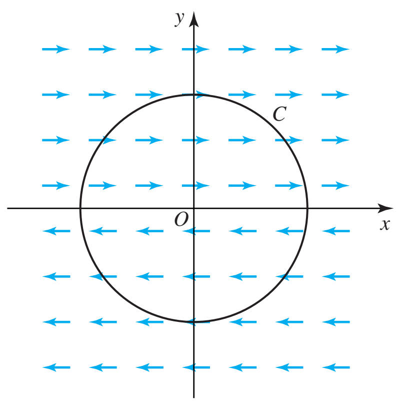
60. $\mathbf{F} = \langle x, y \rangle$ from $A(1, 1)$ to $B(3, -6)$

61. $\mathbf{F} = \langle x, y, z \rangle$ from $A(1, 2, 1)$ to $B(2, 4, 6)$

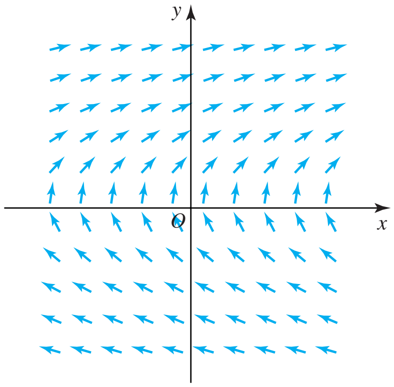
62. $\mathbf{F} = e^{x+y}\langle 1, 1, z \rangle$ from $A(0, 0, 0)$ to $B(-1, 2, -4)$

63. Suppose C is a circle centered at the origin in a vector field $\mathbf{F}$ (see figure).

a. If C is oriented counterclockwise, is $\oint_C \mathbf{F} \cdot d\mathbf{r}$ positive, negative, or zero?
b. If C is oriented clockwise, is $\oint_C \mathbf{F} \cdot d\mathbf{r}$ positive, negative, or zero?
c. Is $\mathbf{F}$ conservative in $\mathbb{R}^2$? Explain.



64. A vector field that is continuous in $\mathbb{R}^2$ is given (see figure). Is it conservative?



65. Work by a constant force Evaluate a line integral to show that the work done in moving an object from point A to point B in the presence of a constant force $\mathbf{F} = \langle a, b, c \rangle$ is $\mathbf{F} \cdot \overrightarrow{AB}$.

Explorations and Challenges

66. Conservation of energy Suppose an object with mass m moves in a region R in a conservative force field given by $\mathbf{F} = -\nabla\varphi$, where φ is a potential function in a region R. The motion of the object is governed by Newton's Second Law of Motion, $\mathbf{F} = m\mathbf{a}$, where $\mathbf{a}$ is the acceleration. Suppose the object moves from point A to point B in R.

a. Show that the equation of motion is $m\dfrac{d\mathbf{v}}{dt} = -\nabla\varphi$.

b. Show that $\dfrac{d\mathbf{v}}{dt} \cdot \mathbf{v} = \dfrac{1}{2}\dfrac{d}{dt}(\mathbf{v} \cdot \mathbf{v})$.

c. Take the dot product of both sides of the equation in part (a) with $\mathbf{v}(t) = \mathbf{r}'(t)$ and integrate along a curve between A and B. Use part (b) and the fact that $\mathbf{F}$ is conservative to show that the total energy (kinetic plus potential) $\dfrac{1}{2}m|\mathbf{v}|^2 + \varphi$ is the same at A and B. Conclude that because A and B are arbitrary, energy is conserved in R.

67. Gravitational potential The gravitational force between two point masses M and m is

$$\mathbf{F} = GMm\frac{\mathbf{r}}{|\mathbf{r}|^3} = GMm\frac{\langle x, y, z \rangle}{(x^2 + y^2 + z^2)^{3/2}},$$

where G is the gravitational constant.

a. Verify that this force field is conservative on any region excluding the origin.

b. Find a potential function φ for this force field such that $\mathbf{F} = -\nabla\varphi$.

c. Suppose the object with mass m is moved from a point A to a point B, where A is a distance r_1 from M, and B is a distance r_2 from M. Show that the work done in moving the object is $GMm\left(\dfrac{1}{r_2} - \dfrac{1}{r_1}\right)$.

d. Does the work depend on the path between A and B? Explain.

68. Radial fields in $\mathbb{R}^3$ are conservative Prove that the radial field $\mathbf{F} = \dfrac{\mathbf{r}}{|\mathbf{r}|^p}$, where $\mathbf{r} = \langle x, y, z \rangle$ and p is a real number, is conservative on any region not containing the origin. For what values of p is $\mathbf{F}$ conservative on a region that contains the origin?

69. Rotation fields are usually not conservative

a. Prove that the rotation field $\mathbf{F} = \dfrac{\langle -y, x \rangle}{|\mathbf{r}|^p}$, where $\mathbf{r} = \langle x, y \rangle$, is not conservative for $p \neq 2$.

b. For $p = 2$, show that $\mathbf{F}$ is conservative on any region not containing the origin.

c. Find a potential function for $\mathbf{F}$ when $p = 2$.

70. Linear and quadratic vector fields

a. For what values of a, b, c, and d is the field $\mathbf{F} = \langle ax + by, cx + dy \rangle$ conservative?

b. For what values of a, b, and c is the field $\mathbf{F} = \langle ax^2 - by^2, cxy \rangle$ conservative?

71. Alternative construction of potential functions in $\mathbb{R}^2$ Assume the vector field $\mathbf{F}$ is conservative on $\mathbb{R}^2$, so that the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path. Use the following procedure to construct a potential function φ for the vector field $\mathbf{F} = \langle f, g \rangle = \langle 2x - y, -x + 2y \rangle$.

a. Let A be $(0, 0)$ and let B be an arbitrary point (x, y). Define $\varphi(x, y)$ to be the work required to move an object from A to B, where $\varphi(A) = 0$. Let C_1 be the path from A to $(x, 0)$ to B, and let C_2 be the path from A to $(0, y)$ to B. Draw a picture.

b. Evaluate $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} f\,dx + g\,dy$ and conclude that $\varphi(x, y) = x^2 - xy + y^2$.

c. Verify that the same potential function is obtained by evaluating the line integral over C_2.

72–75. Alternative construction of potential functions *Use the procedure in Exercise 71 to construct potential functions for the following fields.*

72. $\mathbf{F} = \langle -y, -x \rangle$

73. $\mathbf{F} = \langle x, y \rangle$

74. $\mathbf{F} = \dfrac{\mathbf{r}}{|\mathbf{r}|}$, where $\mathbf{r} = \langle x, y \rangle$

75. $\mathbf{F} = \langle 2x^3 + xy^2, 2y^3 + x^2y \rangle$

QUICK CHECK ANSWERS

1. A figure-8 is closed but not simple; a torus is connected but not simply connected. **2.** The vector field is obtained by differentiating the potential function. So additive constants in the potential give the same vector field: $\nabla(\varphi + C) = \nabla\varphi$, where C is a constant. **3.** Show that $\nabla(e^x \cos y) = \langle e^x \cos y, -e^x \sin y \rangle$, which is the original vector field. A similar calculation may be done for part (b). **4.** The vector field $\nabla(xy + xz - yz)$ is the gradient of $xy + xz - yz$, so the vector field is conservative. ◄

17.4 Green's Theorem

The preceding section gave a version of the Fundamental Theorem of Calculus that applies to line integrals. In this and the remaining sections of the text, you will see additional extensions of the Fundamental Theorem that apply to regions in $\mathbb{R}^2$ and $\mathbb{R}^3$. All these fundamental theorems share a common feature. Part 2 of the Fundamental Theorem of Calculus (Chapter 5) says

$$\int_a^b \frac{df}{dx}\,dx = f(b) - f(a),$$

which relates the integral of $\frac{df}{dx}$ on an interval $[a, b]$ to the values of f on the boundary of $[a, b]$. The Fundamental Theorem for line integrals says

$$\int_C \nabla\varphi \cdot d\mathbf{r} = \varphi(B) - \varphi(A),$$

which relates the integral of $\nabla\varphi$ on a piecewise-smooth oriented curve C to the boundary values of φ. (The boundary consists of the two endpoints A and B.)

The subject of this section is Green's Theorem, which is another step in this progression. It relates the double integral of derivatives of a function over a region in $\mathbb{R}^2$ to function values on the boundary of that region.

Circulation Form of Green's Theorem

Throughout this section, unless otherwise stated, we assume curves in the plane are simple closed piecewise-smooth oriented curves. By a result called the *Jordan Curve Theorem*, such curves have a well-defined interior such that when the curve is traversed in the counterclockwise direction (viewed from above), the interior is on the left. With this orientation, there is a unique outward unit normal vector that points to the right (at points where the curve is smooth). We also assume curves in the plane lie in regions that are both connected and simply connected.

Suppose the vector field $\mathbf{F}$ is defined on a region R whose boundary is the closed curve C. As we have seen, the circulation $\oint_C \mathbf{F} \cdot d\mathbf{r}$ (Section 17.2) measures the net component of $\mathbf{F}$ in the direction tangent to C. It is easiest to visualize the circulation when $\mathbf{F}$ represents the velocity of a fluid moving in two dimensions. For example, let C be the unit circle with a counterclockwise orientation. The vector field $\mathbf{F} = \langle -y, x \rangle$ has a positive circulation of 2π on C (Section 17.2) because the vector field is everywhere tangent to C (**Figure 17.31**). A nonzero circulation on a closed curve says that the vector field must have some property *inside* the curve that produces the circulation. You can think of this property as a *net rotation*.

To visualize the rotation of a vector field, imagine a small paddle wheel, fixed at a point in the vector field, with its axis perpendicular to the xy-plane (Figure 17.31). The strength of the rotation at that point is seen in the speed at which the paddle wheel spins, and the direction of the rotation is the direction in which the paddle wheel spins. At a different point in the vector field, the paddle wheel will, in general, have a different speed and direction of rotation.

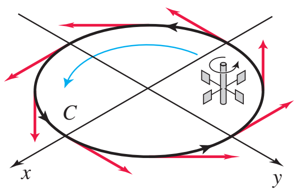


Figure 17.31

The first form of Green's Theorem relates the circulation on C to the double integral, over the region R, of a quantity that measures rotation at each point of R.

> **THEOREM 17.7 Green's Theorem—Circulation Form**
> Let C be a simple closed piecewise-smooth curve, oriented counterclockwise, that encloses a connected and simply connected region R in the plane. Assume $\mathbf{F} = \langle f, g \rangle$, where f and g have continuous first partial derivatives in R. Then
>
> $$\underbrace{\oint_C \mathbf{F} \cdot d\mathbf{r}}_{\text{circulation}} = \underbrace{\oint_C f\,dx + g\,dy}_{\text{circulation}} = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA.$$

➤ The circulation form of Green's Theorem is also called the *tangential*, or *curl*, form.

The proof of a special case of the theorem is given at the end of this section. Notice that the two line integrals on the left side of Green's Theorem give the circulation of the vector field on C. The double integral on the right side involves the quantity $\dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y}$, which describes the rotation of the vector field *within* C that produces the circulation *on* C. This quantity is called the *two-dimensional curl* of the vector field.

Figure 17.32 illustrates how the curl measures the rotation of a particular vector field at a point P. If the horizontal component of the field decreases in the y-direction at P $(f_y < 0)$ and the vertical component increases in the x-direction at P $(g_x > 0)$, then $\dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y} > 0$, and the field has a counterclockwise rotation at P. The double integral in Green's Theorem computes the net rotation of the field throughout R. The theorem says that the net rotation throughout R equals the circulation on the boundary of R.

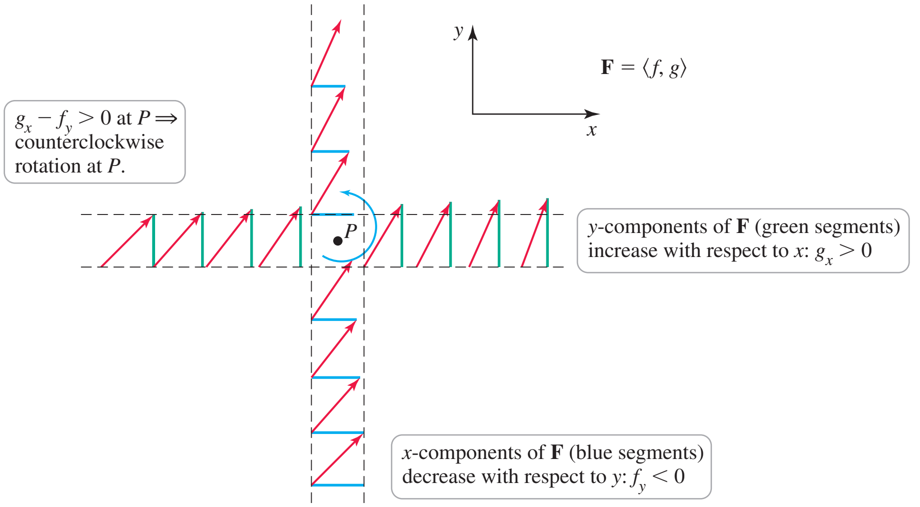


Figure 17.32

QUICK CHECK 1 Compute $\dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y}$ for the radial vector field $\mathbf{F} = \langle x, y \rangle$. What does this tell you about the circulation on a simple closed curve? ◄

Green's Theorem has an important consequence when applied to a conservative vector field. Recall from Theorem 17.3 that if $\mathbf{F} = \langle f, g \rangle$ is conservative, then its components satisfy the condition $f_y = g_x$. If R is a region of $\mathbb{R}^2$ on which the conditions of Green's Theorem are satisfied, then for a conservative field, we have

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \underbrace{\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)}_{0} dA = 0.$$

➤ In some cases, the rotation of a vector field is not obvious. For example, the parallel flow in a channel $\mathbf{F} = \langle 0, 1 - x^2 \rangle$, for $|x| \le 1$, has a nonzero curl for $x \neq 0$. See Exercise 72.

Green's Theorem confirms the fact (Theorem 17.6) that if $\mathbf{F}$ is a conservative vector field in a region, then the circulation $\oint_C \mathbf{F} \cdot d\mathbf{r}$ is zero on any simple closed curve in the region. A two-dimensional vector field $\mathbf{F} = \langle f, g \rangle$ for which $\dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y} = 0$ at all points of a region is said to be *irrotational*, because it produces zero circulation on closed curves in the region. Irrotational vector fields on simply connected regions in $\mathbb{R}^2$ are conservative.

DEFINITION Two-Dimensional Curl

The **two-dimensional curl** of the vector field $\mathbf{F} = \langle f, g \rangle$ is $\dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y}$. If the curl is zero throughout a region, the vector field is **irrotational** on that region.

Evaluating circulation integrals of conservative vector fields on closed curves is easy. The integral is always zero. Green's Theorem provides a way to evaluate circulation integrals for nonconservative vector fields.

EXAMPLE 1 **Circulation of a rotation field** Consider the rotation vector field $\mathbf{F} = \langle -y, x \rangle$ on the unit disk $R = \{(x, y): x^2 + y^2 \le 1\}$ (Figure 17.31). In Example 6 of Section 17.2, we showed that $\oint_C \mathbf{F} \cdot d\mathbf{r} = 2\pi$, where C is the boundary of R oriented counterclockwise. Confirm this result using Green's Theorem.

SOLUTION Note that $f(x, y) = -y$ and $g(x, y) = x$; therefore, the curl of $\mathbf{F}$ is $\dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y} = 2$. By Green's Theorem,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \underbrace{\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)}_{2} dA = \iint_R 2\, dA = 2 \times \text{area of } R = 2\pi.$$

The curl $\dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y}$ is nonzero on R, which results in a nonzero circulation on the boundary of R.

Related Exercises 18, 20 ◄

Calculating Area by Green's Theorem A useful consequence of Green's Theorem arises with the vector fields $\mathbf{F} = \langle f, g \rangle = \langle 0, x \rangle$ and $\mathbf{F} = \langle f, g \rangle = \langle y, 0 \rangle$. In the first case, we have $g_x = 1$ and $f_y = 0$; therefore, by Green's Theorem,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C \underbrace{x\, dy}_{\mathbf{F} \cdot d\mathbf{r}} = \iint_R \underbrace{dA}_{\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 1} = \text{area of } R.$$

In the second case, $g_x = 0$ and $f_y = 1$, and Green's Theorem says

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C y\, dx = -\iint_R dA = -\text{area of } R.$$

These two results may also be combined in one statement to give the following theorem.

THEOREM 17.8 **Area of a Plane Region by Line Integrals**
Under the conditions of Green's Theorem, the area of a region R enclosed by a curve C is

$$\oint_C x\, dy = -\oint_C y\, dx = \frac{1}{2}\oint_C (x\, dy - y\, dx).$$

A remarkably simple calculation of the area of an ellipse follows from this result.

EXAMPLE 2 **Area of an ellipse** Find the area of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

SOLUTION An ellipse with counterclockwise orientation is described parametrically by $\mathbf{r}(t) = \langle x, y \rangle = \langle a \cos t, b \sin t \rangle$, for $0 \le t \le 2\pi$. Noting that $dx = -a \sin t\, dt$ and $dy = b \cos t\, dt$, we have

$$\begin{aligned} x\, dy - y\, dx &= (a \cos t)(b \cos t)\, dt - (b \sin t)(-a \sin t)\, dt \\ &= ab\,(\cos^2 t + \sin^2 t)\, dt \\ &= ab\, dt. \end{aligned}$$

Expressing the line integral as an ordinary integral with respect to t, the area of the ellipse is

$$\frac{1}{2}\oint_C \underbrace{(x\, dy - y\, dx)}_{ab\, dt} = \frac{ab}{2}\int_0^{2\pi} dt = \pi ab.$$

Related Exercises 22–23 ◄

Flux Form of Green's Theorem

Let C be a closed curve enclosing a region R in $\mathbb{R}^2$ and let $\mathbf{F}$ be a vector field defined on R. We assume C and R have the previously stated properties; specifically, C is oriented counterclockwise with an outward normal vector $\mathbf{n}$. Recall that the outward flux of $\mathbf{F}$ across C is $\oint_C \mathbf{F} \cdot \mathbf{n}\, ds$ (Section 17.2). The second form of Green's Theorem relates the flux across C to a property of the vector field within R that produces the flux.

➤ The flux form of Green's Theorem is also called the *normal*, or *divergence*, form.

THEOREM 17.9 Green's Theorem—Flux Form

Let C be a simple closed piecewise-smooth curve, oriented counterclockwise, that encloses a connected and simply connected region R in the plane. Assume $\mathbf{F} = \langle f, g \rangle$, where f and g have continuous first partial derivatives in R. Then

$$\underbrace{\oint_C \mathbf{F} \cdot \mathbf{n}\, ds}_{\text{outward flux}} = \underbrace{\oint_C f\, dy - g\, dx}_{\text{outward flux}} = \iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \right) dA,$$

where $\mathbf{n}$ is the outward unit normal vector on the curve.

➤ The two forms of Green's Theorem are related in the following way: Applying the circulation form of the theorem to $\mathbf{F} = \langle -g, f \rangle$ results in the flux form, and applying the flux form of the theorem to $\mathbf{F} = \langle g, -f \rangle$ results in the circulation form.

The two line integrals on the left side of Theorem 17.9 give the outward flux of the vector field across C. The double integral on the right side involves the quantity $\dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y}$, which is the property of the vector field that produces the flux across C. This quantity is called the *two-dimensional divergence*.

Figure 17.33 illustrates how the divergence measures the flux of a particular vector field at a point P. If $f_x > 0$ at P, it indicates an expansion of the vector field in the x-direction (if f_x is negative, it indicates a contraction). Similarly, if $g_y > 0$ at P, it indicates an expansion of the vector field in the y-direction. The combined effect of $f_x + g_y > 0$ at a point is a net outward flux across a small circle enclosing P.

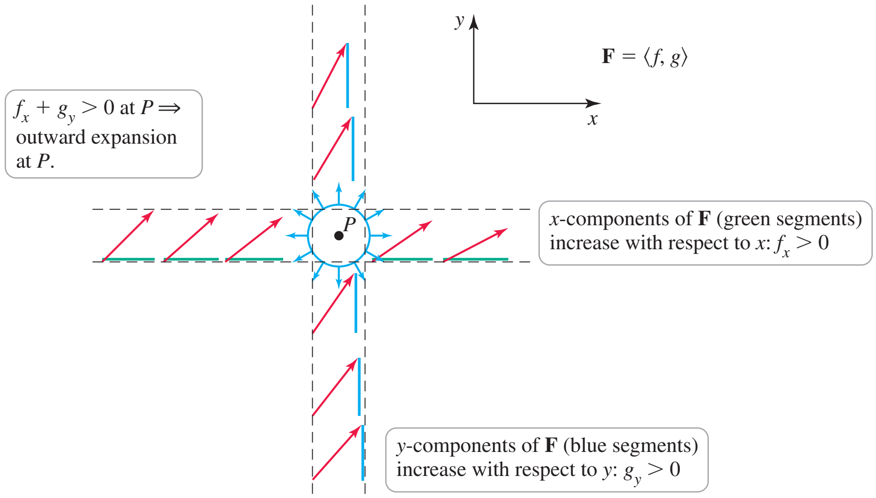


Figure 17.33

If the divergence of $\mathbf{F}$ is zero throughout a region on which $\mathbf{F}$ satisfies the conditions of Theorem 17.9, then the outward flux across the boundary is zero. Vector fields with a zero divergence are said to be *source free*. If the divergence is positive throughout R, the outward flux across C is positive, meaning that the vector field acts as a *source* in R. If the divergence is negative throughout R, the outward flux across C is negative, meaning that the vector field acts as a *sink* in R.

QUICK CHECK 2 Compute $\dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y}$ for the rotation field $\mathbf{F} = \langle -y, x \rangle$. What does this tell you about the outward flux of $\mathbf{F}$ across a simple closed curve? ◄

DEFINITION Two-Dimensional Divergence

The **two-dimensional divergence** of the vector field $\mathbf{F} = \langle f, g \rangle$ is $\dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y}$. If the divergence is zero throughout a region, the vector field is **source free** on that region.

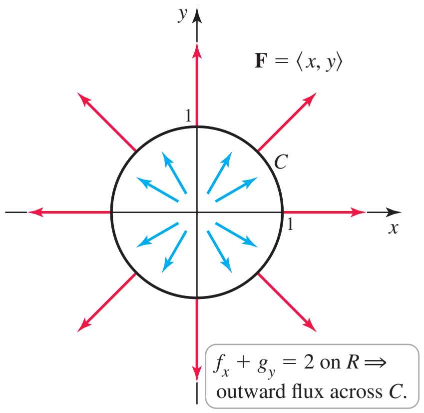


Figure 17.34

EXAMPLE 3 **Outward flux of a radial field** Use Green's Theorem to compute the outward flux of the radial field $\mathbf{F} = \langle x, y \rangle$ across the unit circle $C = \{(x, y): x^2 + y^2 = 1\}$ (**Figure 17.34**). Interpret the result.

SOLUTION We have already calculated the outward flux of the radial field across C as a line integral and found it to be 2π (Example 8, Section 17.2). Computing the outward flux using Green's Theorem, note that $f(x, y) = x$ and $g(x, y) = y$; therefore, the divergence of $\mathbf{F}$ is $\dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y} = 2$. By Green's Theorem, we have

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_R \underbrace{\left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \right)}_{2} dA = \iint_R 2\, dA = 2 \times \text{area of } R = 2\pi.$$

The positive divergence on R results in an outward flux of the vector field across the boundary of R.

Related Exercise 27 ◄

As with the circulation form, the flux form of Green's Theorem can be used in either direction: to simplify line integrals or to simplify double integrals.

EXAMPLE 4 **Line integral as a double integral** Evaluate

$$\oint_C (4x^3 + \sin y^2)\, dy - (4y^3 + \cos x^2)\, dx,$$

where C is the boundary of the disk $R = \{(x, y): x^2 + y^2 \le 4\}$ oriented counterclockwise.

SOLUTION Letting $f(x, y) = 4x^3 + \sin y^2$ and $g(x, y) = 4y^3 + \cos x^2$, Green's Theorem takes the form

$$\oint_C \underbrace{(4x^3 + \sin y^2)}_{f}\, dy - \underbrace{(4y^3 + \cos x^2)}_{g}\, dx$$

$$= \iint_R (\underbrace{12x^2}_{f_x} + \underbrace{12y^2}_{g_y})\, dA \qquad \text{Green's Theorem, flux form}$$

$$= 12 \int_0^{2\pi} \int_0^2 r^2 \underbrace{r\, dr\, d\theta}_{dA} \qquad \text{Polar coordinates; } x^2 + y^2 = r^2$$

$$= 12 \int_0^{2\pi} \frac{r^4}{4}\Big|_0^2\, d\theta \qquad \text{Evaluate inner integral.}$$

$$= 48 \int_0^{2\pi} d\theta = 96\pi. \qquad \text{Evaluate outer integral.}$$

Related Exercises 35–36 ◄

Circulation and Flux on More General Regions

Some ingenuity is required to extend both forms of Green's Theorem to more complicated regions. The next two examples illustrate Green's Theorem on two such regions: a half-annulus and a full annulus.

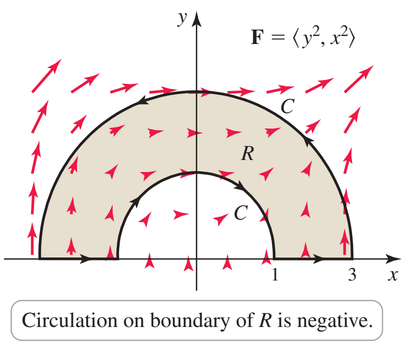


Figure 17.35

EXAMPLE 5 **Circulation on a half-annulus** Consider the vector field $\mathbf{F} = \langle y^2, x^2 \rangle$ on the half-annulus $R = \{(x, y): 1 \le x^2 + y^2 \le 9, y \ge 0\}$, whose boundary is C. Find the circulation on C, assuming it has the orientation shown in **Figure 17.35**.

SOLUTION The circulation on C is

$$\oint_C f\, dx + g\, dy = \oint_C y^2\, dx + x^2\, dy.$$

With the given orientation, the curve runs counterclockwise on the outer semicircle and clockwise on the inner semicircle. Identifying $f(x, y) = y^2$ and $g(x, y) = x^2$, the circulation form of Green's Theorem converts the line integral into a double integral. The double integral is most easily evaluated in polar coordinates using $x = r\cos\theta$ and $y = r\sin\theta$:

$$\oint_C \underbrace{y^2}_{f}\, dx + \underbrace{x^2}_{g}\, dy = \iint_R (\underbrace{2x}_{g_x} - \underbrace{2y}_{f_y})\, dA \qquad \text{Green's Theorem, circulation form}$$

$$= 2\int_0^{\pi}\int_1^3 (r\cos\theta - r\sin\theta)\, \underbrace{r\, dr\, d\theta}_{dA} \qquad \text{Convert to polar coordinates.}$$

$$= 2\int_0^{\pi} (\cos\theta - \sin\theta)\frac{r^3}{3}\bigg|_1^3\, d\theta \qquad \text{Evaluate inner integral.}$$

$$= \frac{52}{3}\int_0^{\pi} (\cos\theta - \sin\theta)\, d\theta \qquad \text{Simplify.}$$

$$= -\frac{104}{3}. \qquad \text{Evaluate outer integral.}$$

The vector field (Figure 17.35) suggests why the circulation is negative. The field is roughly *opposed* to the direction of C on the outer semicircle but roughly aligned with the direction of C on the inner semicircle. Because the outer semicircle is longer and the field has greater magnitudes on the outer curve than on the inner curve, the greater contribution to the circulation is negative.

Related Exercise 41 ◄

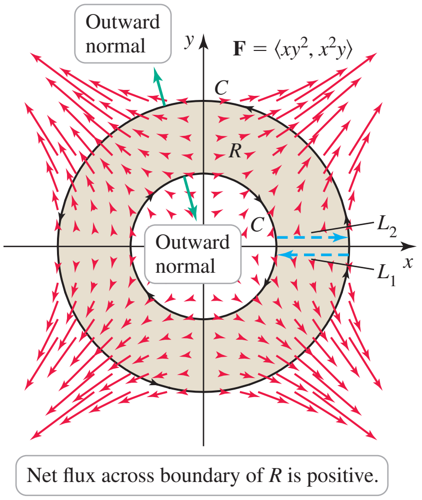


Figure 17.36

➤ Another way to deal with the flux across the annulus is to apply Green's Theorem to the entire disk $|r| \le 2$ and compute the flux across the outer circle. Then apply Green's Theorem to the disk $|r| \le 1$ and compute the flux across the inner circle. Note that the flux *out of* the inner disk is a flux *into* the annulus. Therefore, the difference of the two fluxes gives the net flux for the annulus.

EXAMPLE 6 Flux across the boundary of an annulus Find the outward flux of the vector field $\mathbf{F} = \langle xy^2, x^2y\rangle$ across the boundary of the annulus $R = \{(x, y): 1 \le x^2 + y^2 \le 4\}$, which, when expressed in polar coordinates, is the set $\{(r, \theta): 1 \le r \le 2, 0 \le \theta \le 2\pi\}$ (**Figure 17.36**).

SOLUTION Because the annulus R is not simply connected, Green's Theorem does not apply as stated in Theorem 17.9. This difficulty is overcome by defining the curve C shown in Figure 17.36, which is simple, closed, and piecewise smooth. The connecting links L_1 and L_2 below and above the x-axis are traversed in opposite directions. Letting L_1 and L_2 approach the x-axis, the contributions to the line integral cancel on L_1 and L_2. Because of this cancellation, we take C to be the curve that runs counterclockwise on the outer boundary and clockwise on the inner boundary.

Using the flux form of Green's Theorem and converting to polar coordinates, we have

$$\oint_C \mathbf{F}\cdot\mathbf{n}\, ds = \oint_C f\, dy - g\, dx = \oint_C xy^2\, dy - x^2y\, dx \qquad \text{Substitute for } f \text{ and } g.$$

$$= \iint_R (\underbrace{y^2}_{f_x} + \underbrace{x^2}_{g_y})\, dA \qquad \text{Green's Theorem, flux form}$$

$$= \int_0^{2\pi}\int_1^2 (r^2)\, r\, dr\, d\theta \qquad \text{Polar coordinates; } x^2 + y^2 = r^2$$

$$= \int_0^{2\pi} \frac{r^4}{4}\bigg|_1^2\, d\theta \qquad \text{Evaluate inner integral.}$$

$$= \frac{15}{4}\int_0^{2\pi} d\theta \qquad \text{Simplify.}$$

$$= \frac{15\pi}{2}. \qquad \text{Evaluate outer integral.}$$

Figure 17.36 shows the vector field and explains why the flux across C is positive. Because the field increases in magnitude moving away from the origin, the outward flux across the outer boundary is greater than the inward flux across the inner boundary. Hence, the net outward flux across C is positive.

Related Exercise 42 ◀

➤ Notice that the divergence of the vector field in Example 6 $(x^2 + y^2)$ is positive on R, so we expect an outward flux across C.

Stream Functions

We can now see a wonderful parallel between circulation properties (and conservative vector fields) and flux properties (and source-free fields). We need one more piece to complete the picture; it is the *stream function*, which plays the same role for source-free fields that the potential function plays for conservative fields.

Consider a two-dimensional vector field $\mathbf{F} = \langle f, g \rangle$ that is differentiable on a region R. A **stream function** for the vector field—if it exists—is a function ψ (pronounced *psigh* or *psee*) that satisfies

$$\frac{\partial \psi}{\partial y} = f, \qquad \frac{\partial \psi}{\partial x} = -g.$$

➤ Potential function for $\mathbf{F} = \langle f, g \rangle$:

$$\varphi_x = f \quad \text{and} \quad \varphi_y = g$$

Stream function for $\mathbf{F} = \langle f, g \rangle$:

$$\psi_x = -g \quad \text{and} \quad \psi_y = f$$

If we compute the divergence of a vector field $\mathbf{F} = \langle f, g \rangle$ that has a stream function and use the fact that $\psi_{xy} = \psi_{yx}$, then

$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = \underbrace{\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right)}_{\psi_{yx} = \psi_{xy}} = 0.$$

We see that the existence of a stream function guarantees that the vector field has zero divergence or, equivalently, is source free. The converse is also true on simply connected regions of $\mathbb{R}^2$.

QUICK CHECK 3 Show that $\psi = \frac{1}{2}(y^2 - x^2)$ is a stream function for the vector field $\mathbf{F} = \langle y, x \rangle$. Show that $\mathbf{F}$ has zero divergence. ◀

As discussed in Section 17.1, the level curves of a stream function are called flow curves or streamlines—and for good reason. It can be shown (Exercise 70) that the vector field $\mathbf{F}$ is everywhere tangent to the streamlines, which means that a graph of the streamlines shows the flow of the vector field. Finally, just as circulation integrals of a conservative vector field are independent of path, flux integrals of a source-free field are also independent of path (Exercise 69).

Vector fields that are both conservative and source free are quite interesting mathematically because they have both a potential function and a stream function. It can be shown that the level curves of the potential and stream functions form orthogonal families; that is, at each point of intersection, the line tangent to one level curve is orthogonal to the line tangent to the other level curve (equivalently, the gradient vector of one function is orthogonal to the gradient vector of the other function). Such vector fields have zero curl $(g_x - f_y = 0)$ and zero divergence $(f_x + g_y = 0)$. If we write the zero divergence condition in terms of the potential function φ, we find that

➤ In fluid dynamics, velocity fields that are both conservative and source free are called *ideal flows*. They model fluids that are irrotational and incompressible.

$$0 = f_x + g_y = \varphi_{xx} + \varphi_{yy}.$$

Writing the zero curl condition in terms of the stream function ψ, we find that

$$0 = g_x - f_y = -\psi_{xx} - \psi_{yy}.$$

We see that the potential function and the stream function both satisfy an important equation known as **Laplace's equation**:

$$\varphi_{xx} + \varphi_{yy} = 0 \quad \text{and} \quad \psi_{xx} + \psi_{yy} = 0.$$

➤ Methods for finding solutions of Laplace's equation are discussed in advanced mathematics courses.

Any function satisfying Laplace's equation can be used as a potential function or stream function for a conservative, source-free vector field. These vector fields are used in fluid dynamics, electrostatics, and other modeling applications.

Table 17.1 shows the parallel properties of conservative and source-free vector fields in two dimensions. We assume C is a simple piecewise-smooth oriented curve and either is closed or has endpoints A and B.

Table 17.1

Conservative Fields $\mathbf{F} = \langle f, g \rangle$	Source-Free Fields $\mathbf{F} = \langle f, g \rangle$
• curl $= \dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y} = 0$	• divergence $= \dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y} = 0$
• Potential function φ with $\mathbf{F} = \nabla\varphi$ or $f = \dfrac{\partial \varphi}{\partial x}, \quad g = \dfrac{\partial \varphi}{\partial y}$	• Stream function ψ with $f = \dfrac{\partial \psi}{\partial y}, \quad g = -\dfrac{\partial \psi}{\partial x}$
• Circulation $= \oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ on all closed curves C.	• Flux $= \oint_C \mathbf{F} \cdot \mathbf{n}\, ds = 0$ on all closed curves C.
• Evaluation of line integral $\int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(B) - \varphi(A)$	• Evaluation of line integral $\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \psi(B) - \psi(A)$

With Green's Theorem in the picture, we may also give a concise summary of the various cases that arise with line integrals of both the circulation and flux types (Table 17.2).

Table 17.2

Circulation/work integrals: $\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C f\, dx + g\, dy$

	C **closed**	C **not closed**
F conservative ($\mathbf{F} = \nabla\varphi$)	$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$	$\int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(B) - \varphi(A)$
F not conservative	Green's Theorem $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R (g_x - f_y)\, dA$	Direct evaluation $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b (fx' + gy')\, dt$

Flux integrals: $\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_C f\, dy - g\, dx$

	C **closed**	C **not closed**
F source free ($f = \psi_y, g = -\psi_x$)	$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = 0$	$\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \psi(B) - \psi(A)$
F not source free	Green's Theorem $\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_R (f_x + g_y)\, dA$	Direct evaluation $\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_a^b (fy' - gx')\, dt$

Proof of Green's Theorem on Special Regions

The proof of Green's Theorem is straightforward when restricted to special regions. We consider regions R enclosed by a simple closed smooth curve C oriented in the counterclockwise direction, such that the region can be expressed in two ways (**Figure 17.37**):

➤ This restriction on R means that lines parallel to the coordinate axes intersect the boundary of R at most twice.

- $R = \{(x, y)\colon a \le x \le b, G_1(x) \le y \le G_2(x)\}$ or
- $R = \{(x, y)\colon H_1(y) \le x \le H_2(y), c \le y \le d\}$.

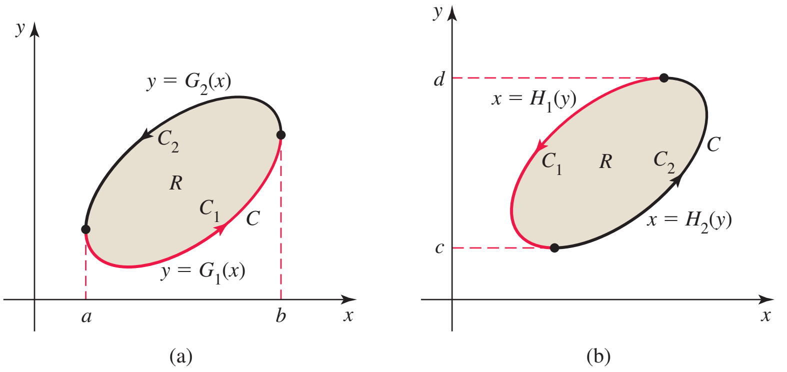


Figure 17.37

Under these conditions, we prove the circulation form of Green's Theorem:

$$\oint_C f\,dx + g\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA.$$

Beginning with the term $\displaystyle\iint_R \frac{\partial f}{\partial y}\,dA$, we write this double integral as an iterated integral, where $G_1(x) \le y \le G_2(x)$ in the inner integral and $a \le x \le b$ in the outer integral (**Figure 17.37a**). The upper curve is labeled C_2 and the lower curve is labeled C_1. Notice that the inner integral of $\dfrac{\partial f}{\partial y}$ with respect to y gives $f(x, y)$. Therefore, the first step of the double integration is

$$\iint_R \frac{\partial f}{\partial y}\,dA = \int_a^b \int_{G_1(x)}^{G_2(x)} \frac{\partial f}{\partial y}\,dy\,dx \qquad \text{Convert to an iterated integral.}$$

$$= \int_a^b \Big(\underbrace{f(x, G_2(x))}_{\text{on } C_2} - \underbrace{f(x, G_1(x))}_{\text{on } C_1}\Big)\,dx.$$

Over the interval $a \le x \le b$, the points $(x, G_2(x))$ trace out the upper part of C (labeled C_2) in the *negative* (clockwise) direction. Similarly, over the interval $a \le x \le b$, the points $(x, G_1(x))$ trace out the lower part of C (labeled C_1) in the *positive* (counterclockwise) direction.

Therefore,

$$\iint_R \frac{\partial f}{\partial y}\,dA = \int_a^b \big(f(x, G_2(x)) - f(x, G_1(x))\big)\,dx$$

$$= \int_{-C_2} f\,dx - \int_{C_1} f\,dx$$

$$= -\int_{C_2} f\,dx - \int_{C_1} f\,dx \qquad \int_{-C_2} f\,dx = -\int_{C_2} f\,dx$$

$$= -\oint_C f\,dx. \qquad \int_C f\,dx = \int_{C_1} f\,dx + \int_{C_2} f\,dx$$

A similar argument applies to the double integral of $\dfrac{\partial g}{\partial x}$, except we use the bounding curves $x = H_1(y)$ and $x = H_2(y)$, where C_1 is now the left curve and C_2 is the right curve (**Figure 17.37b**). We have

$$\iint_R \frac{\partial g}{\partial x}\, dA = \int_c^d \int_{H_1(y)}^{H_2(y)} \frac{\partial g}{\partial x}\, dx\, dy \qquad \text{Convert to an iterated integral.}$$

$$= \int_c^d \big(\underbrace{g(H_2(y), y)}_{C_2} - \underbrace{g(H_1(y), y)}_{-C_1}\big)\, dy \qquad \int \frac{\partial g}{\partial x}\, dx = g$$

$$= \int_{C_2} g\, dy - \int_{-C_1} g\, dy$$

$$= \int_{C_2} g\, dy + \int_{C_1} g\, dy \qquad \int_{-C_1} g\, dy = -\int_{C_1} g\, dy$$

$$= \oint_C g\, dy. \qquad \int_C g\, dy = \int_{C_1} g\, dy + \int_{C_2} g\, dy$$

Combining these two calculations results in

$$\iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA = \oint_C f\, dx + g\, dy.$$

QUICK CHECK 4 Explain why Green's Theorem proves that if $g_x = f_y$, then the vector field $\mathbf{F} = \langle f, g \rangle$ is conservative. ◄

As mentioned earlier, with a change of notation (replace g with f and f with $-g$), the flux form of Green's Theorem is obtained. This proof also completes the list of equivalent properties of conservative fields given in Section 17.3: From Green's Theorem, it follows that if $\dfrac{\partial g}{\partial x} = \dfrac{\partial f}{\partial y}$ on a simply connected region R, then the vector field $\mathbf{F} = \langle f, g \rangle$ is conservative on R.

SECTION 17.4 EXERCISES

Getting Started

1. Explain why the two forms of Green's Theorem are analogs of the Fundamental Theorem of Calculus.

2. Referring to both forms of Green's Theorem, match each idea in Column 1 to an idea in Column 2:

Line integral for flux	Double integral of the curl
Line integral for circulation	Double integral of the divergence

3. How do you use a line integral to compute the area of a plane region?

4. Why does a two-dimensional vector field with zero curl on a region have zero circulation on a closed curve that bounds the region?

5. Why does a two-dimensional vector field with zero divergence on a region have zero outward flux across a closed curve that bounds the region?

6. Sketch a two-dimensional vector field that has zero curl everywhere in the plane.

7. Sketch a two-dimensional vector field that has zero divergence everywhere in the plane.

8. Discuss one of the parallels between a conservative vector field and a source-free vector field.

9–14. *Assume C is a circle centered at the origin, oriented counterclockwise, that encloses disk R in the plane. Complete the following steps for each vector field* $\mathbf{F}$.

a. *Calculate the two-dimensional curl of* $\mathbf{F}$.
b. *Calculate the two-dimensional divergence of* $\mathbf{F}$.
c. *Is* $\mathbf{F}$ *irrotational on R?*
d. *Is* $\mathbf{F}$ *source free on R?*

9. $\mathbf{F} = \langle x, y \rangle$

10. $\mathbf{F} = \langle y, -x \rangle$

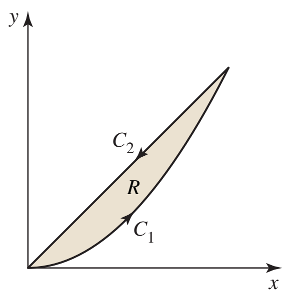
11. $\mathbf{F} = \langle y, -3x \rangle$

12. $\mathbf{F} = \langle x^2 + 2xy, -2xy - y^2 \rangle$

13. $\mathbf{F} = \langle 4x^3y, xy^2 + x^4 \rangle$

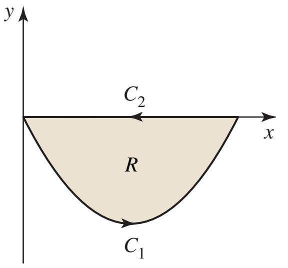
14. $\mathbf{F} = \langle 4x^3 + y, 12xy \rangle$

15. Suppose C is the boundary of region $R = \{(x, y): x^2 \le y \le x \le 1\}$, oriented counterclockwise (see figure); let $\mathbf{F} = \langle 1, x \rangle$.



a. Compute the two-dimensional curl of **F** and determine whether **F** is irrotational.
b. Find parameterizations $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ for C_1 and C_2, respectively.
c. Evaluate both the line integral and the double integral in the circulation form of Green's Theorem and check for consistency.
d. Compute the two-dimensional divergence of **F** and use the flux form of Green's Theorem to explain why the outward flux is 0.

16. Suppose C is the boundary of region $R = \{(x, y): 2x^2 - 2x \le y \le 0\}$, oriented counterclockwise (see figure); let $\mathbf{F} = \langle x, 1 \rangle$.



a. Compute the two-dimensional curl of **F** and use the circulation form of Green's Theorem to explain why the circulation is 0.
b. Compute the two-dimensional divergence of **F** and determine whether **F** is source free.
c. Find parameterizations $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ for C_1 and C_2, respectively.
d. Evaluate both the line integral and the double integral in the flux form of Green's Theorem and check for consistency.

Practice Exercises

17–20. Green's Theorem, circulation form *Consider the following regions R and vector fields* **F**.

a. Compute the two-dimensional curl of the vector field.
b. Evaluate both integrals in Green's Theorem and check for consistency.

17. $\mathbf{F} = \langle 2y, -2x \rangle$; R is the region bounded by $y = \sin x$ and $y = 0$, for $0 \le x \le \pi$.

18. $\mathbf{F} = \langle -3y, 3x \rangle$; R is the triangle with vertices $(0, 0)$, $(1, 0)$, and $(0, 2)$.

19. $\mathbf{F} = \langle -2xy, x^2 \rangle$; R is the region bounded by $y = x(2 - x)$ and $y = 0$.

20. $\mathbf{F} = \langle 0, x^2 + y^2 \rangle$; $R = \{(x, y): x^2 + y^2 \le 1\}$

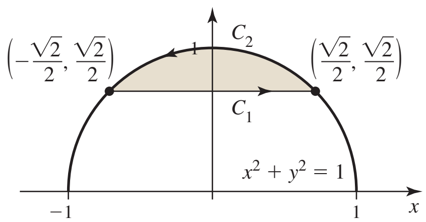
21–26. Area of regions *Use a line integral on the boundary to find the area of the following regions.*

21. A disk of radius 5

22. A region bounded by an ellipse with major and minor axes of lengths 12 and 8, respectively

23. $\{(x, y): x^2 + y^2 \le 16\}$

24. The region shown in the figure



25. The region bounded by the parabolas $\mathbf{r}(t) = \langle t, 2t^2 \rangle$ and $\mathbf{r}(t) = \langle t, 12 - t^2 \rangle$, for $-2 \le t \le 2$

26. The region bounded by the curve $\mathbf{r}(t) = \langle t(1 - t^2), 1 - t^2 \rangle$, for $-1 \le t \le 1$ (*Hint:* Plot the curve.)

27–30. Green's Theorem, flux form *Consider the following regions R and vector fields* **F**.

a. Compute the two-dimensional divergence of the vector field.
b. Evaluate both integrals in Green's Theorem and check for consistency.

27. $\mathbf{F} = \langle x, y \rangle$; $R = \{(x, y): x^2 + y^2 \le 4\}$

28. $\mathbf{F} = \langle x, -3y \rangle$; R is the triangle with vertices $(0, 0)$, $(1, 2)$, and $(0, 2)$.

29. $\mathbf{F} = \langle 2xy, x^2 \rangle$; $R = \{(x, y): 0 \le y \le x(2 - x)\}$

30. $\mathbf{F} = \langle x^2 + y^2, 0 \rangle$; $R = \{(x, y): x^2 + y^2 \le 1\}$

31–40. Line integrals *Use Green's Theorem to evaluate the following line integrals. Assume all curves are oriented counterclockwise. A sketch is helpful.*

31. $\oint_C \langle 3y + 1, 4x^2 + 3 \rangle \cdot d\mathbf{r}$, where C is the boundary of the rectangle with vertices $(0, 0)$, $(4, 0)$, $(4, 2)$, and $(0, 2)$

32. $\oint_C \langle \sin y, x \rangle \cdot d\mathbf{r}$, where C is the boundary of the triangle with vertices $(0, 0)$, $\left(\frac{\pi}{2}, 0\right)$, and $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$

33. $\oint_C xe^y\,dx + x\,dy$, where C is the boundary of the region bounded by the curves $y = x^2$, $x = 2$, and the x-axis

34. $\oint_C \frac{1}{1 + y^2}\,dx + y\,dy$, where C is the boundary of the triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 1)$

35. $\oint_C (2x + e^{y^2})\,dy - (4y^2 + e^{x^2})\,dx$, where C is the boundary of the square with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$

36. $\oint_C (2x - 3y)\,dy - (3x + 4y)\,dx$, where C is the unit circle

37. $\oint_C f\,dy - g\,dx$, where $\langle f, g \rangle = \langle 0, xy \rangle$ and C is the triangle with vertices $(0, 0)$, $(2, 0)$, and $(0, 4)$

38. $\oint_C f\,dy - g\,dx$, where $\langle f, g \rangle = \langle x^2, 2y^2 \rangle$ and C is the upper half of the unit circle and the line segment $-1 \le x \le 1$, oriented clockwise

39. The circulation line integral of $\mathbf{F} = \langle x^2 + y^2, 4x + y^3 \rangle$, where C is the boundary of $\{(x, y): 0 \le y \le \sin x, 0 \le x \le \pi\}$

40. The flux line integral of $\mathbf{F} = \langle e^{x-y}, e^{y-x} \rangle$, where C is the boundary of $\{(x, y): 0 \le y \le x, 0 \le x \le 1\}$

41–48. Circulation and flux *For the following vector fields, compute (a) the circulation on, and (b) the outward flux across, the boundary of the given region. Assume boundary curves are oriented counterclockwise.*

41. $\mathbf{F} = \langle x, y \rangle$; R is the half-annulus $\{(r, \theta): 1 \le r \le 2, 0 \le \theta \le \pi\}$.

42. $\mathbf{F} = \langle -y, x \rangle$; R is the annulus $\{(r, \theta): 1 \le r \le 3, 0 \le \theta \le 2\pi\}$.

43. $\mathbf{F} = \langle 2x + y, x - 4y \rangle$; R is the quarter-annulus $\{(r, \theta): 1 \le r \le 4, 0 \le \theta \le \pi/2\}$.

44. $\mathbf{F} = \langle x - y, -x + 2y \rangle$; R is the parallelogram $\{(x, y): 1 - x \le y \le 3 - x, 0 \le x \le 1\}$.

45. $\mathbf{F} = \nabla\left(\sqrt{x^2 + y^2}\right)$; R is the half-annulus $\{(r, \theta): 1 \le r \le 3, 0 \le \theta \le \pi\}$.

46. $\mathbf{F} = \left\langle \ln(x^2 + y^2), \tan^{-1}\frac{y}{x} \right\rangle$; R is the eighth-annulus $\{(r, \theta): 1 \le r \le 2, 0 \le \theta \le \pi/4\}$.

47. $\mathbf{F} = \langle x + y^2, x^2 - y \rangle$; $R = \{(x, y): y^2 \le x \le 2 - y^2\}$.

48. $\mathbf{F} = \langle y \cos x, -\sin x \rangle$; R is the square $\{(x, y): 0 \le x \le \pi/2, 0 \le y \le \pi/2\}$.

49. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. The work required to move an object around a closed curve C in the presence of a vector force field is the circulation of the force field on the curve.

b. If a vector field has zero divergence throughout a region (on which the conditions of Green's Theorem are met), then the circulation on the boundary of that region is zero.

c. If the two-dimensional curl of a vector field is positive throughout a region (on which the conditions of Green's Theorem are met), then the circulation on the boundary of that region is positive (assuming counterclockwise orientation).

50–51. Special line integrals *Prove the following identities, where C is a simple closed smooth oriented curve.*

50. $\oint_C dx = \oint_C dy = 0$

51. $\oint_C f(x)\,dx + g(y)\,dy = 0$, where f and g have continuous derivatives on the region enclosed by C

52. Double integral to line integral Use the flux form of Green's Theorem to evaluate $\iint_R (2xy + 4y^3)\,dA$, where R is the triangle with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$.

53. Area line integral Show that the value of

$$\oint_C xy^2\,dx + (x^2y + 2x)\,dy$$

depends only on the area of the region enclosed by C.

54. Area line integral In terms of the parameters a and b, how is the value of $\oint_C ay\,dx + bx\,dy$ related to the area of the region enclosed by C, assuming counterclockwise orientation of C?

55–58. Stream function *Recall that if the vector field $\mathbf{F} = \langle f, g \rangle$ is source free (zero divergence), then a stream function ψ exists such that $f = \psi_y$ and $g = -\psi_x$.*

a. *Verify that the given vector field has zero divergence.*

b. *Integrate the relations $f = \psi_y$ and $g = -\psi_x$ to find a stream function for the field.*

55. $\mathbf{F} = \langle 4, 2 \rangle$

56. $\mathbf{F} = \langle y^2, x^2 \rangle$

57. $\mathbf{F} = \langle -e^{-x}\sin y, e^{-x}\cos y \rangle$

58. $\mathbf{F} = \langle x^2, -2xy \rangle$

Explorations and Challenges

59–62. Ideal flow *A two-dimensional vector field describes* ***ideal flow*** *if it has both zero curl and zero divergence on a simply connected region.*

a. *Verify that both the curl and the divergence of the given field are zero.*

b. *Find a potential function φ and a stream function ψ for the field.*

c. *Verify that φ and ψ satisfy Laplace's equation $\varphi_{xx} + \varphi_{yy} = \psi_{xx} + \psi_{yy} = 0$.*

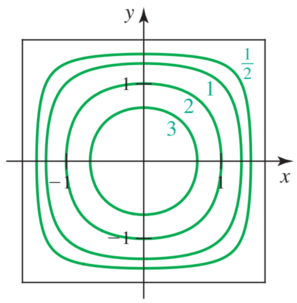
59. $\mathbf{F} = \langle e^x \cos y, -e^x \sin y \rangle$

60. $\mathbf{F} = \langle x^3 - 3xy^2, y^3 - 3x^2y \rangle$

61. $\mathbf{F} = \left\langle \tan^{-1}\frac{y}{x}, \frac{1}{2}\ln(x^2 + y^2) \right\rangle$, for $x > 0$

62. $\mathbf{F} = \frac{\langle x, y \rangle}{x^2 + y^2}$, for $x > 0, y > 0$

63. Flow in an ocean basin An idealized two-dimensional ocean is modeled by the square region $R = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ with boundary C. Consider the stream function $\psi(x, y) = 4\cos x \cos y$ defined on R (see figure).



a. The horizontal (east-west) component of the velocity is $u = \psi_y$ and the vertical (north-south) component of the velocity is $v = -\psi_x$. Sketch a few representative velocity vectors and show that the flow is counterclockwise around the region.

b. Is the velocity field source free? Explain.

c. Is the velocity field irrotational? Explain.

d. Let C be the boundary of R. Find the total outward flux across C.

e. Find the circulation on C assuming counterclockwise orientation.

64. Green's Theorem as a Fundamental Theorem of Calculus Show that if the circulation form of Green's Theorem is applied to the vector field $\left\langle 0, \frac{f(x)}{c} \right\rangle$, where $c > 0$ and $R = \{(x, y): a \le x \le b, 0 \le y \le c\}$, then the result is the Fundamental Theorem of Calculus,

$$\int_a^b \frac{df}{dx}\,dx = f(b) - f(a).$$

65. Green's Theorem as a Fundamental Theorem of Calculus Show that if the flux form of Green's Theorem is applied to the vector field $\left\langle \frac{f(x)}{c}, 0 \right\rangle$, where $c > 0$ and $R = \{(x, y): a \le x \le b, 0 \le y \le c\}$, then the result is the Fundamental Theorem of Calculus,

$$\int_a^b \frac{df}{dx}\,dx = f(b) - f(a).$$

66. What's wrong? Consider the rotation field $\mathbf{F} = \frac{\langle -y, x \rangle}{x^2 + y^2}$.

a. Verify that the two-dimensional curl of $\mathbf{F}$ is zero, which suggests that the double integral in the circulation form of Green's Theorem is zero.
b. Use a line integral to verify that the circulation on the unit circle of the vector field is 2π.
c. Explain why the results of parts (a) and (b) do not agree.

67. What's wrong? Consider the radial field $\mathbf{F} = \frac{\langle x, y \rangle}{x^2 + y^2}$.

a. Verify that the divergence of $\mathbf{F}$ is zero, which suggests that the double integral in the flux form of Green's Theorem is zero.
b. Use a line integral to verify that the outward flux across the unit circle of the vector field is 2π.
c. Explain why the results of parts (a) and (b) do not agree.

68. Conditions for Green's Theorem Consider the radial field $\mathbf{F} = \langle f, g \rangle = \frac{\langle x, y \rangle}{\sqrt{x^2 + y^2}} = \frac{\mathbf{r}}{|\mathbf{r}|}$.

a. Explain why the conditions of Green's Theorem do not apply to $\mathbf{F}$ on a region that includes the origin.
b. Let R be the unit disk centered at the origin and compute $\iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \right) dA$.
c. Evaluate the line integral in the flux form of Green's Theorem on the boundary of R.
d. Do the results of parts (b) and (c) agree? Explain.

69. Flux integrals Assume the vector field $\mathbf{F} = \langle f, g \rangle$ is source free (zero divergence) with stream function ψ. Let C be any smooth simple curve from A to the distinct point B. Show that the flux integral $\int_C \mathbf{F} \cdot \mathbf{n}\,ds$ is independent of path; that is, $\int_C \mathbf{F} \cdot \mathbf{n}\,ds = \psi(B) - \psi(A)$.

70. Streamlines are tangent to the vector field Assume the vector field $\mathbf{F} = \langle f, g \rangle$ is related to the stream function ψ by $\psi_y = f$ and $\psi_x = -g$ on a region R. Prove that at all points of R, the vector field is tangent to the streamlines (the level curves of the stream function).

71. Streamlines and equipotential lines Assume that on $\mathbb{R}^2$, the vector field $\mathbf{F} = \langle f, g \rangle$ has a potential function φ such that $f = \varphi_x$ and $g = \varphi_y$, and it has a stream function ψ such that $f = \psi_y$ and $g = -\psi_x$. Show that the equipotential curves (level curves of φ) and the streamlines (level curves of ψ) are everywhere orthogonal.

72. Channel flow The flow in a long shallow channel is modeled by the velocity field $\mathbf{F} = \langle 0, 1 - x^2 \rangle$, where $R = \{(x, y): |x| \le 1 \text{ and } |y| < 5\}$.

a. Sketch R and several streamlines of $\mathbf{F}$.
b. Evaluate the curl of $\mathbf{F}$ on the lines $x = 0$, $x = 1/4$, $x = 1/2$, and $x = 1$.
c. Compute the circulation on the boundary of the region R.
d. How do you explain the fact that the curl of $\mathbf{F}$ is nonzero at points of R, but the circulation is zero?

QUICK CHECK ANSWERS

1. $g_x - f_y = 0$, which implies zero circulation on a closed curve. **2.** $f_x + g_y = 0$, which implies zero flux across a closed curve. **3.** $\psi_y = y$ is the x-component of $\mathbf{F} = \langle y, x \rangle$, and $-\psi_x = x$ is the y-component of $\mathbf{F}$. Also, the divergence of $\mathbf{F}$ is $y_x + x_y = 0$. **4.** If the curl is zero on a region, then all closed-path integrals are zero, which is a condition (Section 17.3) for a conservative field. ➤

17.5 Divergence and Curl

Green's Theorem sets the stage for the final act in our exploration of calculus. The last four sections of this chapter have the following goal: to lift both forms of Green's Theorem out of the plane ($\mathbb{R}^2$) and into space ($\mathbb{R}^3$). It is done as follows.

- The circulation form of Green's Theorem relates a line integral over a simple closed oriented curve in the plane to a double integral over the enclosed region. In an analogous manner, we will see that *Stokes' Theorem* (Section 17.7) relates a line integral over a simple closed oriented curve in $\mathbb{R}^3$ to a double integral over a surface whose boundary is that curve.
- The flux form of Green's Theorem relates a line integral over a simple closed oriented curve in the plane to a double integral over the enclosed region. Similarly, the *Divergence Theorem* (Section 17.8) relates an integral over a closed oriented surface in $\mathbb{R}^3$ to a triple integral over the region enclosed by that surface.

In order to make these extensions, we need a few more tools.

- The two-dimensional divergence and two-dimensional curl must be extended to three dimensions (this section).
- The idea of a *surface integral* must be introduced (Section 17.6).

> Review: The divergence measures the expansion or contraction of a vector field at each point. The flux form of Green's Theorem implies that if the two-dimensional divergence of a vector field is zero throughout a simply connected plane region, then the outward flux across the boundary of the region is zero. If the divergence is nonzero, Green's Theorem gives the outward flux across the boundary.

The Divergence

Recall that in two dimensions, the divergence of the vector field $\mathbf{F} = \langle f, g \rangle$ is $\dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y}$. The extension to three dimensions is straightforward. If $\mathbf{F} = \langle f, g, h \rangle$ is a differentiable vector field defined on a region of $\mathbb{R}^3$, the divergence of $\mathbf{F}$ is $\dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y} + \dfrac{\partial h}{\partial z}$. The interpretation of the three-dimensional divergence is much the same as it is in two dimensions. It measures the expansion or contraction of the vector field at each point. If the divergence is zero at all points of a region, the vector field is *source free* on that region.

Recall the *del operator* ∇ that was introduced in Section 15.5 to define the gradient:

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle.$$

This object is not really a vector; it is an operation that is applied to a function or a vector field. Applying it directly to a scalar function f results in the gradient of f:

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} = \langle f_x, f_y, f_z \rangle.$$

> In evaluating $\nabla \cdot \mathbf{F}$ as a dot product, each component of ∇ is applied to the corresponding component of $\mathbf{F}$, producing $f_x + g_y + h_z$.

However, if we form the *dot product* of ∇ and a vector field $\mathbf{F} = \langle f, g, h \rangle$, the result is

$$\nabla \cdot \mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle f, g, h \rangle = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z},$$

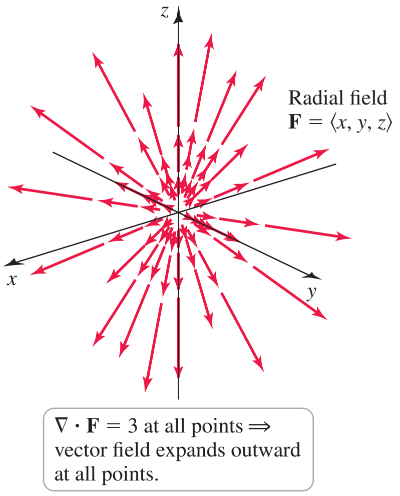
which is the divergence of $\mathbf{F}$, also denoted div $\mathbf{F}$. Like all dot products, the divergence is a scalar; in this case, it is a scalar-valued function.

DEFINITION Divergence of a Vector Field

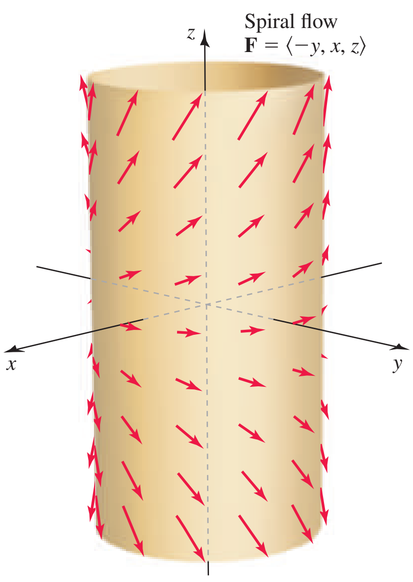
The **divergence** of a vector field $\mathbf{F} = \langle f, g, h \rangle$ that is differentiable on a region of $\mathbb{R}^3$ is

$$\text{div}\,\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}.$$

If $\nabla \cdot \mathbf{F} = 0$, the vector field is **source free**.



(a)



(b)

Figure 17.38

EXAMPLE 1 **Computing the divergence** Compute the divergence of the following vector fields.

a. $\mathbf{F} = \langle x, y, z \rangle$ (a radial field)

b. $\mathbf{F} = \langle -y, x - z, y \rangle$ (a rotation field)

c. $\mathbf{F} = \langle -y, x, z \rangle$ (a spiral flow)

SOLUTION

a. The divergence is $\nabla \cdot \mathbf{F} = \nabla \cdot \langle x, y, z \rangle = \dfrac{\partial x}{\partial x} + \dfrac{\partial y}{\partial y} + \dfrac{\partial z}{\partial z} = 1 + 1 + 1 = 3.$

Because the divergence is positive, the flow expands outward at all points (Figure 17.38a).

b. The divergence is

$$\nabla \cdot \mathbf{F} = \nabla \cdot \langle -y, x - z, y \rangle = \frac{\partial(-y)}{\partial x} + \frac{\partial(x - z)}{\partial y} + \frac{\partial y}{\partial z} = 0 + 0 + 0 = 0,$$

so the field is source free.

c. This field is a combination of the two-dimensional rotation field $\mathbf{F} = \langle -y, x \rangle$ and a vertical flow in the z-direction; the net effect is a field that spirals upward for $z > 0$ and spirals downward for $z < 0$ (Figure 17.38b). The divergence is

$$\nabla \cdot \mathbf{F} = \nabla \cdot \langle -y, x, z \rangle = \frac{\partial(-y)}{\partial x} + \frac{\partial x}{\partial y} + \frac{\partial z}{\partial z} = 0 + 0 + 1 = 1.$$

QUICK CHECK 1 Show that if a vector field has the form $\mathbf{F} = \langle f(y, z), g(x, z), h(x, y) \rangle$, then $\text{div}\,\mathbf{F} = 0$. ◄

The rotational part of the field in x and y does not contribute to the divergence. However, the z-component of the field produces a nonzero divergence.

Related Exercises 10–11 ◄

Divergence of a Radial Vector Field The vector field considered in Example 1a is just one of many radial fields that have important applications (for example, the inverse square laws of gravitation and electrostatics). The following example leads to a general result for the divergence of radial vector fields.

EXAMPLE 2 **Divergence of a radial field** Compute the divergence of the radial vector field

$$\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|} = \frac{\langle x, y, z \rangle}{\sqrt{x^2 + y^2 + z^2}}.$$

SOLUTION This radial field has the property that it is directed outward from the origin and all vectors have unit length $(|\mathbf{F}| = 1)$. Let's compute one piece of the divergence; the others follow the same pattern. Using the Quotient Rule, the derivative with respect to x of the first component of $\mathbf{F}$ is

$$\begin{aligned} \frac{\partial}{\partial x}\left(\frac{x}{(x^2 + y^2 + z^2)^{1/2}}\right) &= \frac{(x^2 + y^2 + z^2)^{1/2} - x^2(x^2 + y^2 + z^2)^{-1/2}}{x^2 + y^2 + z^2} && \text{Quotient Rule} \\ &= \frac{|\mathbf{r}| - x^2|\mathbf{r}|^{-1}}{|\mathbf{r}|^2} && \sqrt{x^2 + y^2 + z^2} = |\mathbf{r}| \\ &= \frac{|\mathbf{r}|^2 - x^2}{|\mathbf{r}|^3}. && \text{Simplify.} \end{aligned}$$

A similar calculation of the y- and z-derivatives yields $\dfrac{|\mathbf{r}|^2 - y^2}{|\mathbf{r}|^3}$ and $\dfrac{|\mathbf{r}|^2 - z^2}{|\mathbf{r}|^3}$, respectively. Adding the three terms, we find that

$$\begin{aligned} \nabla \cdot \mathbf{F} &= \frac{|\mathbf{r}|^2 - x^2}{|\mathbf{r}|^3} + \frac{|\mathbf{r}|^2 - y^2}{|\mathbf{r}|^3} + \frac{|\mathbf{r}|^2 - z^2}{|\mathbf{r}|^3} \\ &= 3\frac{|\mathbf{r}|^2}{|\mathbf{r}|^3} - \frac{x^2 + y^2 + z^2}{|\mathbf{r}|^3} && \text{Collect terms.} \\ &= \frac{2}{|\mathbf{r}|}. && x^2 + y^2 + z^2 = |\mathbf{r}|^2 \end{aligned}$$

Related Exercise 18 ◄

Examples 1a and 2 give two special cases of the following theorem about the divergence of radial vector fields (Exercise 73).

> **THEOREM 17.10** **Divergence of Radial Vector Fields**
> For a real number p, the divergence of the radial vector field
> $$\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|^p} = \frac{\langle x, y, z \rangle}{(x^2 + y^2 + z^2)^{p/2}} \quad \text{is} \quad \nabla \cdot \mathbf{F} = \frac{3 - p}{|\mathbf{r}|^p}.$$

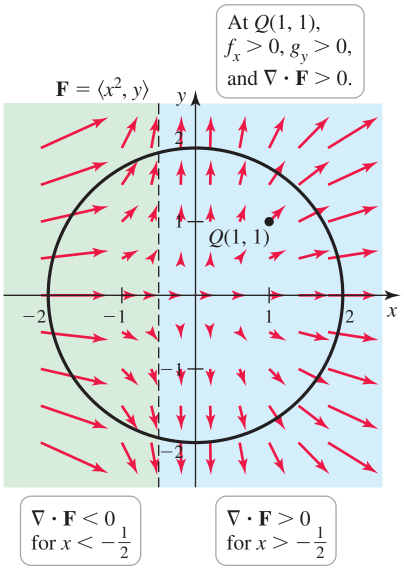


Figure 17.39

EXAMPLE 3 **Divergence from a graph** To gain some intuition about the divergence, consider the two-dimensional vector field $\mathbf{F} = \langle f, g \rangle = \langle x^2, y \rangle$ and a circle C of radius 2 centered at the origin (**Figure 17.39**).

a. Without computing it, determine whether the two-dimensional divergence is positive or negative at the point $Q(1, 1)$. Why?

b. Confirm your conjecture in part (a) by computing the two-dimensional divergence at Q.

c. Based on part (b), over what regions within the circle is the divergence positive and over what regions within the circle is the divergence negative?

d. By inspection of the figure, on what part of the circle is the flux across the boundary outward? Is the net flux out of the circle positive or negative?

➤ To understand the conclusion of Example 3a, note that as you move through the point Q from left to right, the horizontal components of the vectors increase in length ($f_x > 0$). As you move through the point Q in the upward direction, the vertical components of the vectors also increase in length ($g_y > 0$).

SOLUTION

a. At $Q(1, 1)$ the x-component and the y-component of the field are increasing ($f_x > 0$ and $g_y > 0$), so the field is expanding at that point and the two-dimensional divergence is positive.

b. Calculating the two-dimensional divergence, we find that

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y) = 2x + 1.$$

At $Q(1, 1)$ the divergence is 3, confirming part (a).

c. From part (b), we see that $\nabla \cdot \mathbf{F} = 2x + 1 > 0$, for $x > -\frac{1}{2}$, and $\nabla \cdot \mathbf{F} < 0$, for $x < -\frac{1}{2}$. To the left of the line $x = -\frac{1}{2}$ the field is contracting, and to the right of the line the field is expanding.

QUICK CHECK 2 Verify the claim made in Example 3d by showing that the net outward flux of $\mathbf{F}$ across C is positive. (*Hint:* If you use Green's Theorem to evaluate the integral $\int_C f\,dy - g\,dx$, convert to polar coordinates.) ◀

d. Using Figure 17.39, it appears that the field is tangent to the circle at two points with $x \approx -1$. For points on the circle with $x < -1$, the flow is into the circle; for points on the circle with $x > -1$, the flow is out of the circle. It appears that the net outward flux across C is positive. The points where the field changes from inward to outward may be determined exactly (Exercise 46). *Related Exercises 21–22* ◀

The Curl

➤ Review: The *two-dimensional curl* $g_x - f_y$ measures the rotation of a vector field at a point. The circulation form of Green's Theorem implies that if the two-dimensional curl of a vector field is zero throughout a simply connected region, then the circulation on the boundary of the region is also zero. If the curl is nonzero, Green's Theorem gives the circulation along the curve.

Just as the divergence $\nabla \cdot \mathbf{F}$ is the dot product of the *del operator* and $\mathbf{F}$, the three-dimensional curl is the cross product $\nabla \times \mathbf{F}$. If we formally use the notation for the cross product in terms of a 3×3 determinant, we obtain the definition of the curl:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f & g & h \end{vmatrix} \begin{array}{l} \leftarrow \text{Unit Vectors} \\ \leftarrow \text{Components of } \nabla \\ \leftarrow \text{Components of } \mathbf{F} \end{array}$$

$$= \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\mathbf{i} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\mathbf{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\mathbf{k}.$$

The curl of a vector field, also denoted curl $\mathbf{F}$, is a vector with three components. Notice that the $\mathbf{k}$-component of the curl $(g_x - f_y)$ is the two-dimensional curl, which gives the rotation in the xy-plane at a point. The $\mathbf{i}$- and $\mathbf{j}$-components of the curl correspond to the rotation of the vector field in planes parallel to the yz-plane (orthogonal to $\mathbf{i}$) and in planes parallel to the xz-plane (orthogonal to $\mathbf{j}$) (Figure 17.40).

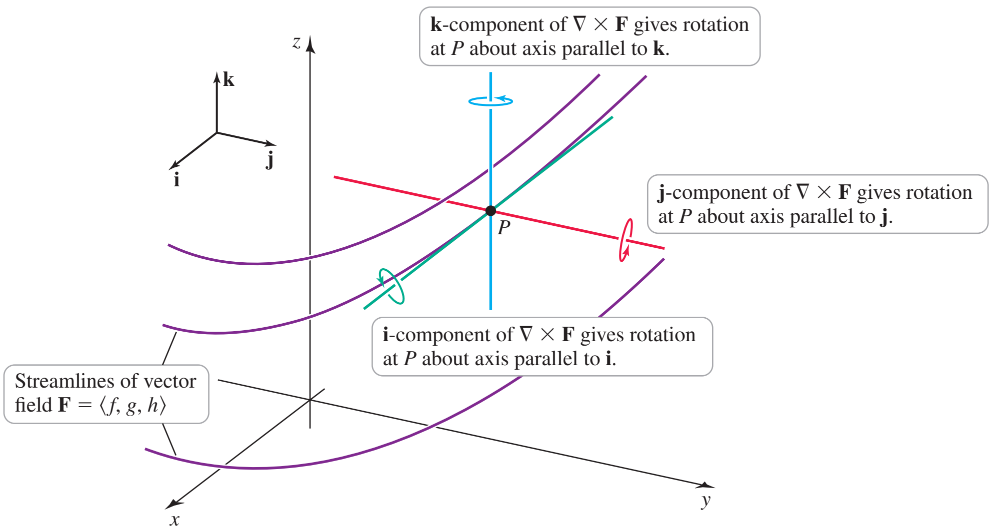


Figure 17.40

> **DEFINITION Curl of a Vector Field**
>
> The **curl** of a vector field $\mathbf{F} = \langle f, g, h\rangle$ that is differentiable on a region of $\mathbb{R}^3$ is
>
> $$\begin{aligned}\nabla \times \mathbf{F} &= \text{curl } \mathbf{F}\\ &= \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\mathbf{i} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\mathbf{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\mathbf{k}.\end{aligned}$$
>
> If $\nabla \times \mathbf{F} = \mathbf{0}$, the vector field is **irrotational**.

Curl of a General Rotation Vector Field We can clarify the physical meaning of the curl by considering the vector field $\mathbf{F} = \mathbf{a} \times \mathbf{r}$, where $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ is a nonzero constant vector and $\mathbf{r} = \langle x, y, z\rangle$. Writing out its components, we see that

$$\mathbf{F} = \mathbf{a} \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix} = (a_2 z - a_3 y)\,\mathbf{i} + (a_3 x - a_1 z)\,\mathbf{j} + (a_1 y - a_2 x)\,\mathbf{k}.$$

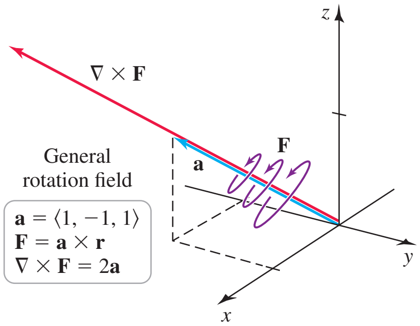


Figure 17.41

This vector field is a *general rotation field* in three dimensions. With $a_1 = a_2 = 0$ and $a_3 = 1$, we have the familiar two-dimensional rotation field $\langle -y, x\rangle$ with its axis in the $\mathbf{k}$-direction. More generally, $\mathbf{F}$ is the superposition of three rotation fields with axes in the $\mathbf{i}$-, $\mathbf{j}$-, and $\mathbf{k}$-directions. The result is a single rotation field with an axis in the direction of $\mathbf{a}$ (Figure 17.41).

Three calculations tell us a lot about the general rotation field. The first calculation confirms that $\nabla \cdot \mathbf{F} = 0$ (Exercise 42). Just as with rotation fields in two dimensions, the divergence of a general rotation field is zero.

The second calculation (Exercises 43–44) uses the right-hand rule for cross products to show that the vector field $\mathbf{F} = \mathbf{a} \times \mathbf{r}$ is indeed a rotation field that circles the vector $\mathbf{a}$ in a counterclockwise direction looking along the length of $\mathbf{a}$ from head to tail (Figure 17.41).

The third calculation (Exercise 45) says that $\nabla \times \mathbf{F} = 2\mathbf{a}$. Therefore, the curl of the general rotation field is in the direction of the axis of rotation $\mathbf{a}$ (Figure 17.41). The magnitude of the curl is $|\nabla \times \mathbf{F}| = 2|\mathbf{a}|$. It can be shown (Exercise 52) that if $\mathbf{F}$ is a velocity field, then $|\mathbf{a}|$ is the constant angular speed of rotation of the field, denoted ω. The angular speed is the rate (radians per unit time) at which a small particle in the vector field rotates about the axis of the field. Therefore, the angular speed is half the magnitude of the curl, or

$$\omega = |\mathbf{a}| = \frac{1}{2}|\nabla \times \mathbf{F}|.$$

➤ Just as $\nabla f \cdot \mathbf{n}$ is the directional derivative in the direction $\mathbf{n}$, $(\nabla \times \mathbf{F}) \cdot \mathbf{n}$ is the directional spin in the direction $\mathbf{n}$.

The rotation field $\mathbf{F} = \mathbf{a} \times \mathbf{r}$ suggests a related question. Suppose a paddle wheel is placed in the vector field $\mathbf{F}$ at a point P with the axis of the wheel in the direction of a unit vector $\mathbf{n}$ (Figure 17.42). How should $\mathbf{n}$ be chosen so the paddle wheel spins fastest? The scalar component of $\nabla \times \mathbf{F}$ in the direction of $\mathbf{n}$ is

$$(\nabla \times \mathbf{F}) \cdot \mathbf{n} = |\nabla \times \mathbf{F}| \cos\theta, \quad (|\mathbf{n}| = 1)$$

where θ is the angle between $\nabla \times \mathbf{F}$ and $\mathbf{n}$. The scalar component is greatest in magnitude and the paddle wheel spins fastest when $\theta = 0$ or $\theta = \pi$; that is, when $\mathbf{n}$ and $\nabla \times \mathbf{F}$ are parallel. If the axis of the paddle wheel is orthogonal to $\nabla \times \mathbf{F}$ $(\theta = \pm\pi/2)$, the wheel doesn't spin.

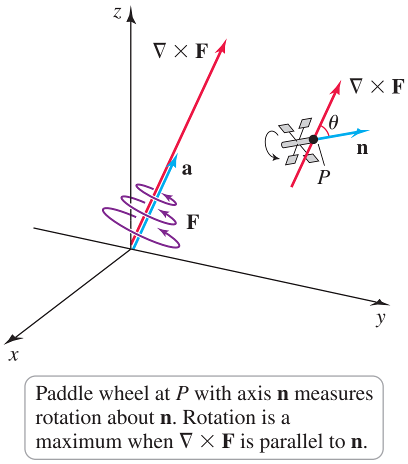


Figure 17.42

> **General Rotation Vector Field**
>
> The **general rotation vector field** is $\mathbf{F} = \mathbf{a} \times \mathbf{r}$, where the nonzero constant vector $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ is the axis of rotation and $\mathbf{r} = \langle x, y, z\rangle$. For all nonzero choices of $\mathbf{a}$, $|\nabla \times \mathbf{F}| = 2|\mathbf{a}|$ and $\nabla \cdot \mathbf{F} = 0$. If $\mathbf{F}$ is a velocity field, then the constant angular speed of the field is
>
> $$\omega = |\mathbf{a}| = \frac{1}{2}|\nabla \times \mathbf{F}|.$$

QUICK CHECK 3 Show that if a vector field has the form $\mathbf{F} = \langle f(x), g(y), h(z) \rangle$, then $\nabla \times \mathbf{F} = \mathbf{0}$. ◄

EXAMPLE 4 **Curl of a rotation field** Compute the curl of the rotation field $\mathbf{F} = \mathbf{a} \times \mathbf{r}$, where $\mathbf{a} = \langle 2, -1, 1 \rangle$ and $\mathbf{r} = \langle x, y, z \rangle$ (Figure 17.41). What are the direction and the magnitude of the curl?

SOLUTION A quick calculation shows that

$$\mathbf{F} = \mathbf{a} \times \mathbf{r} = (-y - z)\,\mathbf{i} + (x - 2z)\,\mathbf{j} + (x + 2y)\,\mathbf{k}.$$

The curl of the vector field is

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -y - z & x - 2z & x + 2y \end{vmatrix} = 4\mathbf{i} - 2\mathbf{j} + 2\mathbf{k} = 2\mathbf{a}.$$

We have confirmed that $\nabla \times \mathbf{F} = 2\mathbf{a}$ and that the direction of the curl is the direction of $\mathbf{a}$, which is the axis of rotation. The magnitude of $\nabla \times \mathbf{F}$ is $|2\mathbf{a}| = 2\sqrt{6}$, which is twice the angular speed of rotation.

Related Exercises 25–26 ◄

Working with Divergence and Curl

The divergence and curl satisfy some of the same properties that ordinary derivatives satisfy. For example, given a real number c and differentiable vector fields $\mathbf{F}$ and $\mathbf{G}$, we have the following properties.

Divergence Properties	**Curl Properties**
$\nabla \cdot (\mathbf{F} + \mathbf{G}) = \nabla \cdot \mathbf{F} + \nabla \cdot \mathbf{G}$	$\nabla \times (\mathbf{F} + \mathbf{G}) = (\nabla \times \mathbf{F}) + (\nabla \times \mathbf{G})$
$\nabla \cdot (c\mathbf{F}) = c(\nabla \cdot \mathbf{F})$	$\nabla \times (c\mathbf{F}) = c(\nabla \times \mathbf{F})$

These and other properties are explored in Exercises 65–72.

Additional properties that have importance in theory and applications are presented in the following theorems and examples.

THEOREM 17.11 **Curl of a Conservative Vector Field**
Suppose $\mathbf{F}$ is a conservative vector field on an open region D of $\mathbb{R}^3$. Let $\mathbf{F} = \nabla \varphi$, where φ is a potential function with continuous second partial derivatives on D. Then $\nabla \times \mathbf{F} = \nabla \times \nabla \varphi = \mathbf{0}$: The curl of the gradient is the zero vector and $\mathbf{F}$ is irrotational.

Proof: We must calculate $\nabla \times \nabla \varphi$:

$$\nabla \times \nabla \varphi = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \varphi_x & \varphi_y & \varphi_z \end{vmatrix} = \underbrace{(\varphi_{zy} - \varphi_{yz})}_{0}\,\mathbf{i} + \underbrace{(\varphi_{xz} - \varphi_{zx})}_{0}\,\mathbf{j} + \underbrace{(\varphi_{yx} - \varphi_{xy})}_{0}\,\mathbf{k} = \mathbf{0}.$$

The mixed partial derivatives are equal by Clairaut's Theorem (Theorem 15.4).

The converse of this theorem (if $\nabla \times \mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is a conservative field) is handled in Section 17.7 by means of Stokes' Theorem. ◄

➤ First note that $\nabla \times \mathbf{F}$ is a vector, so it makes sense to take the divergence of the curl.

THEOREM 17.12 **Divergence of the Curl**
Suppose $\mathbf{F} = \langle f, g, h \rangle$, where f, g, and h have continuous second partial derivatives. Then $\nabla \cdot (\nabla \times \mathbf{F}) = 0$: The divergence of the curl is zero.

Proof: Again, a calculation is needed:

$$\begin{aligned}
&\nabla \cdot (\nabla \times \mathbf{F}) \\
&\quad = \frac{\partial}{\partial x}\left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) \\
&\quad = \underbrace{(h_{yx} - h_{xy})}_{0} + \underbrace{(g_{xz} - g_{zx})}_{0} + \underbrace{(f_{zy} - f_{yz})}_{0} = 0.
\end{aligned}$$

Clairaut's Theorem (Theorem 15.4) ensures that the mixed partial derivatives are equal. ◄

The gradient, the divergence, and the curl may be combined in many ways—some of which are undefined. For example, the gradient of the curl $(\nabla(\nabla \times \mathbf{F}))$ and the curl of the divergence $(\nabla \times (\nabla \cdot \mathbf{F}))$ are undefined. However, a combination that *is* defined and is important is the divergence of the gradient $\nabla \cdot \nabla u$, where u is a scalar-valued function. This combination is denoted $\nabla^2 u$ and is called the **Laplacian** of u; it arises in many physical situations (Exercises 56–58, 62). Carrying out the calculation, we find that

$$\nabla \cdot \nabla u = \frac{\partial}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial}{\partial y}\frac{\partial u}{\partial y} + \frac{\partial}{\partial z}\frac{\partial u}{\partial z} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$

We close with a result that is useful in its own right but is also intriguing because it parallels the Product Rule from single-variable calculus.

THEOREM 17.13 Product Rule for the Divergence
Let u be a scalar-valued function that is differentiable on a region D and let $\mathbf{F}$ be a vector field that is differentiable on D. Then

$$\nabla \cdot (u\mathbf{F}) = \nabla u \cdot \mathbf{F} + u(\nabla \cdot \mathbf{F}).$$

The rule says that the "derivative" of the product is the "derivative" of the first function multiplied by the second function plus the first function multiplied by the "derivative" of the second function. However, in each instance, "derivative" must be interpreted correctly for the operations to make sense. The proof of the theorem requires a direct calculation (Exercise 67). Other similar vector calculus identities are presented in Exercises 68–72.

QUICK CHECK 4 Is $\nabla \cdot (u\mathbf{F})$ a vector function or a scalar function? ◄

EXAMPLE 5 More properties of radial fields Let $\mathbf{r} = \langle x, y, z \rangle$ and let $\varphi = \dfrac{1}{|\mathbf{r}|} = (x^2 + y^2 + z^2)^{-1/2}$ be a potential function.

a. Find the associated gradient field $\mathbf{F} = \nabla\left(\dfrac{1}{|\mathbf{r}|}\right)$.

b. Compute $\nabla \cdot \mathbf{F}$.

SOLUTION

a. The gradient has three components. Computing the first component reveals a pattern:

$$\frac{\partial \varphi}{\partial x} = \frac{\partial}{\partial x}(x^2 + y^2 + z^2)^{-1/2} = -\frac{1}{2}(x^2 + y^2 + z^2)^{-3/2}\,2x = -\frac{x}{|\mathbf{r}|^3}.$$

Making a similar calculation for the y- and z-derivatives, the gradient is

$$\mathbf{F} = \nabla\left(\frac{1}{|\mathbf{r}|}\right) = -\frac{\langle x, y, z\rangle}{|\mathbf{r}|^3} = -\frac{\mathbf{r}}{|\mathbf{r}|^3}.$$

This result reveals that $\mathbf{F}$ is an inverse square vector field (for example, a gravitational or electric field), and its potential function is $\varphi = \dfrac{1}{|\mathbf{r}|}$.

b. The divergence $\nabla \cdot \mathbf{F} = \nabla \cdot \left(-\frac{\mathbf{r}}{|\mathbf{r}|^3} \right)$ involves a product of the vector function $\mathbf{r} = \langle x, y, z \rangle$ and the scalar function $|\mathbf{r}|^{-3}$. Applying Theorem 17.13, we find that

$$\nabla \cdot \mathbf{F} = \nabla \cdot \left(-\frac{\mathbf{r}}{|\mathbf{r}|^3} \right) = -\nabla \frac{1}{|\mathbf{r}|^3} \cdot \mathbf{r} - \frac{1}{|\mathbf{r}|^3} \nabla \cdot \mathbf{r}.$$

A calculation similar to part (a) shows that $\nabla \frac{1}{|\mathbf{r}|^3} = -\frac{3\mathbf{r}}{|\mathbf{r}|^5}$ (Exercise 35). Therefore,

$$\begin{aligned} \nabla \cdot \mathbf{F} = \nabla \cdot \left(-\frac{\mathbf{r}}{|\mathbf{r}|^3} \right) &= -\underbrace{\nabla \frac{1}{|\mathbf{r}|^3}}_{-3\mathbf{r}/|\mathbf{r}|^5} \cdot \mathbf{r} - \frac{1}{|\mathbf{r}|^3} \underbrace{\nabla \cdot \mathbf{r}}_{3} \\ &= \frac{3\mathbf{r}}{|\mathbf{r}|^5} \cdot \mathbf{r} - \frac{3}{|\mathbf{r}|^3} && \text{Substitute for } \nabla \frac{1}{|\mathbf{r}|^3}. \\ &= \frac{3|\mathbf{r}|^2}{|\mathbf{r}|^5} - \frac{3}{|\mathbf{r}|^3} && \mathbf{r} \cdot \mathbf{r} = |\mathbf{r}|^2 \\ &= 0. \end{aligned}$$

The result is consistent with Theorem 17.10 (with $p = 3$): The divergence of an inverse square vector field in $\mathbb{R}^3$ is zero. It does not happen for any other radial fields of this form.

Related Exercises 35–36 ◄

Summary of Properties of Conservative Vector Fields

We can now extend the list of equivalent properties of conservative vector fields **F** defined on an open connected region. Theorem 17.11 is added to the list given at the end of Section 17.3.

> **Properties of a Conservative Vector Field**
>
> Let **F** be a conservative vector field whose components have continuous second partial derivatives on an open connected region D in $\mathbb{R}^3$. Then **F** has the following equivalent properties.
>
> **1.** There exists a potential function φ such that $\mathbf{F} = \nabla \varphi$ (definition).
>
> **2.** $\int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(B) - \varphi(A)$ for all points A and B in D and all piecewise-smooth oriented curves C in D from A to B.
>
> **3.** $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ on all simple piecewise-smooth closed oriented curves C in D.
>
> **4.** $\nabla \times \mathbf{F} = \mathbf{0}$ at all points of D.

SECTION 17.5 EXERCISES

Getting Started

1. Explain how to compute the divergence of the vector field $\mathbf{F} = \langle f, g, h \rangle$.

2. Interpret the divergence of a vector field.

3. What does it mean if the divergence of a vector field is zero throughout a region?

4. Explain how to compute the curl of the vector field $\mathbf{F} = \langle f, g, h \rangle$.

5. Interpret the curl of a general rotation vector field.

6. What does it mean if the curl of a vector field is zero throughout a region?

7. What is the value of $\nabla \cdot (\nabla \times \mathbf{F})$?

8. What is the value of $\nabla \times \nabla u$?

Practice Exercises

9–16. Divergence of vectors fields *Find the divergence of the following vector fields.*

9. $\mathbf{F} = \langle 2x, 4y, -3z \rangle$

10. $\mathbf{F} = \langle -2y, 3x, z \rangle$

11. $\mathbf{F} = \langle 12x, -6y, -6z \rangle$ **12.** $\mathbf{F} = \langle x^2yz, -xy^2z, -xyz^2 \rangle$

13. $\mathbf{F} = \langle x^2 - y^2, y^2 - z^2, z^2 - x^2 \rangle$

14. $\mathbf{F} = \langle e^{-x+y}, e^{-y+z}, e^{-z+x} \rangle$

15. $\mathbf{F} = \dfrac{\langle x, y, z \rangle}{1 + x^2 + y^2}$

16. $\mathbf{F} = \langle yz \sin x, xz \cos y, xy \cos z \rangle$

17–20. Divergence of radial fields *Calculate the divergence of the following radial fields. Express the result in terms of the position vector* $\mathbf{r}$ *and its length* $|\mathbf{r}|$. *Check for agreement with Theorem 17.10.*

17. $\mathbf{F} = \dfrac{\langle x, y, z \rangle}{x^2 + y^2 + z^2} = \dfrac{\mathbf{r}}{|\mathbf{r}|^2}$

18. $\mathbf{F} = \dfrac{\langle x, y, z \rangle}{(x^2 + y^2 + z^2)^{3/2}} = \dfrac{\mathbf{r}}{|\mathbf{r}|^3}$

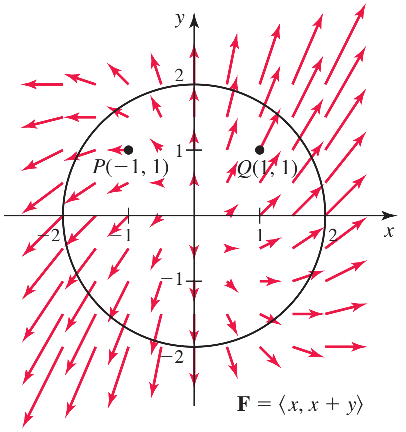
19. $\mathbf{F} = \dfrac{\langle x, y, z \rangle}{(x^2 + y^2 + z^2)^2} = \dfrac{\mathbf{r}}{|\mathbf{r}|^4}$

20. $\mathbf{F} = \langle x, y, z \rangle (x^2 + y^2 + z^2) = \mathbf{r}|\mathbf{r}|^2$

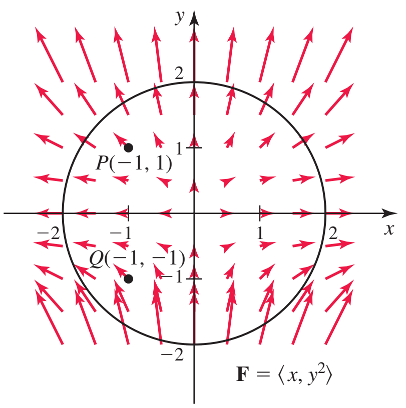
21–22. Divergence and flux from graphs *Consider the following vector fields, the circle C, and two points P and Q.*

a. *Without computing the divergence, does the graph suggest that the divergence is positive or negative at P and Q? Justify your answer.*
b. *Compute the divergence and confirm your conjecture in part (a).*
c. *On what part of C is the flux outward? Inward?*
d. *Is the net outward flux across C positive or negative?*

21. $\mathbf{F} = \langle x, x + y \rangle$



22. $\mathbf{F} = \langle x, y^2 \rangle$



23–26. Curl of a rotation field *Consider the following vector fields, where* $\mathbf{r} = \langle x, y, z \rangle$.

a. *Compute the curl of the field and verify that it has the same direction as the axis of rotation.*
b. *Compute the magnitude of the curl of the field.*

23. $\mathbf{F} = \langle 1, 0, 0 \rangle \times \mathbf{r}$ **24.** $\mathbf{F} = \langle 1, -1, 0 \rangle \times \mathbf{r}$

25. $\mathbf{F} = \langle 1, -1, 1 \rangle \times \mathbf{r}$ **26.** $\mathbf{F} = \langle 1, -2, -3 \rangle \times \mathbf{r}$

27–34. Curl of a vector field *Compute the curl of the following vector fields.*

27. $\mathbf{F} = \langle x^2 - y^2, xy, z \rangle$ **28.** $\mathbf{F} = \langle 0, z^2 - y^2, -yz \rangle$

29. $\mathbf{F} = \langle x^2 - z^2, 1, 2xz \rangle$ **30.** $\mathbf{F} = \mathbf{r} = \langle x, y, z \rangle$

31. $\mathbf{F} = \dfrac{\langle x, y, z \rangle}{(x^2 + y^2 + z^2)^{3/2}} = \dfrac{\mathbf{r}}{|\mathbf{r}|^3}$

32. $\mathbf{F} = \dfrac{\langle x, y, z \rangle}{(x^2 + y^2 + z^2)^{1/2}} = \dfrac{\mathbf{r}}{|\mathbf{r}|}$

33. $\mathbf{F} = \langle z^2 \sin y, xz^2 \cos y, 2xz \sin y \rangle$

34. $\mathbf{F} = \langle 3xz^3e^{y^2}, 2xz^3e^{y^2}, 3xz^2e^{y^2} \rangle$

35–38. Derivative rules *Prove the following identities. Use Theorem 17.13 (Product Rule) whenever possible.*

35. $\nabla\left(\dfrac{1}{|\mathbf{r}|^3}\right) = -\dfrac{3\mathbf{r}}{|\mathbf{r}|^5}$ (used in Example 5)

36. $\nabla\left(\dfrac{1}{|\mathbf{r}|^2}\right) = -\dfrac{2\mathbf{r}}{|\mathbf{r}|^4}$

37. $\nabla \cdot \nabla\left(\dfrac{1}{|\mathbf{r}|^2}\right) = \dfrac{2}{|\mathbf{r}|^4}$ (*Hint:* Use Exercise 36.)

38. $\nabla(\ln |\mathbf{r}|) = \dfrac{\mathbf{r}}{|\mathbf{r}|^2}$

39. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. For a function f of a single variable, if $f'(x) = 0$ for all x in the domain, then f is a constant function. If $\nabla \cdot \mathbf{F} = 0$ for all points in the domain, then $\mathbf{F}$ is constant.
b. If $\nabla \times \mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is constant.
c. A vector field consisting of parallel vectors has zero curl.
d. A vector field consisting of parallel vectors has zero divergence.
e. curl $\mathbf{F}$ is orthogonal to $\mathbf{F}$.

40. Another derivative combination Let $\mathbf{F} = \langle f, g, h \rangle$ and let u be a differentiable scalar-valued function.

a. Take the dot product of $\mathbf{F}$ and the del operator; then apply the result to u to show that

$$(\mathbf{F} \cdot \nabla)u = \left(f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y} + h\frac{\partial}{\partial z}\right)u = f\frac{\partial u}{\partial x} + g\frac{\partial u}{\partial y} + h\frac{\partial u}{\partial z}.$$

b. Evaluate $(\mathbf{F} \cdot \nabla)(xy^2z^3)$ at $(1, 1, 1)$, where $\mathbf{F} = \langle 1, 1, 1 \rangle$.

41. Does it make sense? Are the following expressions defined? If so, state whether the result is a scalar or a vector. Assume **F** is a sufficiently differentiable vector field and φ is a sufficiently differentiable scalar-valued function.

a. $\nabla \cdot \varphi$ **b.** $\nabla \mathbf{F}$ **c.** $\nabla \cdot \nabla\varphi$
d. $\nabla(\nabla \cdot \varphi)$ **e.** $\nabla(\nabla \times \varphi)$ **f.** $\nabla \cdot (\nabla \cdot \mathbf{F})$
g. $\nabla \times \nabla\varphi$ **h.** $\nabla \times (\nabla \cdot \mathbf{F})$ **i.** $\nabla \times (\nabla \times \mathbf{F})$

42. Zero divergence of the rotation field Show that the general rotation field $\mathbf{F} = \mathbf{a} \times \mathbf{r}$, where **a** is a nonzero constant vector and $\mathbf{r} = \langle x, y, z\rangle$, has zero divergence.

43. General rotation fields

a. Let $\mathbf{a} = \langle 0, 1, 0\rangle$, let $\mathbf{r} = \langle x, y, z\rangle$, and consider the rotation field $\mathbf{F} = \mathbf{a} \times \mathbf{r}$. Use the right-hand rule for cross products to find the direction of **F** at the points $(0, 1, 1)$, $(1, 1, 0)$, $(0, 1, -1)$, and $(-1, 1, 0)$.
b. With $\mathbf{a} = \langle 0, 1, 0\rangle$, explain why the rotation field $\mathbf{F} = \mathbf{a} \times \mathbf{r}$ circles the y-axis in the counterclockwise direction looking along **a** from head to tail (that is, in the negative y-direction).

44. General rotation fields Generalize Exercise 43 to show that the rotation field $\mathbf{F} = \mathbf{a} \times \mathbf{r}$ circles the vector **a** in the counterclockwise direction looking along **a** from head to tail.

45. Curl of the rotation field For the general rotation field $\mathbf{F} = \mathbf{a} \times \mathbf{r}$, where **a** is a nonzero constant vector and $\mathbf{r} = \langle x, y, z\rangle$, show that curl $\mathbf{F} = 2\mathbf{a}$.

46. Inward to outward Find the exact points on the circle $x^2 + y^2 = 2$ at which the field $\mathbf{F} = \langle f, g\rangle = \langle x^2, y\rangle$ switches from pointing inward to pointing outward on the circle, or vice versa.

47. Maximum divergence Within the cube $\{(x, y, z): |x| \le 1, |y| \le 1, |z| \le 1\}$, where does div **F** have the greatest magnitude when $\mathbf{F} = \langle x^2 - y^2, xy^2 z, 2xz\rangle$?

48. Maximum curl Let $\mathbf{F} = \langle z, 0, -y\rangle$.

a. Find the scalar component of curl **F** in the direction of the unit vector $\mathbf{n} = \langle 1, 0, 0\rangle$.
b. Find the scalar component of curl **F** in the direction of the unit vector $\mathbf{n} = \left\langle \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$.
c. Find the unit vector **n** that maximizes $\text{scal}_{\mathbf{n}}\langle -1, 1, 0\rangle$ and state the value of $\text{scal}_{\mathbf{n}}\langle -1, 1, 0\rangle$ in this direction.

49. Zero component of the curl For what vectors **n** is $(\text{curl } \mathbf{F}) \cdot \mathbf{n} = 0$ when $\mathbf{F} = \langle y, -2z, -x\rangle$?

50–51. Find a vector field *Find a vector field* **F** *with the given curl. In each case, is the vector field you found unique?*

50. curl $\mathbf{F} = \langle 0, 1, 0\rangle$ **51.** curl $\mathbf{F} = \langle 0, z, -y\rangle$

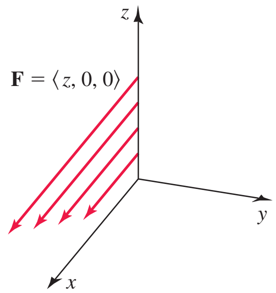
Explorations and Challenges

52. Curl and angular speed Consider the rotational velocity field $\mathbf{v} = \mathbf{a} \times \mathbf{r}$, where **a** is a nonzero constant vector and $\mathbf{r} = \langle x, y, z\rangle$. Use the fact that an object moving in a circular path of radius R with speed $|\mathbf{v}|$ has an angular speed of $\omega = |\mathbf{v}|/R$.

a. Sketch a position vector **a**, which is the axis of rotation for the vector field, and a position vector **r** of a point P in $\mathbb{R}^3$. Let θ be the angle between the two vectors. Show that the perpendicular distance from P to the axis of rotation is $R = |\mathbf{r}|\sin\theta$.
b. Show that the speed of a particle in the velocity field is $|\mathbf{a} \times \mathbf{r}|$ and that the angular speed of the object is $|\mathbf{a}|$.
c. Conclude that $\omega = \frac{1}{2}|\nabla \times \mathbf{v}|$.

53. Paddle wheel in a vector field Let $\mathbf{F} = \langle z, 0, 0\rangle$ and let **n** be a unit vector aligned with the axis of a paddle wheel located on the x-axis (see figure).

a. If the paddle wheel is oriented with $\mathbf{n} = \langle 1, 0, 0\rangle$, in what direction (if any) does the wheel spin?
b. If the paddle wheel is oriented with $\mathbf{n} = \langle 0, 1, 0\rangle$, in what direction (if any) does the wheel spin?
c. If the paddle wheel is oriented with $\mathbf{n} = \langle 0, 0, 1\rangle$, in what direction (if any) does the wheel spin?



54. Angular speed Consider the rotational velocity field $\mathbf{v} = \langle -2y, 2z, 0\rangle$.

a. If a paddle wheel is placed in the xy-plane with its axis normal to this plane, what is its angular speed?
b. If a paddle wheel is placed in the xz-plane with its axis normal to this plane, what is its angular speed?
c. If a paddle wheel is placed in the yz-plane with its axis normal to this plane, what is its angular speed?

55. Angular speed Consider the rotational velocity field $\mathbf{v} = \langle 0, 10z, -10y\rangle$. If a paddle wheel is placed in the plane $x + y + z = 1$ with its axis normal to this plane, how fast does the paddle wheel spin (in revolutions per unit time)?

56–58. Heat flux *Suppose a solid object in* $\mathbb{R}^3$ *has a temperature distribution given by* $T(x, y, z)$. *The heat flow vector field in the object is* $\mathbf{F} = -k\nabla T$, *where the conductivity* $k > 0$ *is a property of the material. Note that the heat flow vector points in the direction opposite to that of the gradient, which is the direction of greatest temperature decrease. The divergence of the heat flow vector is* $\nabla \cdot \mathbf{F} = -k\nabla \cdot \nabla T = -k\nabla^2 T$ *(the Laplacian of* T*). Compute the heat flow vector field and its divergence for the following temperature distributions.*

56. $T(x, y, z) = 100e^{-\sqrt{x^2+y^2+z^2}}$

57. $T(x, y, z) = 100e^{-x^2+y^2+z^2}$

58. $T(x, y, z) = 100\left(1 + \sqrt{x^2 + y^2 + z^2}\right)$

59. Gravitational potential The potential function for the gravitational force field due to a mass M at the origin acting on a mass m is $\varphi = GMm/|\mathbf{r}|$, where $\mathbf{r} = \langle x, y, z\rangle$ is the position vector of the mass m, and G is the gravitational constant.

a. Compute the gravitational force field $\mathbf{F} = -\nabla\varphi$.
b. Show that the field is irrotational; that is, show that $\nabla \times \mathbf{F} = \mathbf{0}$.

60. Electric potential The potential function for the force field due to a charge q at the origin is $\varphi = \dfrac{1}{4\pi\varepsilon_0}\dfrac{q}{|\mathbf{r}|}$, where $\mathbf{r} = \langle x, y, z\rangle$ is the position vector of a point in the field, and ε_0 is the permittivity of free space.

a. Compute the force field $\mathbf{F} = -\nabla\varphi$.
b. Show that the field is irrotational; that is, show that $\nabla \times \mathbf{F} = \mathbf{0}$.

61. Navier-Stokes equation The Navier-Stokes equation is the fundamental equation of fluid dynamics that models the flow in everything from bathtubs to oceans. In one of its many forms (incompressible, viscous flow), the equation is

$$\rho\left(\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V}\right) = -\nabla p + \mu(\nabla\cdot\nabla)\mathbf{V}.$$

In this notation, $\mathbf{V} = \langle u, v, w\rangle$ is the three-dimensional velocity field, p is the (scalar) pressure, ρ is the constant density of the fluid, and μ is the constant viscosity. Write out the three component equations of this vector equation. (See Exercise 40 for an interpretation of the operations.)

T 62. Stream function and vorticity The rotation of a three-dimensional velocity field $\mathbf{V} = \langle u, v, w\rangle$ is measured by the **vorticity** $\omega = \nabla \times \mathbf{V}$. If $\omega = \mathbf{0}$ at all points in the domain, the flow is irrotational.

a. Which of the following velocity fields is irrotational: $\mathbf{V} = \langle 2, -3y, 5z\rangle$ or $\mathbf{V} = \langle y, x - z, -y\rangle$?
b. Recall that for a two-dimensional source-free flow $\mathbf{V} = \langle u, v, 0\rangle$, a stream function $\psi(x, y)$ may be defined such that $u = \psi_y$ and $v = -\psi_x$. For such a two-dimensional flow, let $\zeta = \mathbf{k}\cdot\nabla\times\mathbf{V}$ be the $\mathbf{k}$-component of the vorticity. Show that $\nabla^2\psi = \nabla\cdot\nabla\psi = -\zeta$.
c. Consider the stream function $\psi(x, y) = \sin x \sin y$ on the square region $R = \{(x, y): 0 \le x \le \pi, 0 \le y \le \pi\}$. Find the velocity components u and v; then sketch the velocity field.
d. For the stream function in part (c), find the vorticity function ζ as defined in part (b). Plot several level curves of the vorticity function. Where on R is it a maximum? A minimum?

63. Ampère's Law One of Maxwell's equations for electromagnetic waves is $\nabla \times \mathbf{B} = C\dfrac{\partial \mathbf{E}}{\partial t}$, where $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic field, and C is a constant.

a. Show that the fields

$$\mathbf{E}(z, t) = A\sin(kz - \omega t)\,\mathbf{i} \text{ and } \mathbf{B}(z, t) = A\sin(kz - \omega t)\,\mathbf{j}$$

satisfy the equation for constants A, k, and ω, provided $\omega = k/C$.
b. Make a rough sketch showing the directions of $\mathbf{E}$ and $\mathbf{B}$.

64. Splitting a vector field Express the vector field $\mathbf{F} = \langle xy, 0, 0\rangle$ in the form $\mathbf{V} + \mathbf{W}$, where $\nabla\cdot\mathbf{V} = 0$ and $\nabla \times \mathbf{W} = \mathbf{0}$.

65. Properties of div and curl Prove the following properties of the divergence and curl. Assume $\mathbf{F}$ and $\mathbf{G}$ are differentiable vector fields and c is a real number.

a. $\nabla\cdot(\mathbf{F} + \mathbf{G}) = \nabla\cdot\mathbf{F} + \nabla\cdot\mathbf{G}$
b. $\nabla \times (\mathbf{F} + \mathbf{G}) = (\nabla \times \mathbf{F}) + (\nabla \times \mathbf{G})$
c. $\nabla\cdot(c\mathbf{F}) = c(\nabla\cdot\mathbf{F})$
d. $\nabla \times (c\mathbf{F}) = c(\nabla \times \mathbf{F})$

66. Equal curls If two functions of one variable, f and g, have the property that $f' = g'$, then f and g differ by a constant. Prove or disprove: If $\mathbf{F}$ and $\mathbf{G}$ are nonconstant vector fields in $\mathbb{R}^2$ with curl $\mathbf{F}$ = curl $\mathbf{G}$ and div $\mathbf{F}$ = div $\mathbf{G}$ at all points of $\mathbb{R}^2$, then $\mathbf{F}$ and $\mathbf{G}$ differ by a constant vector.

67–72. Identities *Prove the following identities. Assume φ is a differentiable scalar-valued function and* $\mathbf{F}$ *and* $\mathbf{G}$ *are differentiable vector fields, all defined on a region of* $\mathbb{R}^3$.

67. $\nabla\cdot(\varphi\mathbf{F}) = \nabla\varphi\cdot\mathbf{F} + \varphi\nabla\cdot\mathbf{F}$ (Product Rule)

68. $\nabla \times (\varphi\mathbf{F}) = (\nabla\varphi \times \mathbf{F}) + (\varphi\nabla \times \mathbf{F})$ (Product Rule)

69. $\nabla\cdot(\mathbf{F} \times \mathbf{G}) = \mathbf{G}\cdot(\nabla \times \mathbf{F}) - \mathbf{F}\cdot(\nabla \times \mathbf{G})$

70. $\nabla \times (\mathbf{F} \times \mathbf{G}) = (\mathbf{G}\cdot\nabla)\mathbf{F} - \mathbf{G}(\nabla\cdot\mathbf{F}) - (\mathbf{F}\cdot\nabla)\mathbf{G} + \mathbf{F}(\nabla\cdot\mathbf{G})$

71. $\nabla(\mathbf{F}\cdot\mathbf{G}) = (\mathbf{G}\cdot\nabla)\mathbf{F} + (\mathbf{F}\cdot\nabla)\mathbf{G} + \mathbf{G} \times (\nabla \times \mathbf{F}) + \mathbf{F} \times (\nabla \times \mathbf{G})$

72. $\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla\cdot\mathbf{F}) - (\nabla\cdot\nabla)\mathbf{F}$

73. Divergence of radial fields Prove that for a real number p, with $\mathbf{r} = \langle x, y, z\rangle$, $\nabla\cdot\dfrac{\langle x, y, z\rangle}{|\mathbf{r}|^p} = \dfrac{3 - p}{|\mathbf{r}|^p}$.

74. Gradients and radial fields Prove that for a real number p, with $\mathbf{r} = \langle x, y, z\rangle$, $\nabla\left(\dfrac{1}{|\mathbf{r}|^p}\right) = \dfrac{-p\mathbf{r}}{|\mathbf{r}|^{p+2}}$.

75. Divergence of gradient fields Prove that for a real number p, with $\mathbf{r} = \langle x, y, z\rangle$, $\nabla\cdot\nabla\left(\dfrac{1}{|\mathbf{r}|^p}\right) = \dfrac{p(p - 1)}{|\mathbf{r}|^{p+2}}$.

QUICK CHECK ANSWERS

1. The x-derivative of the divergence is applied to $f(y, z)$, which gives zero. Similarly, the y- and z-derivatives are zero. **2.** The net outward flux is 4π. **3.** In the curl, the first component of $\mathbf{F}$ is differentiated only with respect to y and z, so the contribution from the first component is zero. Similarly, the second and third components of $\mathbf{F}$ make no contribution to the curl. **4.** The divergence is a scalar-valued function. ➤

17.6 Surface Integrals

We have studied integrals on the real line, on regions in the plane, on solid regions in space, and along curves in space. One situation is still unexplored. Suppose a sphere has a known temperature distribution; perhaps it is cold near the poles and warm near the equator. How do you find the average temperature over the entire sphere? In analogy with other average value calculations, we should expect to "add up" the temperature values over the sphere and divide by the surface area of the sphere. Because the temperature varies continuously over the sphere, adding up means integrating. How do you integrate a function over a surface? This question leads to *surface integrals*.

Parallel Concepts	
Curves	**Surfaces**
Arc length	Surface area
Line integrals	Surface integrals
One-parameter description	Two-parameter description

It helps to keep curves, arc length, and line integrals in mind as we discuss surfaces, surface area, and surface integrals. What we discover about surfaces parallels what we already know about curves—all "lifted" up one dimension.

Parameterized Surfaces

A curve in $\mathbb{R}^2$ is defined parametrically by $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, for $a \le t \le b$; it requires one parameter and two dependent variables. Stepping up one dimension to define a surface in $\mathbb{R}^3$, we need *two* parameters and *three* dependent variables. Letting u and v be parameters, the general parametric description of a surface has the form

$$\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle.$$

We make the assumption that the parameters vary over a rectangle $R = \{(u, v): a \le u \le b, c \le v \le d\}$ (**Figure 17.43**). As the parameters (u, v) vary over R, the vector $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$ sweeps out a surface S in $\mathbb{R}^3$.

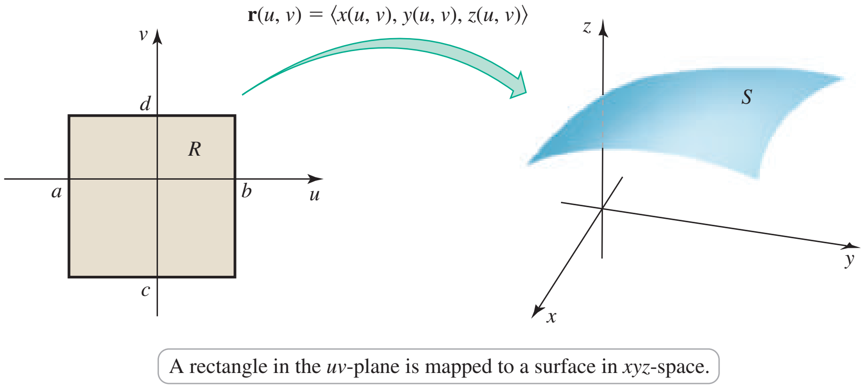


Figure 17.43

We work extensively with three surfaces that are easily described in parametric form. As with parameterized curves, a parametric description of a surface is not unique.

Cylinders In Cartesian coordinates, the set

$$\{(x, y, z): x = a \cos \theta, y = a \sin \theta, 0 \le \theta \le 2\pi, 0 \le z \le h\},$$

where $a > 0$, is a cylindrical surface of radius a and height h with its axis along the z-axis. Using the parameters $u = \theta$ and $v = z$, a parametric description of the cylinder is

$$\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle = \langle a \cos u, a \sin u, v \rangle,$$

where $0 \le u \le 2\pi$ and $0 \le v \le h$ (**Figure 17.44**).

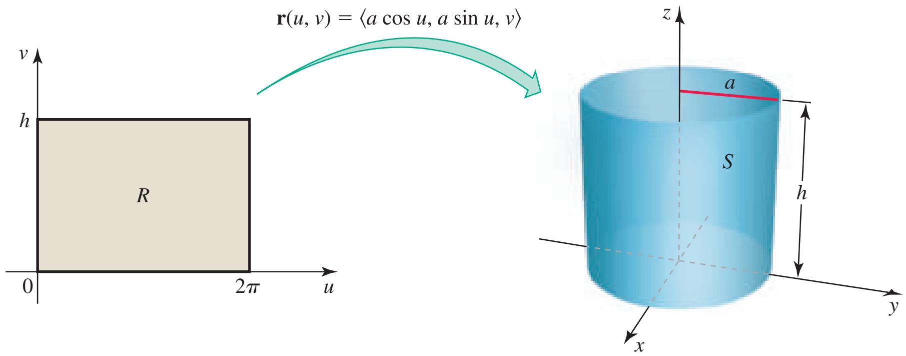


Figure 17.44

QUICK CHECK 1 Describe the surface $\mathbf{r}(u, v) = \langle 2 \cos u, 2 \sin u, v \rangle$, for $0 \le u \le \pi$ and $0 \le v \le 1$. ◄

Cones The surface of a cone of height h and radius a with its vertex at the origin is described in cylindrical coordinates by

$$\{(r, \theta, z): 0 \le r \le a, 0 \le \theta \le 2\pi, z = rh/a\}.$$

➤ Note that when $r = 0$, $z = 0$ and when $r = a$, $z = h$.

For a fixed value of z, we have $r = az/h$; therefore, on the surface of the cone,

➤ Recall the relationships between polar and rectangular coordinates:

$x = r\cos\theta$, $y = r\sin\theta$, and $x^2 + y^2 = r^2$.

$$x = r\cos\theta = \frac{az}{h}\cos\theta \quad \text{and} \quad y = r\sin\theta = \frac{az}{h}\sin\theta.$$

Using the parameters $u = \theta$ and $v = z$, the parametric description of the conical surface is

$$\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle = \left\langle \frac{av}{h}\cos u, \frac{av}{h}\sin u, v \right\rangle,$$

where $0 \le u \le 2\pi$ and $0 \le v \le h$ (Figure 17.45).

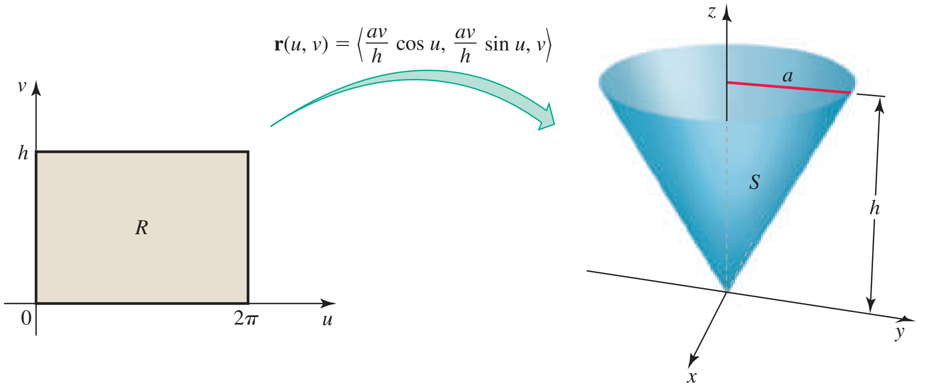


Figure 17.45

QUICK CHECK 2 Describe the surface $\mathbf{r}(u, v) = \langle v\cos u, v\sin u, v \rangle$, for $0 \le u \le \pi$ and $0 \le v \le 10$. ◄

Spheres The parametric description of a sphere of radius a centered at the origin comes directly from spherical coordinates:

$$\{(\rho, \varphi, \theta): \rho = a, 0 \le \varphi \le \pi, 0 \le \theta \le 2\pi\}.$$

➤ The complete cylinder, cone, and sphere are generated as the angle variable θ varies over the half-open interval $[0, 2\pi)$. As in previous chapters, we will use the closed interval $[0, 2\pi]$.

Recall the following relationships among spherical and rectangular coordinates (Section 16.5):

$$x = a\sin\varphi\cos\theta, \qquad y = a\sin\varphi\sin\theta, \qquad z = a\cos\varphi.$$

When we define the parameters $u = \varphi$ and $v = \theta$, a parametric description of the sphere is

$$\mathbf{r}(u, v) = \langle a\sin u\cos v, a\sin u\sin v, a\cos u \rangle,$$

where $0 \le u \le \pi$ and $0 \le v \le 2\pi$ (Figure 17.46).

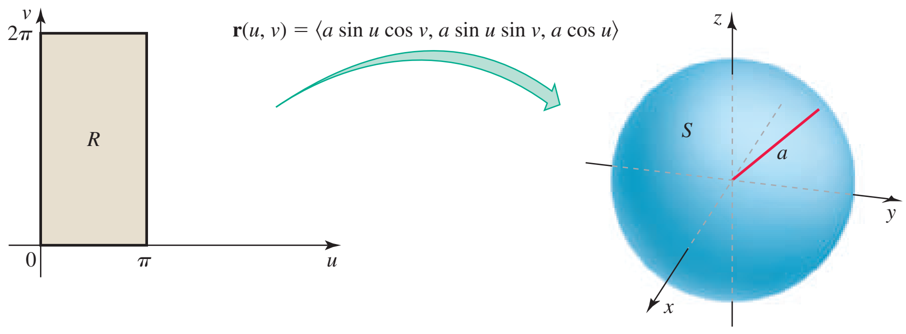


Figure 17.46

QUICK CHECK 3 Describe the surface $\mathbf{r}(u, v) = \langle 4\sin u\cos v, 4\sin u\sin v, 4\cos u \rangle$, for $0 \le u \le \pi/2$ and $0 \le v \le \pi$. ◄

EXAMPLE 1 **Parametric surfaces** Find parametric descriptions for the following surfaces.

a. The plane $3x - 2y + z = 2$

b. The paraboloid $z = x^2 + y^2$, for $0 \le z \le 9$

SOLUTION

a. Defining the parameters $u = x$ and $v = y$, we find that

$$z = 2 - 3x + 2y = 2 - 3u + 2v.$$

Therefore, a parametric description of the plane is

$$\mathbf{r}(u, v) = \langle u, v, 2 - 3u + 2v \rangle,$$

for $-\infty < u < \infty$ and $-\infty < v < \infty$.

b. Thinking in terms of polar coordinates, we let $u = \theta$ and $v = \sqrt{z}$, which means that $z = v^2$. The equation of the paraboloid is $x^2 + y^2 = z = v^2$, so v plays the role of the polar coordinate r. Therefore, $x = v \cos \theta = v \cos u$ and $y = v \sin \theta = v \sin u$. A parametric description for the paraboloid is

$$\mathbf{r}(u, v) = \langle v \cos u, v \sin u, v^2 \rangle,$$

where $0 \le u \le 2\pi$ and $0 \le v \le 3$.

Alternatively, we could choose $u = \theta$ and $v = z$. The resulting description is

$$\mathbf{r}(u, v) = \langle \sqrt{v} \cos u, \sqrt{v} \sin u, v \rangle,$$

where $0 \le u \le 2\pi$ and $0 \le v \le 9$. *Related Exercises 9, 12* ◄

Surface Integrals of Scalar-Valued Functions

We now develop the surface integral of a scalar-valued function f defined on a smooth parameterized surface S described by the equation

$$\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle,$$

where the parameters vary over a rectangle $R = \{(u, v): a \le u \le b, c \le v \le d\}$. The functions x, y, and z are assumed to have continuous partial derivatives with respect to u and v. The rectangular region R in the uv-plane is partitioned into rectangles, with sides of length Δu and Δv, that are ordered in some convenient way, for $k = 1, \ldots, n$. The kth rectangle R_k, which has area $\Delta A = \Delta u \Delta v$, corresponds to a curved patch S_k on the surface S (**Figure 17.47**), which has area ΔS_k. We let (u_k, v_k) be the lower-left corner point of R_k. The parameterization then assigns (u_k, v_k) to a point $P(x(u_k, v_k), y(u_k, v_k), z(u_k, v_k))$, or more simply, $P(x_k, y_k, z_k)$, on S_k. To construct the surface integral, we define a Riemann sum, which adds up function values multiplied by areas of the respective patches:

➤ A more general approach allows (u_k, v_k) to be an arbitrary point in the kth rectangle. The outcome of the two approaches is the same.

$$\sum_{k=1}^{n} f(x(u_k, v_k), y(u_k, v_k), z(u_k, v_k)) \Delta S_k.$$

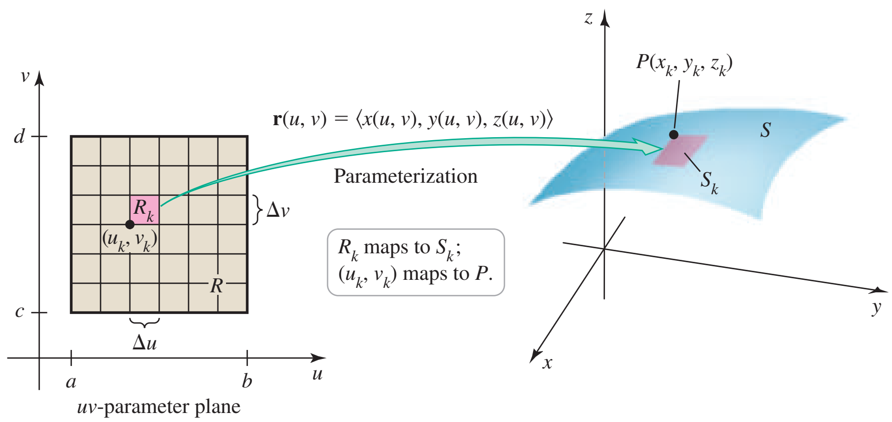


Figure 17.47

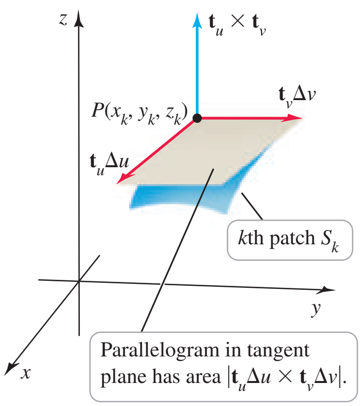


Figure 17.48

➤ In general, the vectors $\mathbf{t}_u$ and $\mathbf{t}_v$ are different for each patch, so they should carry a subscript k. To keep the notation as simple as possible, we have suppressed the subscripts on these vectors with the understanding that they change with k. These tangent vectors are given by partial derivatives because in each case, either u or v is held constant, while the other variable changes.

The crucial step is computing ΔS_k, the area of the kth patch S_k.

Figure 17.48 shows the patch S_k and the point $P(x_k, y_k, z_k)$. Two special vectors are tangent to the surface at P; these vectors lie in the plane tangent to S at P.

- $\mathbf{t}_u$ is a vector tangent to the surface corresponding to a change in u with v constant in the uv-plane.
- $\mathbf{t}_v$ is a vector tangent to the surface corresponding to a change in v with u constant in the uv-plane.

Because the surface S may be written $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$, a tangent vector corresponding to a change in u with v fixed is

$$\mathbf{t}_u = \frac{\partial \mathbf{r}}{\partial u} = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle.$$

Similarly, a tangent vector corresponding to a change in v with u fixed is

$$\mathbf{t}_v = \frac{\partial \mathbf{r}}{\partial v} = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle.$$

Now consider an increment Δu in u with v fixed; the corresponding change in $\mathbf{r}$, which is $\mathbf{r}(u + \Delta u, v) - \mathbf{r}(u, v)$, can be approximated using the definition of the partial derivative of $\mathbf{r}$ with respect to u. Specifically, when Δu is small, we have

$$\frac{\partial \mathbf{r}}{\partial u} \approx \frac{1}{\Delta u}\big(\mathbf{r}(u + \Delta u, v) - \mathbf{r}(u, v)\big).$$

Multiplying both sides of this equation by Δu and recalling that $\mathbf{t}_u = \frac{\partial \mathbf{r}}{\partial u}$, we see that the change in $\mathbf{r}$ corresponding to the increment Δu is approximated by the vector

$$\mathbf{t}_u \Delta u \approx \underbrace{\mathbf{r}(u + \Delta u, v) - \mathbf{r}(u, v)}_{\text{change in } \mathbf{r} \text{ corresponding to } \Delta u}.$$

Using a similar line of reasoning, the change in $\mathbf{r}$ corresponding to the increment Δv (with u fixed) is approximated by the vector

$$\mathbf{t}_v \Delta v \approx \underbrace{\mathbf{r}(u, v + \Delta v) - \mathbf{r}(u, v)}_{\text{change in } \mathbf{r} \text{ corresponding to } \Delta v}.$$

As nonzero scalar multiples of $\mathbf{t}_u$ and $\mathbf{t}_v$, the vectors $\mathbf{t}_u\Delta u$ and $\mathbf{t}_v\Delta v$ are also tangent to the surface. They determine a parallelogram that lies in the plane tangent to S at P (Figure 17.48); the area of this parallelogram approximates the area of the kth patch S_k, which is ΔS_k.

Appealing to the cross product (Section 13.4), the area of the parallelogram is

$$|\mathbf{t}_u\Delta u \times \mathbf{t}_v\Delta v| = |\mathbf{t}_u \times \mathbf{t}_v|\,\Delta u\,\Delta v \approx \Delta S_k.$$

Note that $\mathbf{t}_u \times \mathbf{t}_v$ is evaluated at (u_k, v_k) and is a vector normal to the surface at P, which we assume to be nonzero at all points of S.

We write the Riemann sum with the observation that the areas of the parallelograms approximate the areas of the patches S_k:

$$\begin{aligned}&\sum_{k=1}^{n} f(x(u_k, v_k), y(u_k, v_k), z(u_k, v_k))\Delta S_k \\ &\quad\approx \sum_{k=1}^{n} f(x(u_k, v_k), y(u_k, v_k), z(u_k, v_k))\underbrace{|\mathbf{t}_u \times \mathbf{t}_v|\,\Delta u\,\Delta v}_{\approx \Delta S_k}.\end{aligned}$$

We now assume f is continuous on S. As Δu and Δv approach zero, the areas of the parallelograms approach the areas of the corresponding patches on S. We define the limit

➤ The role that the factor $|\mathbf{t}_u \times \mathbf{t}_v|\, dA$ plays in surface integrals is analogous to the role played by $|\mathbf{r}'(t)|\, dt$ in line integrals.

of this Riemann sum to be the surface integral of f over S, which we write $\iint_S f(x, y, z)\, dS$. The surface integral is evaluated as an ordinary double integral over the region R in the uv-plane:

$$\iint_S f(x, y, z)\, dS = \lim_{\Delta u, \Delta v \to 0} \sum_{k=1}^{n} f(x(u_k, v_k), y(u_k, v_k), z(u_k, v_k))|\mathbf{t}_u \times \mathbf{t}_v|\,\Delta u\, \Delta v$$

$$= \iint_R f(x(u, v), y(u, v), z(u, v))|\mathbf{t}_u \times \mathbf{t}_v|\, dA.$$

If R is a rectangular region, as we have assumed, the double integral becomes an iterated integral with respect to u and v with constant limits. In the special case that $f(x, y, z) = 1$, the integral gives the surface area of S.

➤ The condition that $\mathbf{t}_u \times \mathbf{t}_v$ be nonzero means $\mathbf{t}_u$ and $\mathbf{t}_v$ are nonzero and not parallel. If $\mathbf{t}_u \times \mathbf{t}_v \neq \mathbf{0}$ at all points, then the surface is *smooth*. The value of the integral is independent of the parameterization of S.

DEFINITION Surface Integral of Scalar-Valued Functions on Parameterized Surfaces

Let f be a continuous scalar-valued function on a smooth surface S given parametrically by $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$, where u and v vary over $R = \{(u, v): a \le u \le b, c \le v \le d\}$. Assume also that the tangent vectors $\mathbf{t}_u = \dfrac{\partial \mathbf{r}}{\partial u} = \left\langle \dfrac{\partial x}{\partial u}, \dfrac{\partial y}{\partial u}, \dfrac{\partial z}{\partial u}\right\rangle$ and $\mathbf{t}_v = \dfrac{\partial \mathbf{r}}{\partial v} = \left\langle \dfrac{\partial x}{\partial v}, \dfrac{\partial y}{\partial v}, \dfrac{\partial z}{\partial v}\right\rangle$ are continuous on R and the normal vector $\mathbf{t}_u \times \mathbf{t}_v$ is nonzero on R. Then the **surface integral of f over S** is

$$\iint_S f(x, y, z)\, dS = \iint_R f(x(u, v), y(u, v), z(u, v))|\mathbf{t}_u \times \mathbf{t}_v|\, dA.$$

If $f(x, y, z) = 1$, this integral equals the surface area of S.

EXAMPLE 2 Surface area of a cylinder and sphere Find the surface area of the following surfaces.

a. A cylinder with radius $a > 0$ and height h (excluding the circular ends)

b. A sphere of radius a

SOLUTION The critical step is evaluating the normal vector $\mathbf{t}_u \times \mathbf{t}_v$. It needs to be done only once for any given surface.

a. As shown before, a parametric description of the cylinder is

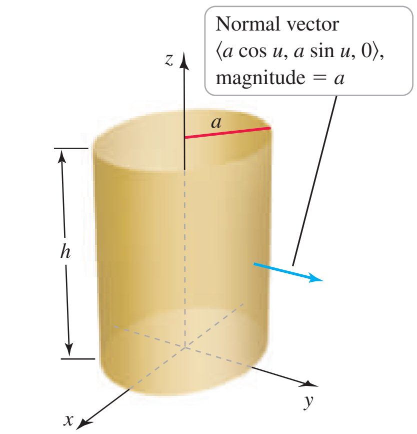
$$\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle = \langle a\cos u, a\sin u, v\rangle,$$

where $0 \le u \le 2\pi$ and $0 \le v \le h$. The required normal vector is

$$\mathbf{t}_u \times \mathbf{t}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & \dfrac{\partial z}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & \dfrac{\partial z}{\partial v} \end{vmatrix} \quad \text{Definition of cross product}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin u & a\cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} \quad \text{Evaluate derivatives.}$$

$$= \langle a\cos u, a\sin u, 0\rangle. \quad \text{Compute cross product.}$$



Cylinder: $\mathbf{r}(u, v) = \langle a\cos u, a\sin u, v\rangle$, $0 \le u \le 2\pi$ and $0 \le v \le h$

Figure 17.49

Notice that this normal vector points outward from the cylinder, away from the z-axis (Figure 17.49). It follows that

$$|\mathbf{t}_u \times \mathbf{t}_v| = \sqrt{a^2\cos^2 u + a^2\sin^2 u} = a.$$

Setting $f(x, y, z) = 1$, the surface area of the cylinder is

$$\iint_S 1\, dS = \iint_R \underbrace{|\mathbf{t}_u \times \mathbf{t}_v|}_{a}\, dA = \int_0^{2\pi}\int_0^h a\, dv\, du = 2\pi ah,$$

confirming the formula for the surface area of a cylinder (excluding the ends).

b. A parametric description of the sphere is

$$\mathbf{r}(u, v) = \langle a \sin u \cos v, a \sin u \sin v, a \cos u \rangle,$$

where $0 \le u \le \pi$ and $0 \le v \le 2\pi$. The required normal vector is

➤ Recall that for the sphere, $u = \varphi$ and $v = \theta$, where φ and θ are spherical coordinates. The element of surface area in spherical coordinates is $dS = a^2 \sin \varphi\, d\varphi\, d\theta$.

$$\mathbf{t}_u \times \mathbf{t}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a \cos u \cos v & a \cos u \sin v & -a \sin u \\ -a \sin u \sin v & a \sin u \cos v & 0 \end{vmatrix}$$
$$= \langle a^2 \sin^2 u \cos v, a^2 \sin^2 u \sin v, a^2 \sin u \cos u \rangle.$$

Computing $|\mathbf{t}_u \times \mathbf{t}_v|$ requires several steps (Exercise 70). However, the needed result is quite simple: $|\mathbf{t}_u \times \mathbf{t}_v| = a^2 \sin u$ and the normal vector $\mathbf{t}_u \times \mathbf{t}_v$ points outward from the surface of the sphere (Figure 17.50). With $f(x, y, z) = 1$, the surface area of the sphere is

$$\iint_S 1\, dS = \iint_R \underbrace{|\mathbf{t}_u \times \mathbf{t}_v|}_{a^2 \sin u}\, dA = \int_0^{2\pi}\int_0^{\pi} a^2 \sin u\, du\, dv = 4\pi a^2,$$

confirming the formula for the surface area of a sphere.

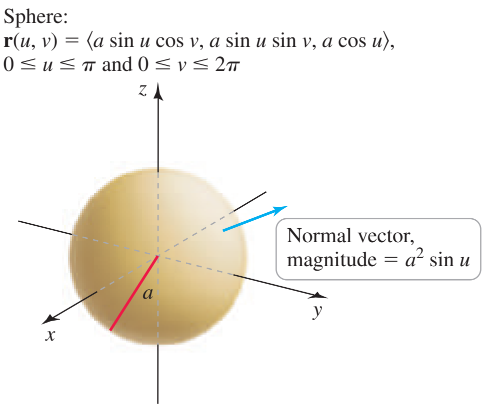


Figure 17.50

Related Exercises 19, 22 ◄

EXAMPLE 3 **Surface area of a partial cylinder** Find the surface area of the cylinder $\{(r, \theta): r = 4, 0 \le \theta \le 2\pi\}$ between the planes $z = 0$ and $z = 16 - 2x$ (excluding the top and bottom surfaces).

SOLUTION Figure 17.51 shows the cylinder bounded by the two planes. With $u = \theta$ and $v = z$, a parametric description of the cylinder is

$$\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle = \langle 4 \cos u, 4 \sin u, v \rangle.$$

The challenge is finding the limits on v, which is the z-coordinate. The plane $z = 16 - 2x$ intersects the cylinder in an ellipse; along this ellipse, as u varies between 0 and 2π, the parameter v also changes. To find the relationship between u and v along this intersection curve, notice that at any point on the cylinder, we have $x = 4 \cos u$ (remember that $u = \theta$). Making this substitution in the equation of the plane, we have

$$z = 16 - 2x = 16 - 2(4 \cos u) = 16 - 8 \cos u.$$

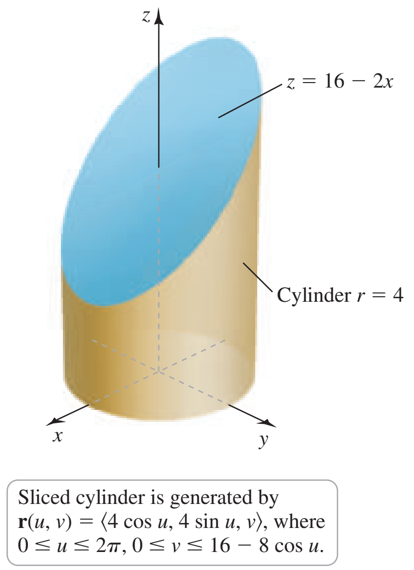
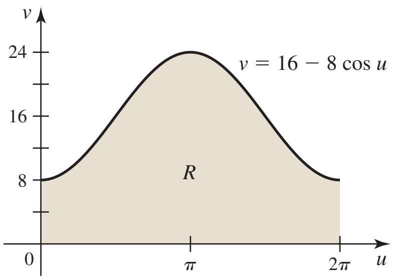


Figure 17.51



Region of integration in the uv-plane is $R = \{(u, v): 0 \le u \le 2\pi, 0 \le v \le 16 - 8\cos u\}$.

Figure 17.52

Substituting $v = z$, the relationship between u and v is $v = 16 - 8\cos u$ (**Figure 17.52**). Therefore, the region of integration in the uv-plane is

$$R = \{(u, v): 0 \le u \le 2\pi, 0 \le v \le 16 - 8\cos u\}.$$

Recall from Example 2a that for the cylinder, $|\mathbf{t}_u \times \mathbf{t}_v| = a = 4$. Setting $f(x, y, z) = 1$, the surface integral for the area is

$$\begin{aligned}
\iint_S 1\, dS &= \iint_R \underbrace{|\mathbf{t}_u \times \mathbf{t}_v|}_{4}\, dA \\
&= \int_0^{2\pi}\int_0^{16-8\cos u} 4\, dv\, du \\
&= 4\int_0^{2\pi}(16 - 8\cos u)\, du && \text{Evaluate inner integral.} \\
&= 4(16u - 8\sin u)\Big|_0^{2\pi} && \text{Evaluate outer integral.} \\
&= 128\pi. && \text{Simplify.}
\end{aligned}$$

Related Exercise 24 ◄

EXAMPLE 4 **Average temperature on a sphere** The temperature on the surface of a sphere of radius a varies with latitude according to the function $T(\varphi, \theta) = 10 + 50\sin\varphi$, for $0 \le \varphi \le \pi$ and $0 \le \theta \le 2\pi$ (φ and θ are spherical coordinates, so the temperature is 10° at the poles, increasing to 60° at the equator). Find the average temperature over the sphere.

SOLUTION We use the parametric description of a sphere. With $u = \varphi$ and $v = \theta$, the temperature function becomes $f(u, v) = 10 + 50\sin u$. Integrating the temperature over the sphere using the fact that $|\mathbf{t}_u \times \mathbf{t}_v| = a^2 \sin u$ (Example 2b), we have

$$\begin{aligned}
\iint_S (10 + 50\sin u)\, dS &= \iint_R (10 + 50\sin u)\underbrace{|\mathbf{t}_u \times \mathbf{t}_v|}_{a^2\sin u}\, dA \\
&= \int_0^{\pi}\int_0^{2\pi}(10 + 50\sin u)a^2\sin u\, dv\, du \\
&= 2\pi a^2\int_0^{\pi}(10 + 50\sin u)\sin u\, du && \text{Evaluate inner integral.} \\
&= 10\pi a^2(4 + 5\pi). && \text{Evaluate outer integral.}
\end{aligned}$$

The average temperature is the integrated temperature $10\pi a^2(4 + 5\pi)$ divided by the surface area of the sphere $4\pi a^2$, so the average temperature is $(20 + 25\pi)/2 \approx 49.3°$.

Related Exercise 42 ◄

Surface Integrals on Explicitly Defined Surfaces Suppose a smooth surface S is defined not parametrically, but explicitly, in the form $z = g(x, y)$ over a region R in the xy-plane. Such a surface may be treated as a parameterized surface. We simply define the parameters to be $u = x$ and $v = y$. Making these substitutions into the expression for $\mathbf{t}_u$ and $\mathbf{t}_v$, a short calculation (Exercise 71) reveals that $\mathbf{t}_u = \mathbf{t}_x = \langle 1, 0, z_x\rangle$, $\mathbf{t}_v = \mathbf{t}_y = \langle 0, 1, z_y\rangle$, and the required normal vector is

$$\mathbf{t}_x \times \mathbf{t}_y = \langle -z_x, -z_y, 1\rangle.$$

It follows that

$$|\mathbf{t}_x \times \mathbf{t}_y| = |\langle -z_x, -z_y, 1\rangle| = \sqrt{z_x^2 + z_y^2 + 1}.$$

With these observations, the surface integral over S can be expressed as a double integral over a region R in the xy-plane.

➤ This is a familiar result: A normal to the surface $z = g(x, y)$ at a point is a constant multiple of the gradient of $z - g(x, y)$, which is $\langle -g_x, -g_y, 1\rangle = \langle -z_x, -z_y, 1\rangle$. The factor $\sqrt{z_x^2 + z_y^2 + 1}$ is analogous to the factor $\sqrt{f'(x)^2 + 1}$ that appears in arc length integrals.

> If the surface S in Theorem 17.14 is generated by revolving a curve in the xy-plane about the x-axis, the theorem gives the standard surface area formula for surfaces of revolution (Exercise 75).

THEOREM 17.14 Evaluation of Surface Integrals of Scalar-Valued Functions on Explicitly Defined Surfaces

Let f be a continuous function on a smooth surface S given by $z = g(x, y)$, for (x, y) in a region R. The surface integral of f over S is

$$\iint_S f(x, y, z)\, dS = \iint_R f(x, y, g(x, y))\sqrt{z_x^2 + z_y^2 + 1}\, dA.$$

If $f(x, y, z) = 1$, the surface integral equals the area of the surface.

EXAMPLE 5 Area of a roof over an ellipse Find the area of the surface S that lies in the plane $z = 12 - 4x - 3y$ directly above the region R bounded by the ellipse $x^2/4 + y^2 = 1$ (Figure 17.53).

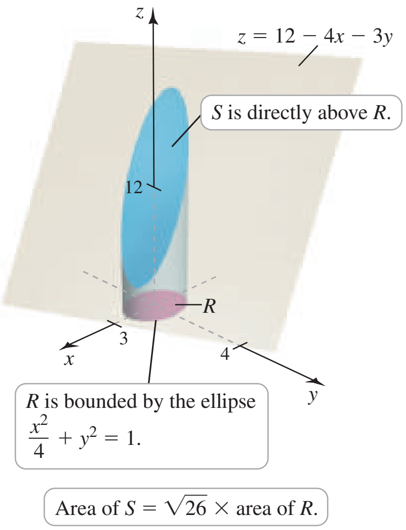


Figure 17.53

SOLUTION Because we are computing the area of the surface, we take $f(x, y, z) = 1$. Note that $z_x = -4$ and $z_y = -3$, so the factor $\sqrt{z_x^2 + z_y^2 + 1}$ has the value $\sqrt{(-4)^2 + (-3)^2 + 1} = \sqrt{26}$ (a constant because the surface is a plane). The relevant surface integral is

$$\iint_S 1\, dS = \iint_R \underbrace{\sqrt{z_x^2 + z_y^2 + 1}}_{\sqrt{26}}\, dA = \sqrt{26}\iint_R dA.$$

The double integral that remains is simply the area of the region R bounded by the ellipse. Because the ellipse has semiaxes of length $a = 2$ and $b = 1$, its area is $\pi ab = 2\pi$. Therefore, the area of S is $2\pi\sqrt{26}$.

This result has a useful interpretation. The plane surface S is not horizontal, so it has a greater area than the horizontal region R beneath it. The factor that converts the area of R to the area of S is $\sqrt{26}$. Notice that if the roof *were* horizontal, then the surface would be $z = c$, the area conversion factor would be 1, and the area of the roof would equal the area of the floor beneath it.

Related Exercises 29–30 ◄

QUICK CHECK 4 The plane $z = y$ forms a 45° angle with the xy-plane. Suppose the plane is the roof of a room and the xy-plane is the floor of the room. Then 1 ft^2 on the floor becomes how many square feet when projected on the roof? ◄

EXAMPLE 6 Mass of a conical sheet A thin conical sheet is described by the surface $z = (x^2 + y^2)^{1/2}$, for $0 \le z \le 4$. The density of the sheet in g/cm^2 is $\rho = f(x, y, z) = (8 - z)$ (decreasing from 8 g/cm^2 at the vertex to 4 g/cm^2 at the top of the cone; Figure 17.54). What is the mass of the cone?

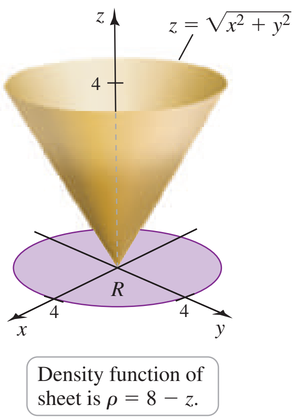


Figure 17.54

SOLUTION We find the mass by integrating the density function over the surface of the cone. The projection of the cone on the xy-plane is found by setting $z = 4$ (the top of the cone) in the equation of the cone. We find that $(x^2 + y^2)^{1/2} = 4$; therefore, the region of integration is the disk $R = \{(x, y): x^2 + y^2 \le 16\}$. The next step is to compute z_x and z_y in order to evaluate $\sqrt{z_x^2 + z_y^2 + 1}$. Differentiating $z^2 = x^2 + y^2$ implicitly gives $2zz_x = 2x$, or $z_x = x/z$. Similarly, $z_y = y/z$. Using the fact that $z^2 = x^2 + y^2$, we have

$$\sqrt{z_x^2 + z_y^2 + 1} = \sqrt{(x/z)^2 + (y/z)^2 + 1} = \sqrt{\underbrace{\frac{x^2 + y^2}{z^2}}_{1} + 1} = \sqrt{2}.$$

To integrate the density over the conical surface, we set $f(x, y, z) = 8 - z$. Replacing z in the integrand by $r = (x^2 + y^2)^{1/2}$ and using polar coordinates, the mass in grams is given by

$$\iint_S f(x, y, z)\, dS = \iint_R f(x, y, z)\underbrace{\sqrt{z_x^2 + z_y^2 + 1}}_{\sqrt{2}}\, dA$$

$$= \sqrt{2}\iint_R (8 - z)\, dA \qquad \text{Substitute.}$$

$$= \sqrt{2}\iint_R (8 - \sqrt{x^2 + y^2})\, dA \qquad z = \sqrt{x^2 + y^2}$$

$$= \sqrt{2}\int_0^{2\pi}\int_0^4 (8 - r)\, r\, dr\, d\theta \qquad \text{Polar coordinates}$$

$$= \sqrt{2}\int_0^{2\pi}\left(4r^2 - \frac{r^3}{3}\right)\bigg|_0^4 d\theta \qquad \text{Evaluate inner integral.}$$

$$= \frac{128\sqrt{2}}{3}\int_0^{2\pi} d\theta \qquad \text{Simplify.}$$

$$= \frac{256\pi\sqrt{2}}{3} \approx 379. \qquad \text{Evaluate outer integral.}$$

As a check, note that the surface area of the cone is $\pi r\sqrt{r^2 + h^2} \approx 71\text{ cm}^2$. If the entire cone had the maximum density $\rho = 8\text{ g/cm}^2$, its mass would be approximately 568 g. If the entire cone had the minimum density $\rho = 4\text{ g/cm}^2$, its mass would be approximately 284 g. The actual mass is between these extremes and closer to the low value because the cone is lighter at the top, where the surface area is greater.

Related Exercise 36 ◀

Table 17.3 summarizes the essential relationships for the explicit and parametric descriptions of cylinders, cones, spheres, and paraboloids. The listed normal vectors are chosen to point away from the z-axis.

Table 17.3

	Explicit Description $z = g(x, y)$		**Parametric Description**	
Surface	**Equation**	**Normal vector; magnitude** $\pm\langle -z_x, -z_y, 1\rangle; \lvert\langle -z_x, -z_y, 1\rangle\rvert$	**Equation**	**Normal vector; magnitude** $\mathbf{t}_u \times \mathbf{t}_v; \lvert\mathbf{t}_u \times \mathbf{t}_v\rvert$
Cylinder	$x^2 + y^2 = a^2$, $0 \le z \le h$	$\langle x, y, 0\rangle; a$	$\mathbf{r} = \langle a\cos u, a\sin u, v\rangle$, $0 \le u \le 2\pi, 0 \le v \le h$	$\langle a\cos u, a\sin u, 0\rangle; a$
Cone	$z^2 = x^2 + y^2$, $0 \le z \le h$	$\langle x/z, y/z, -1\rangle; \sqrt{2}$	$\mathbf{r} = \langle v\cos u, v\sin u, v\rangle$, $0 \le u \le 2\pi, 0 \le v \le h$	$\langle v\cos u, v\sin u, -v\rangle; \sqrt{2}v$
Sphere	$x^2 + y^2 + z^2 = a^2$	$\langle x/z, y/z, 1\rangle; a/z$	$\mathbf{r} = \langle a\sin u\cos v$, $a\sin u\sin v, a\cos u\rangle$, $0 \le u \le \pi, 0 \le v \le 2\pi$	$\langle a^2\sin^2 u\cos v, a^2\sin^2 u\sin v$, $a^2\sin u\cos u\rangle; a^2\sin u$
Paraboloid	$z = x^2 + y^2$, $0 \le z \le h$	$\langle 2x, 2y, -1\rangle; \sqrt{1 + 4(x^2 + y^2)}$	$\mathbf{r} = \langle v\cos u, v\sin u, v^2\rangle$, $0 \le u \le 2\pi, 0 \le v \le \sqrt{h}$	$\langle 2v^2\cos u, 2v^2\sin u, -v\rangle; v\sqrt{1 + 4v^2}$

QUICK CHECK 5 Explain why the explicit description for a cylinder $x^2 + y^2 = a^2$ cannot be used for a surface integral over a cylinder, and a parametric description must be used. ◀

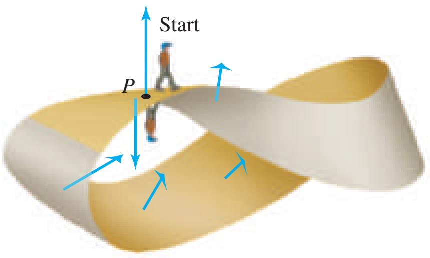


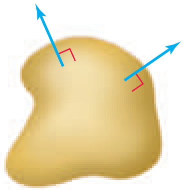
Figure 17.55

Surface Integrals of Vector Fields

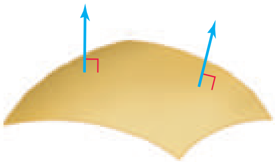
Before beginning a discussion of surface integrals of vector fields, we must address two technical issues about surfaces and normal vectors.

The surfaces we consider in this text are called **two-sided**, or **orientable**, surfaces. To be orientable, a surface must have the property that the normal vectors vary continuously over the surface. In other words, when you walk on any closed path on an orientable surface and return to your starting point, your head must point in the same direction it did when you started. A well-known example of a *nonorientable* surface is the Möbius strip (Figure 17.55). Suppose you start walking the length of the Möbius strip at a point P with your head pointing upward. When you return to P, your head points in the opposite direction, or downward. Therefore, the Möbius strip is not orientable.

At any point of a parameterized orientable surface, there are two unit normal vectors. Therefore, the second point concerns the orientation of the surface or, equivalently, the direction of the normal vector. Once the direction of the normal vector is determined, the surface becomes **oriented**.



Surfaces that enclose a region are oriented so normal vectors point in the outward direction.



For other surfaces, the orientation of the surface must be specified.

Figure 17.56

We make the common assumption that—unless specified otherwise—a closed orientable surface that fully encloses a region (such as a sphere) is oriented so that the normal vectors point in the *outward direction*. For a surface that does not enclose a region in $\mathbb{R}^3$, the orientation must be specified in some way. For example, we might specify that the normal vectors for a particular surface point in the general direction of the positive z-axis; that is, in an upward direction (Figure 17.56).

Now recall that the parameterization of a surface defines a normal vector $\mathbf{t}_u \times \mathbf{t}_v$ at each point. In many cases, the normal vectors are consistent with the specified orientation, in which case no adjustments need to be made. If the direction of $\mathbf{t}_u \times \mathbf{t}_v$ is not consistent with the specified orientation, then the sign of $\mathbf{t}_u \times \mathbf{t}_v$ must be reversed before doing calculations. This process is demonstrated in the following examples.

Flux Integrals It turns out that the most common surface integral of a vector field is a *flux integral*. Consider a vector field $\mathbf{F} = \langle f, g, h \rangle$, continuous on a region in $\mathbb{R}^3$, that represents the flow of a fluid or the transport of a substance. Given a smooth oriented surface S, we aim to compute the net flux of the vector field across the surface. In a small region containing a point P, the flux across the surface is proportional to the component of $\mathbf{F}$ in the direction of the unit normal vector $\mathbf{n}$ at P. If θ is the angle between $\mathbf{F}$ and $\mathbf{n}$, then this component is $\mathbf{F} \cdot \mathbf{n} = |\mathbf{F}||\mathbf{n}| \cos \theta = |\mathbf{F}| \cos \theta$ (because $|\mathbf{n}| = 1$; Figure 17.57a). We have the following special cases.

- If $\mathbf{F}$ and the unit normal vector are aligned at P ($\theta = 0$), then the component of $\mathbf{F}$ in the direction $\mathbf{n}$ is $\mathbf{F} \cdot \mathbf{n} = |\mathbf{F}|$; that is, all of $\mathbf{F}$ flows across the surface in the direction of $\mathbf{n}$ (Figure 17.57b).
- If $\mathbf{F}$ and the unit normal vector point in opposite directions at P ($\theta = \pi$), then the component of $\mathbf{F}$ in the direction $\mathbf{n}$ is $\mathbf{F} \cdot \mathbf{n} = -|\mathbf{F}|$; that is, all of $\mathbf{F}$ flows across the surface in the direction opposite that of $\mathbf{n}$ (Figure 17.57c).
- If $\mathbf{F}$ and the unit normal vector are orthogonal at P ($\theta = \pi/2$), then the component of $\mathbf{F}$ in the direction $\mathbf{n}$ is $\mathbf{F} \cdot \mathbf{n} = 0$; that is, none of $\mathbf{F}$ flows across the surface at that point (Figure 17.57d).

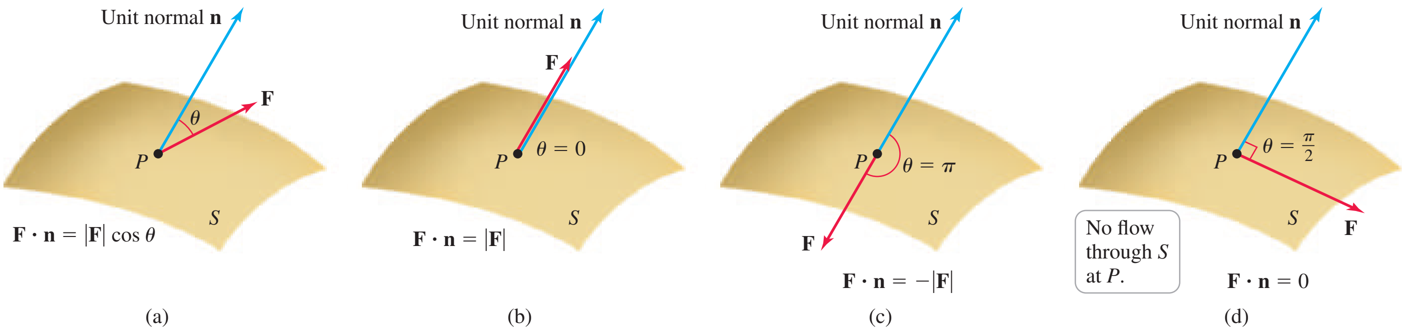


Figure 17.57

The flux integral, denoted $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$ or $\iint_S \mathbf{F} \cdot d\mathbf{S}$, simply adds up the components of $\mathbf{F}$ normal to the surface at all points of the surface. Notice that $\mathbf{F} \cdot \mathbf{n}$ is a scalar-valued function. Here is how the flux integral is computed.

Suppose the smooth oriented surface S is parameterized in the form

$$\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle,$$

where u and v vary over a region R in the uv-plane. The required vector normal to the surface at a point is $\mathbf{t}_u \times \mathbf{t}_v$, which we assume to be consistent with the orientation of S.

➤ If $\mathbf{t}_u \times \mathbf{t}_v$ is not consistent with the specified orientation, its sign must be reversed.

Therefore, the *unit* normal vector consistent with the orientation is $\mathbf{n} = \dfrac{\mathbf{t}_u \times \mathbf{t}_v}{|\mathbf{t}_u \times \mathbf{t}_v|}$. Appealing to the definition of the surface integral for parameterized surfaces, the flux integral is

$$\begin{aligned}
\iint_S \mathbf{F}\cdot\mathbf{n}\, dS &= \iint_R \mathbf{F}\cdot\mathbf{n}|\mathbf{t}_u \times \mathbf{t}_v|\, dA && \text{Definition of surface integral}\\
&= \iint_R \mathbf{F}\cdot\underbrace{\frac{\mathbf{t}_u \times \mathbf{t}_v}{|\mathbf{t}_u \times \mathbf{t}_v|}}_{\mathbf{n}}|\mathbf{t}_u \times \mathbf{t}_v|\, dA && \text{Substitute for } \mathbf{n}.\\
&= \iint_R \mathbf{F}\cdot(\mathbf{t}_u \times \mathbf{t}_v)\, dA. && \text{Convenient cancellation}
\end{aligned}$$

The remarkable occurrence in the flux integral is the cancellation of the factor $|\mathbf{t}_u \times \mathbf{t}_v|$.

The special case in which the surface S is specified in the form $z = s(x, y)$ follows directly by recalling that the required normal vector is $\mathbf{t}_u \times \mathbf{t}_v = \langle -z_x, -z_y, 1\rangle$. In this case, with $\mathbf{F} = \langle f, g, h\rangle$, the integrand of the surface integral is $\mathbf{F}\cdot(\mathbf{t}_u \times \mathbf{t}_v) = -fz_x - gz_y + h$.

➤ The value of the surface integral is independent of the parameterization. However, in contrast to a surface integral of a scalar-valued function, the value of a surface integral of a vector field depends on the orientation of the surface. Changing the orientation changes the sign of the result.

DEFINITION Surface Integral of a Vector Field

Suppose $\mathbf{F} = \langle f, g, h\rangle$ is a continuous vector field on a region of $\mathbb{R}^3$ containing a smooth oriented surface S. If S is defined parametrically as $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$, for (u, v) in a region R, then

$$\iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \iint_R \mathbf{F}\cdot(\mathbf{t}_u \times \mathbf{t}_v)\, dA,$$

where $\mathbf{t}_u = \dfrac{\partial \mathbf{r}}{\partial u} = \left\langle \dfrac{\partial x}{\partial u}, \dfrac{\partial y}{\partial u}, \dfrac{\partial z}{\partial u}\right\rangle$ and $\mathbf{t}_v = \dfrac{\partial \mathbf{r}}{\partial v} = \left\langle \dfrac{\partial x}{\partial v}, \dfrac{\partial y}{\partial v}, \dfrac{\partial z}{\partial v}\right\rangle$ are continuous on R, the normal vector $\mathbf{t}_u \times \mathbf{t}_v$ is nonzero on R, and the direction of the normal vector is consistent with the orientation of S. If S is defined in the form $z = s(x, y)$, for (x, y) in a region R, then

$$\iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \iint_R (-fz_x - gz_y + h)\, dA.$$

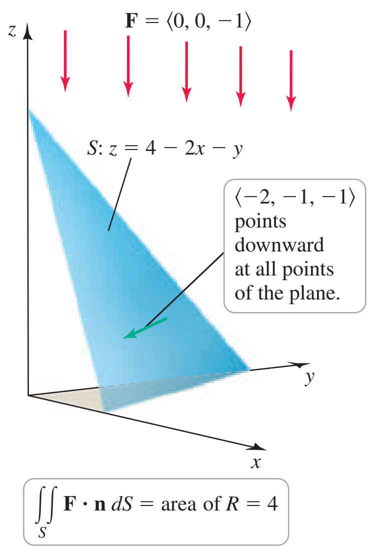


Figure 17.58

EXAMPLE 7 Rain on a roof Consider the vertical vector field $\mathbf{F} = \langle 0, 0, -1\rangle$, corresponding to a constant downward flow. Find the flux in the downward direction across the surface S, which is the plane $z = 4 - 2x - y$ in the first octant.

SOLUTION In this case, the surface is given explicitly. With $z = 4 - 2x - y$, we have $z_x = -2$ and $z_y = -1$. Therefore, the required normal vector is $\langle -z_x, -z_y, 1\rangle = \langle 2, 1, 1\rangle$, which points *upward* (the z-component of the vector is positive). Because we are interested in the *downward* flux of $\mathbf{F}$ across S, the surface must be oriented such that the normal vectors point downward. So we take the normal vector to be $\langle -2, -1, -1\rangle$ (Figure 17.58). Letting R be the region in the xy-plane beneath S and noting that $\mathbf{F} = \langle f, g, h\rangle = \langle 0, 0, -1\rangle$, the flux integral is

$$\iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \iint_R \langle 0, 0, -1\rangle\cdot\langle -2, -1, -1\rangle\, dA = \iint_R dA = \text{area of } R.$$

The base R is a triangle in the xy-plane with vertices $(0, 0)$, $(2, 0)$, and $(0, 4)$, so its area is 4. Therefore, the *downward* flux across S is 4. This flux integral has an interesting interpretation. If the vector field $\mathbf{F}$ represents the rate of rainfall with units of, say, $\mathrm{g/m^2}$

per unit time, then the flux integral gives the mass of rain (in grams) that falls on the surface in a unit of time. This result says that (because the vector field is vertical) the mass of rain that falls on the roof equals the mass that would fall on the floor beneath the roof if the roof were not there. This property is explored further in Exercise 73.

Related Exercises 43–44 ◄

EXAMPLE 8 **Flux of the radial field** Consider the radial vector field $\mathbf{F} = \langle f, g, h \rangle = \langle x, y, z \rangle$. Is the upward flux of the field greater across the hemisphere $x^2 + y^2 + z^2 = 1$, for $z \geq 0$, or across the paraboloid $z = 1 - x^2 - y^2$, for $z \geq 0$? Note that the two surfaces have the same base in the xy-plane and the same high point $(0, 0, 1)$. Use the explicit description for the hemisphere and a parametric description for the paraboloid.

SOLUTION The base of both surfaces in the xy-plane is the unit disk $R = \{(x, y): x^2 + y^2 \leq 1\}$, which, when expressed in polar coordinates, is the set $\{(r, \theta): 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi\}$. To use the explicit description for the hemisphere, we must compute z_x and z_y. Differentiating $x^2 + y^2 + z^2 = 1$ implicitly, we find that $z_x = -x/z$ and $z_y = -y/z$. Therefore, the required normal vector is $\langle x/z, y/z, 1 \rangle$, which points upward on the surface. The flux integral is evaluated by substituting for f, g, h, z_x, and z_y; eliminating z from the integrand; and converting the integral in x and y to an integral in polar coordinates:

➤ Recall that the required normal vector for an explicitly defined surface $z = s(x, y)$ is $\langle -z_x, -z_y, 1 \rangle$.

$$\begin{aligned}
\iint_S \mathbf{F} \cdot \mathbf{n}\, dS &= \iint_R (-f z_x - g z_y + h)\, dA \\
&= \iint_R \left(x\frac{x}{z} + y\frac{y}{z} + z \right) dA && \text{Substitute.} \\
&= \iint_R \left(\frac{x^2 + y^2 + z^2}{z} \right) dA && \text{Simplify.} \\
&= \iint_R \left(\frac{1}{z} \right) dA && x^2 + y^2 + z^2 = 1 \\
&= \iint_R \left(\frac{1}{\sqrt{1 - x^2 - y^2}} \right) dA && z = \sqrt{1 - x^2 - y^2} \\
&= \int_0^{2\pi} \int_0^1 \left(\frac{1}{\sqrt{1 - r^2}} \right) r\, dr\, d\theta && \text{Polar coordinates} \\
&= \int_0^{2\pi} \left(-\sqrt{1 - r^2} \right) \Big|_0^1 d\theta && \text{Evaluate inner integral as an improper integral.} \\
&= \int_0^{2\pi} d\theta = 2\pi. && \text{Evaluate outer integral.}
\end{aligned}$$

For the paraboloid $z = 1 - x^2 - y^2$, we use the parametric description (Example 1b or Table 17.3)

$$\mathbf{r}(u, v) = \langle x, y, z \rangle = \langle v \cos u, v \sin u, 1 - v^2 \rangle,$$

for $0 \leq u \leq 2\pi$ and $0 \leq v \leq 1$. The required vector normal to the surface is

$$\begin{aligned}
\mathbf{t}_u \times \mathbf{t}_v &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -v \sin u & v \cos u & 0 \\ \cos u & \sin u & -2v \end{vmatrix} \\
&= \langle -2v^2 \cos u, -2v^2 \sin u, -v \rangle.
\end{aligned}$$

Notice that the normal vectors point downward on the surface (because the z-component is negative for $0 < v \leq 1$). In order to find the upward flux, we negate the normal vector and use the upward normal vector

$$-(\mathbf{t}_u \times \mathbf{t}_v) = \langle 2v^2 \cos u, 2v^2 \sin u, v \rangle.$$

The flux integral is evaluated by substituting for $\mathbf{F} = \langle x, y, z \rangle$ and $-(\mathbf{t}_u \times \mathbf{t}_v)$ and then evaluating an iterated integral in u and v:

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \int_0^1 \int_0^{2\pi} \langle v \cos u, v \sin u, 1 - v^2 \rangle \cdot \langle 2v^2 \cos u, 2v^2 \sin u, v \rangle\, du\, dv \quad \text{Substitute for F and } -(\mathbf{t}_u \times \mathbf{t}_v).$$

$$= \int_0^1 \int_0^{2\pi} (v^3 + v)\, du\, dv \quad \text{Simplify.}$$

$$= 2\pi \left(\frac{v^4}{4} + \frac{v^2}{2} \right) \bigg|_0^1 = \frac{3\pi}{2}. \quad \text{Evaluate integrals.}$$

We see that the upward flux is greater for the hemisphere than for the paraboloid.

Related Exercises 45, 47 ◄

QUICK CHECK 6 Explain why the upward flux for the radial field in Example 8 is greater for the hemisphere than for the paraboloid. ◄

SECTION 17.6 EXERCISES

Getting Started

1. Give a parametric description for a cylinder with radius a and height h, including the intervals for the parameters.
2. Give a parametric description for a cone with radius a and height h, including the intervals for the parameters.
3. Give a parametric description for a sphere with radius a, including the intervals for the parameters.
4. Explain how to compute the surface integral of a scalar-valued function f over a cone using an explicit description of the cone.
5. Explain how to compute the surface integral of a scalar-valued function f over a sphere using a parametric description of the sphere.
6. Explain what it means for a surface to be orientable.
7. Describe the usual orientation of a closed surface such as a sphere.
8. Why is the upward flux of a vertical vector field $\mathbf{F} = \langle 0, 0, 1 \rangle$ across a surface equal to the area of the projection of the surface in the xy-plane?

Practice Exercises

9–14. Parametric descriptions *Give a parametric description of the form $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$ for the following surfaces. The descriptions are not unique. Specify the required rectangle in the uv-plane.*

9. The plane $2x - 4y + 3z = 16$
10. The cap of the sphere $x^2 + y^2 + z^2 = 16$, for $2\sqrt{2} \le z \le 4$
11. The frustum of the cone $z^2 = x^2 + y^2$, for $2 \le z \le 8$
12. The cone $z^2 = 4(x^2 + y^2)$, for $0 \le z \le 4$
13. The portion of the cylinder $x^2 + y^2 = 9$ in the first octant, for $0 \le z \le 3$
14. The cylinder $y^2 + z^2 = 36$, for $0 \le x \le 9$

15–18. Identify the surface *Describe the surface with the given parametric representation.*

15. $\mathbf{r}(u, v) = \langle u, v, 2u + 3v - 1 \rangle$, for $1 \le u \le 3, 2 \le v \le 4$
16. $\mathbf{r}(u, v) = \langle u, u + v, 2 - u - v \rangle$, for $0 \le u \le 2, 0 \le v \le 2$
17. $\mathbf{r}(u, v) = \langle v \cos u, v \sin u, 4v \rangle$, for $0 \le u \le \pi, 0 \le v \le 3$
18. $\mathbf{r}(u, v) = \langle v, 6 \cos u, 6 \sin u \rangle$, for $0 \le u \le 2\pi, 0 \le v \le 2$

19–24. Surface area using a parametric description *Find the area of the following surfaces using a parametric description of the surface.*

19. The half-cylinder $\{(r, \theta, z): r = 4, 0 \le \theta \le \pi, 0 \le z \le 7\}$
20. The plane $z = 3 - x - 3y$ in the first octant
21. The plane $z = 10 - x - y$ above the square $|x| \le 2, |y| \le 2$
22. The hemisphere $x^2 + y^2 + z^2 = 100$, for $z \ge 0$
23. A cone with base radius r and height h, where r and h are positive constants
24. The cap of the sphere $x^2 + y^2 + z^2 = 4$, for $1 \le z \le 2$

25–28. Surface integrals using a parametric description *Evaluate the surface integral $\iint_S f\, dS$ using a parametric description of the surface.*

25. $f(x, y, z) = x^2 + y^2$, where S is the hemisphere $x^2 + y^2 + z^2 = 36$, for $z \ge 0$
26. $f(x, y, z) = y$, where S is the cylinder $x^2 + y^2 = 9, 0 \le z \le 3$
27. $f(x, y, z) = x$, where S is the cylinder $x^2 + z^2 = 1, 0 \le y \le 3$
28. $f(\rho, \varphi, \theta) = \cos \varphi$, where S is the part of the unit sphere in the first octant

29–34. Surface area using an explicit description *Find the area of the following surfaces using an explicit description of the surface.*

29. The part of the plane $z = 2x + 2y + 4$ over the region R bounded by the triangle with vertices $(0, 0)$, $(2, 0)$, and $(2, 4)$
30. The part of the plane $z = x + 3y + 5$ over the region $R = \{(x, y): 1 \le x^2 + y^2 \le 4\}$
31. The cone $z^2 = 4(x^2 + y^2)$, for $0 \le z \le 4$
32. **T** The trough $z = \frac{1}{2}x^2$, for $-1 \le x \le 1, 0 \le y \le 4$
33. The paraboloid $z = 2(x^2 + y^2)$, for $0 \le z \le 8$
34. The part of the hyperbolic paraboloid $z = 3 + x^2 - y^2$ above the sector $R = \{(r, \theta): 0 \le r \le \sqrt{2}, 0 \le \theta \le \pi/2\}$

35–38. Surface integrals using an explicit description *Evaluate the surface integral $\iint_S f(x, y, z)\, dS$ using an explicit representation of the surface.*

35. $f(x, y, z) = xy$; S is the plane $z = 2 - x - y$ in the first octant.

36. $f(x, y, z) = x^2 + y^2$; S is the paraboloid $z = x^2 + y^2$, for $0 \le z \le 1$.

37. $f(x, y, z) = 25 - x^2 - y^2$; S is the hemisphere centered at the origin with radius 5, for $z \ge 0$.

38. $f(x, y, z) = e^z$; S is the plane $z = 8 - x - 2y$ in the first octant.

39–42. Average values

39. Find the average temperature on that part of the plane $2x + 2y + z = 4$ over the square $0 \le x \le 1, 0 \le y \le 1$, where the temperature is given by $T(x, y, z) = e^{2x+y+z-3}$.

T **40.** Find the average squared distance between the origin and the points on the paraboloid $z = 4 - x^2 - y^2$, for $z \ge 0$.

41. Find the average value of the function $f(x, y, z) = xyz$ on the unit sphere in the first octant.

42. Find the average value of the temperature function $T(x, y, z) = 100 - 25z$ on the cone $z^2 = x^2 + y^2$, for $0 \le z \le 2$.

43–48. Surface integrals of vector fields *Find the flux of the following vector fields across the given surface with the specified orientation. You may use either an explicit or a parametric description of the surface.*

43. $\mathbf{F} = \langle 0, 0, -1 \rangle$ across the slanted face of the tetrahedron $z = 4 - x - y$ in the first octant; normal vectors point upward.

44. $\mathbf{F} = \langle x, y, z \rangle$ across the slanted face of the tetrahedron $z = 10 - 2x - 5y$ in the first octant; normal vectors point upward.

45. $\mathbf{F} = \langle x, y, z \rangle$ across the slanted surface of the cone $z^2 = x^2 + y^2$, for $0 \le z \le 1$; normal vectors point upward.

46. $\mathbf{F} = \langle e^{-y}, 2z, xy \rangle$ across the curved sides of the surface $S = \{(x, y, z): z = \cos y, |y| \le \pi, 0 \le x \le 4\}$; normal vectors point upward.

47. $\mathbf{F} = \mathbf{r}/|\mathbf{r}|^3$ across the sphere of radius a centered at the origin, where $\mathbf{r} = \langle x, y, z \rangle$; normal vectors point outward.

48. $\mathbf{F} = \langle -y, x, 1 \rangle$ across the cylinder $y = x^2$, for $0 \le x \le 1$, $0 \le z \le 4$; normal vectors point in the general direction of the positive y-axis.

49. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. If the surface S is given by $\{(x, y, z): 0 \le x \le 1, 0 \le y \le 1, z = 10\}$, then $\iint_S f(x, y, z)\, dS = \int_0^1 \int_0^1 f(x, y, 10)\, dx\, dy$.

b. If the surface S is given by $\{(x, y, z): 0 \le x \le 1, 0 \le y \le 1, z = x\}$, then $\iint_S f(x, y, z)\, dS = \int_0^1 \int_0^1 f(x, y, x)\, dx\, dy$.

c. The surface $\mathbf{r} = \langle v \cos u, v \sin u, v^2 \rangle$, for $0 \le u \le \pi$, $0 \le v \le 2$, is the same as the surface $\mathbf{r} = \langle \sqrt{v} \cos 2u, \sqrt{v} \sin 2u, v \rangle$, for $0 \le u \le \pi/2, 0 \le v \le 4$.

d. Given the standard parameterization of a sphere, the normal vectors $\mathbf{t}_u \times \mathbf{t}_v$ are outward normal vectors.

50–53. Miscellaneous surface integrals *Evaluate the following integrals using the method of your choice. Assume normal vectors point either outward or upward.*

50. $\displaystyle\iint_S \nabla \ln |\mathbf{r}| \cdot \mathbf{n}\, dS$, where S is the hemisphere $x^2 + y^2 + z^2 = a^2$, for $z \ge 0$, and where $\mathbf{r} = \langle x, y, z \rangle$

51. $\displaystyle\iint_S |\mathbf{r}|\, dS$, where S is the cylinder $x^2 + y^2 = 4$, for $0 \le z \le 8$, where $\mathbf{r} = \langle x, y, z \rangle$

52. $\displaystyle\iint_S xyz\, dS$, where S is that part of the plane $z = 6 - y$ that lies in the cylinder $x^2 + y^2 = 4$

53. $\displaystyle\iint_S \frac{\langle x, 0, z \rangle}{\sqrt{x^2 + z^2}} \cdot \mathbf{n}\, dS$, where S is the cylinder $x^2 + z^2 = a^2$, $|y| \le 2$

54. Cone and sphere The cone $z^2 = x^2 + y^2$, for $z \ge 0$, cuts the sphere $x^2 + y^2 + z^2 = 16$ along a curve C.

a. Find the surface area of the sphere below C, for $z \ge 0$.
b. Find the surface area of the sphere above C.
c. Find the surface area of the cone below C, for $z \ge 0$.

T **55. Cylinder and sphere** Consider the sphere $x^2 + y^2 + z^2 = 4$ and the cylinder $(x - 1)^2 + y^2 = 1$, for $z \ge 0$. Find the surface area of the cylinder inside the sphere.

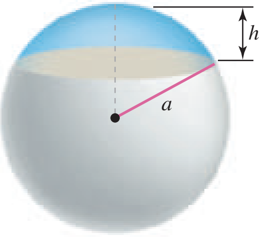
56. Flux on a tetrahedron Find the upward flux of the field $\mathbf{F} = \langle x, y, z \rangle$ across the plane $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$ in the first octant, where a, b, and c are positive real numbers. Show that the flux equals c times the area of the base of the region. Interpret the result physically.

57. Flux across a cone Consider the field $\mathbf{F} = \langle x, y, z \rangle$ and the cone $z^2 = \dfrac{x^2 + y^2}{a^2}$, for $0 \le z \le 1$.

a. Show that when $a = 1$, the outward flux across the cone is zero. Interpret the result.
b. Find the outward flux (away from the z-axis), for any $a > 0$. Interpret the result.

58. Surface area formula for cones Find the general formula for the surface area of a cone with height h and base radius a (excluding the base).

59. Surface area formula for spherical cap A sphere of radius a is sliced parallel to the equatorial plane at a distance $a - h$ from the equatorial plane (see figure). Find the general formula for the surface area of the resulting spherical cap (excluding the base) with thickness h.



Explorations and Challenges

60. Radial fields and spheres Consider the radial field $\mathbf{F} = \mathbf{r}/|\mathbf{r}|^p$, where $\mathbf{r} = \langle x, y, z \rangle$ and p is a real number. Let S be the sphere of radius a centered at the origin. Show that the outward flux of $\mathbf{F}$ across the sphere is $4\pi/a^{p-3}$. It is instructive to do the calculation using both an explicit and a parametric description of the sphere.

61–63. Heat flux *The heat flow vector field for conducting objects is $\mathbf{F} = -k\nabla T$, where $T(x, y, z)$ is the temperature in the object and $k > 0$ is a constant that depends on the material. Compute the outward flux of $\mathbf{F}$ across the following surfaces S for the given temperature distributions. Assume $k = 1$.*

61. $T(x, y, z) = 100e^{-x-y}$; S consists of the faces of the cube $|x| \le 1$, $|y| \le 1$, $|z| \le 1$.

62. $T(x, y, z) = 100e^{-x^2-y^2-z^2}$; S is the sphere $x^2 + y^2 + z^2 = a^2$.

63. $T(x, y, z) = -\ln(x^2 + y^2 + z^2)$; S is the sphere $x^2 + y^2 + z^2 = a^2$.

64. Flux across a cylinder Let S be the cylinder $x^2 + y^2 = a^2$, for $-L \le z \le L$.

a. Find the outward flux of the field $\mathbf{F} = \langle x, y, 0 \rangle$ across S.

b. Find the outward flux of the field $\mathbf{F} = \dfrac{\langle x, y, 0 \rangle}{(x^2 + y^2)^{p/2}} = \dfrac{\mathbf{r}}{|\mathbf{r}|^p}$ across S, where $|\mathbf{r}|$ is the distance from the z-axis and p is a real number.

c. In part (b), for what values of p is the outward flux finite as $a \to \infty$ (with L fixed)?

d. In part (b), for what values of p is the outward flux finite as $L \to \infty$ (with a fixed)?

65. Flux across concentric spheres Consider the radial fields $\mathbf{F} = \dfrac{\langle x, y, z \rangle}{(x^2 + y^2 + z^2)^{p/2}} = \dfrac{\mathbf{r}}{|\mathbf{r}|^p}$, where p is a real number. Let S consist of the spheres A and B centered at the origin with radii $0 < a < b$, respectively. The total outward flux across S consists of the flux out of S across the outer sphere B minus the flux into S across the inner sphere A.

a. Find the total flux across S with $p = 0$. Interpret the result.

b. Show that for $p = 3$ (an inverse square law), the flux across S is independent of a and b.

66–69. Mass and center of mass *Let S be a surface that represents a thin shell with density ρ. The moments about the coordinate planes (see Section 16.6) are $M_{yz} = \iint_S x\rho(x, y, z)\, dS$, $M_{xz} = \iint_S y\rho(x, y, z)\, dS$, and $M_{xy} = \iint_S z\rho(x, y, z)\, dS$. The coordinates of the center of mass of the shell are $\bar{x} = \dfrac{M_{yz}}{m}$, $\bar{y} = \dfrac{M_{xz}}{m}$, and $\bar{z} = \dfrac{M_{xy}}{m}$, where m is the mass of the shell. Find the mass and center of mass of the following shells. Use symmetry whenever possible.*

66. The constant-density hemispherical shell $x^2 + y^2 + z^2 = a^2$, $z \ge 0$

67. The constant-density cone with radius a, height h, and base in the xy-plane

68. The constant-density half-cylinder $x^2 + z^2 = a^2$, $-\dfrac{h}{2} \le y \le \dfrac{h}{2}$, $z \ge 0$

69. The cylinder $x^2 + y^2 = a^2$, $0 \le z \le 2$, with density $\rho(x, y, z) = 1 + z$

70. Outward normal to a sphere Show that $|\mathbf{t}_u \times \mathbf{t}_v| = a^2 \sin u$ for a sphere of radius a defined parametrically by $\mathbf{r}(u, v) = \langle a \sin u \cos v, a \sin u \sin v, a \cos u \rangle$, where $0 \le u \le \pi$ and $0 \le v \le 2\pi$.

71. Special case of surface integrals of scalar-valued functions Suppose a surface S is defined as $z = g(x, y)$ on a region R. Show that $\mathbf{t}_x \times \mathbf{t}_y = \langle -z_x, -z_y, 1 \rangle$ and that $\iint_S f(x, y, z)\, dS = \iint_R f(x, y, g(x, y))\sqrt{z_x^2 + z_y^2 + 1}\, dA$.

72. Surfaces of revolution Suppose $y = f(x)$ is a continuous and positive function on $[a, b]$. Let S be the surface generated when the graph of f on $[a, b]$ is revolved about the x-axis.

a. Show that S is described parametrically by $\mathbf{r}(u, v) = \langle u, f(u) \cos v, f(u) \sin v \rangle$, for $a \le u \le b$, $0 \le v \le 2\pi$.

b. Find an integral that gives the surface area of S.

c. Apply the result of part (b) to the surface generated with $f(x) = x^3$, for $1 \le x \le 2$.

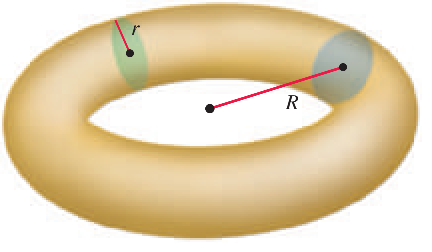
d. Apply the result of part (b) to the surface generated with $f(x) = (25 - x^2)^{1/2}$, for $3 \le x \le 4$.

73. Rain on roofs Let $z = s(x, y)$ define a surface over a region R in the xy-plane, where $z \ge 0$ on R. Show that the downward flux of the vertical vector field $\mathbf{F} = \langle 0, 0, -1 \rangle$ across S equals the area of R. Interpret the result physically.

74. Surface area of a torus

a. Show that a torus with radii $R > r$ (see figure) may be described parametrically by $\mathbf{r}(u, v) = \langle (R + r \cos u) \cos v, (R + r \cos u) \sin v, r \sin u \rangle$, for $0 \le u \le 2\pi$, $0 \le v \le 2\pi$.

b. Show that the surface area of the torus is $4\pi^2 Rr$.



75. Surfaces of revolution—single variable Let f be differentiable and positive on the interval $[a, b]$. Let S be the surface generated when the graph of f on $[a, b]$ is revolved about the x-axis. Use Theorem 17.14 to show that the area of S (as given in Section 6.6) is

$$\int_a^b 2\pi f(x)\sqrt{1 + f'(x)^2}\, dx.$$

QUICK CHECK ANSWERS

1. A half-cylinder with height 1 and radius 2 with its axis along the z-axis **2.** A half-cone with height 10 and radius 10 **3.** A quarter-sphere with radius 4 **4.** $\sqrt{2}$ **5.** The cylinder $x^2 + y^2 = a^2$ does not represent a function, so z_x and z_y cannot be computed. **6.** The vector field is everywhere orthogonal to the hemisphere, so the hemisphere has maximum flux at every point. ➤

17.7 Stokes' Theorem

With the divergence, the curl, and surface integrals in hand, we are ready to present two of the crowning results of calculus. Fortunately, all the heavy lifting has been done. In this section, you will see Stokes' Theorem, and in the next section, we present the Divergence Theorem.

➤ Born in Ireland, George Gabriel Stokes (1819–1903) led a long and distinguished life as one of the prominent mathematicians and physicists of his day. He entered Cambridge University as a student and remained there as a professor for most of his life, taking the Lucasian chair of mathematics once held by Sir Isaac Newton. The first statement of Stokes' Theorem was given by William Thomson (Lord Kelvin).

Stokes' Theorem

Stokes' Theorem is the three-dimensional version of the circulation form of Green's Theorem. Recall that if C is a closed simple piecewise-smooth oriented curve in the xy-plane enclosing a simply connected region R, and $\mathbf{F} = \langle f, g \rangle$ is a differentiable vector field on R, then Green's Theorem says that

$$\underbrace{\oint_C \mathbf{F} \cdot d\mathbf{r}}_{\text{circulation}} = \iint_R \underbrace{(g_x - f_y)}_{\text{curl or rotation}} \, dA.$$

The line integral on the left gives the circulation along the boundary of R. The double integral on the right sums the curl of the vector field over all points of R. If $\mathbf{F}$ represents a fluid flow, the theorem says that the cumulative rotation of the flow within R equals the circulation along the boundary.

In Stokes' Theorem, the plane region R in Green's Theorem becomes an oriented surface S in $\mathbb{R}^3$. The circulation integral in Green's Theorem remains a circulation integral, but now over the closed simple piecewise-smooth oriented curve C that forms the boundary of S. The double integral of the curl in Green's Theorem becomes a surface integral of the three-dimensional curl (Figure 17.59).

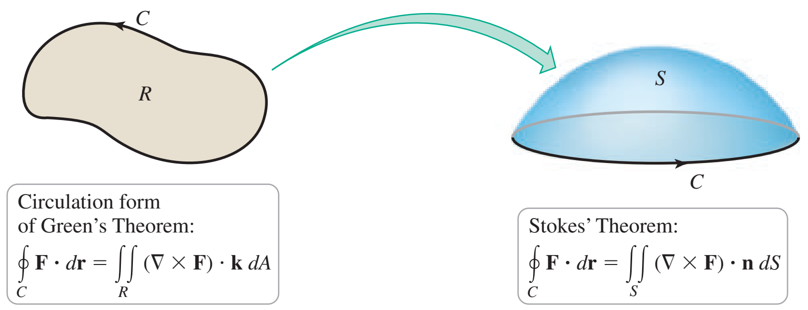


Figure 17.59

Stokes' Theorem involves an oriented curve C and an oriented surface S on which there are two unit normal vectors at every point. These orientations must be consistent and the normal vectors must be chosen correctly. Here is the right-hand rule that relates the orientations of S and C and determines the choice of the normal vectors:

> If the fingers of your right hand curl in the positive direction around C, then your right thumb points in the (general) direction of the vectors normal to S (Figure 17.60).

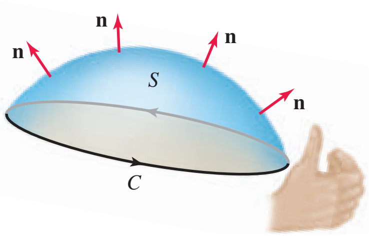


Figure 17.60

A common situation occurs when C has a counterclockwise orientation when viewed from above; then the vectors normal to S point upward.

➤ The right-hand rule tells you which of two normal vectors at a point of S to use. Remember that the direction of normal vectors changes continuously on an oriented surface.

THEOREM 17.15 Stokes' Theorem
Let S be an oriented surface in $\mathbb{R}^3$ with a piecewise-smooth closed boundary C whose orientation is consistent with that of S. Assume $\mathbf{F} = \langle f, g, h \rangle$ is a vector field whose components have continuous first partial derivatives on S. Then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS,$$

where $\mathbf{n}$ is the unit vector normal to S determined by the orientation of S.

QUICK CHECK 1 Suppose S is a region in the xy-plane with a boundary oriented counterclockwise. What is the normal to S? Explain why Stokes' Theorem becomes the circulation form of Green's Theorem. ◂

The meaning of Stokes' Theorem is much the same as for the circulation form of Green's Theorem: Under the proper conditions, the accumulated rotation of the vector field over the surface S (as given by the normal component of the curl) equals the net circulation on the boundary of S. An outline of the proof of Stokes' Theorem is given at the end of this section. First, we look at some special cases that give further insight into the theorem.

If $\mathbf{F}$ is a conservative vector field on a domain D, then it has a potential function φ such that $\mathbf{F} = \nabla\varphi$. Because $\nabla \times \nabla\varphi = \mathbf{0}$, it follows that $\nabla \times \mathbf{F} = \mathbf{0}$ (Theorem 17.11); therefore, the circulation integral is zero on all closed curves in D. Recall that the circulation integral is also a work integral for the force field $\mathbf{F}$, which emphasizes the fact that no work is done in moving an object on a closed path in a conservative force field. Among the important conservative vector fields are the radial fields $\mathbf{F} = \mathbf{r}/|\mathbf{r}|^p$, which generally have zero curl and zero circulation on closed curves.

➤ Recall that for a constant nonzero vector $\mathbf{a}$ and the position vector $\mathbf{r} = \langle x, y, z\rangle$, the field $\mathbf{F} = \mathbf{a} \times \mathbf{r}$ is a rotation field. In Example 1,

$$\mathbf{F} = \langle 0, 1, 1\rangle \times \langle x, y, z\rangle.$$

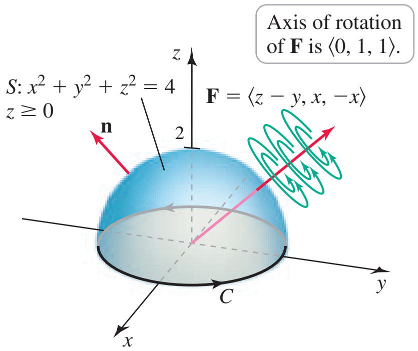


Figure 17.61

EXAMPLE 1 Verifying Stokes' Theorem Confirm that Stokes' Theorem holds for the vector field $\mathbf{F} = \langle z - y, x, -x\rangle$, where S is the hemisphere $x^2 + y^2 + z^2 = 4$, for $z \geq 0$, and C is the circle $x^2 + y^2 = 4$ oriented counterclockwise.

SOLUTION The orientation of C implies that vectors normal to S should point in the outward direction. The vector field is a rotation field $\mathbf{a} \times \mathbf{r}$, where $\mathbf{a} = \langle 0, 1, 1\rangle$ and $\mathbf{r} = \langle x, y, z\rangle$; so the axis of rotation points in the direction of the vector $\langle 0, 1, 1\rangle$ (Figure 17.61). We first compute the circulation integral in Stokes' Theorem. The curve C with the given orientation is parameterized as $\mathbf{r}(t) = \langle 2\cos t, 2\sin t, 0\rangle$, for $0 \leq t \leq 2\pi$; therefore, $\mathbf{r}'(t) = \langle -2\sin t, 2\cos t, 0\rangle$. The circulation integral is

$$\begin{aligned}
\oint_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^{2\pi} \mathbf{F}\cdot\mathbf{r}'(t)\,dt && \text{Definition of line integral}\\
&= \int_0^{2\pi} \langle \underbrace{z - y}_{-2\sin t}, x, -x\rangle \cdot \langle -2\sin t, 2\cos t, 0\rangle\,dt && \text{Substitute.}\\
&= \int_0^{2\pi} 4(\sin^2 t + \cos^2 t)\,dt && \text{Simplify.}\\
&= 4\int_0^{2\pi} dt && \sin^2 t + \cos^2 t = 1\\
&= 8\pi. && \text{Evaluate integral.}
\end{aligned}$$

The surface integral requires computing the curl of the vector field:

$$\nabla \times \mathbf{F} = \nabla \times \langle z - y, x, -x\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ z - y & x & -x \end{vmatrix} = \langle 0, 2, 2\rangle.$$

Recall from Section 17.6 (Table 17.3) that the required outward normal to the hemisphere is $\langle x/z, y/z, 1\rangle$. The region of integration is the base of the hemisphere in the xy-plane, which is

$$R = \{(x, y): x^2 + y^2 \leq 4\}, \text{ or, in polar coordinates, } \{(r, \theta): 0 \leq r \leq 2, 0 \leq \theta \leq 2\pi\}.$$

Combining these results, the surface integral in Stokes' Theorem is

$$\begin{aligned}
\iint_S \underbrace{(\nabla \times \mathbf{F})}_{\langle 0,2,2\rangle}\cdot\mathbf{n}\,dS &= \iint_R \langle 0, 2, 2\rangle \cdot \left\langle \frac{x}{z}, \frac{y}{z}, 1\right\rangle dA && \text{Substitute and convert to a double integral over } R.\\
&= \iint_R \left(\frac{2y}{\sqrt{4 - x^2 - y^2}} + 2\right) dA && \text{Simplify and use } z = \sqrt{4 - x^2 - y^2}.\\
&= \int_0^{2\pi}\int_0^2 \left(\frac{2r\sin\theta}{\sqrt{4 - r^2}} + 2\right) r\,dr\,d\theta. && \text{Convert to polar coordinates.}
\end{aligned}$$

We integrate first with respect to θ because the integral of $\sin\theta$ from 0 to 2π is zero and the first term in the integral is eliminated. Therefore, the surface integral reduces to

> In eliminating the first term of this double integral, we note that the improper integral $\int_0^2 \frac{r^2}{\sqrt{4-r^2}}\,dr$ has a finite value.

$$\iint_S (\nabla \times \mathbf{F})\cdot\mathbf{n}\,dS = \int_0^2\int_0^{2\pi}\left(\frac{2r^2\sin\theta}{\sqrt{4-r^2}} + 2r\right)d\theta\,dr$$

$$= \int_0^2\int_0^{2\pi} 2r\,d\theta\,dr \qquad \int_0^{2\pi}\sin\theta\,d\theta = 0$$

$$= 4\pi\int_0^2 r\,dr \qquad \text{Evaluate inner integral.}$$

$$= 8\pi. \qquad \text{Evaluate outer integral.}$$

Computed either as a line integral or as a surface integral, the vector field has a positive circulation along the boundary of S, which is produced by the net rotation of the field over the surface S.

Related Exercises 5–6 ◀

In Example 1, it was possible to evaluate both the line integral and the surface integral that appear in Stokes' Theorem. Often the theorem provides an easier way to evaluate difficult line integrals.

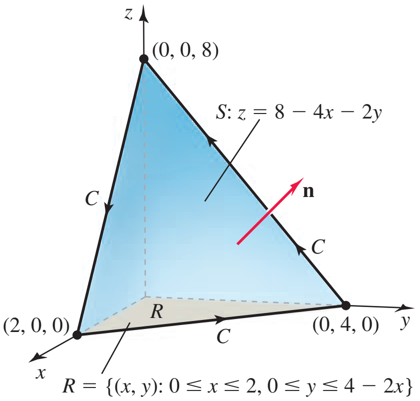


Figure 17.62

EXAMPLE 2 **Using Stokes' Theorem to evaluate a line integral** Evaluate the line integral $\oint_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F} = z\,\mathbf{i} - z\,\mathbf{j} + (x^2 - y^2)\,\mathbf{k}$ and C consists of the three line segments that bound the plane $z = 8 - 4x - 2y$ in the first octant, oriented as shown in Figure 17.62.

SOLUTION Evaluating the line integral directly involves parameterizing the three line segments. Instead, we use Stokes' Theorem to convert the line integral to a surface integral, where S is that portion of the plane $z = 8 - 4x - 2y$ that lies in the first octant. The curl of the vector field is

$$\nabla \times \mathbf{F} = \nabla \times \langle z, -z, x^2 - y^2\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ z & -z & x^2 - y^2 \end{vmatrix} = \langle 1 - 2y, 1 - 2x, 0\rangle.$$

> Recall that for an explicitly defined surface S given by $z = s(x, y)$ over a region R with $\mathbf{F} = \langle f, g, h\rangle$,
> $$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_R (-fz_x - gz_y + h)\,dA.$$
> In Example 2, $\mathbf{F}$ is replaced with $\nabla \times \mathbf{F}$.

The appropriate vector normal to the plane $z = 8 - 4x - 2y$ is $\langle -z_x, -z_y, 1\rangle = \langle 4, 2, 1\rangle$, which points upward, consistent with the orientation of C. The triangular region R in the xy-plane beneath S is found by setting $z = 0$ in the equation of the plane; we find that $R = \{(x, y): 0 \le x \le 2, 0 \le y \le 4 - 2x\}$. The surface integral in Stokes' Theorem may now be evaluated:

$$\iint_S \underbrace{(\nabla \times \mathbf{F})}_{\langle 1-2y,\,1-2x,\,0\rangle}\cdot\mathbf{n}\,dS = \iint_R \langle 1 - 2y, 1 - 2x, 0\rangle \cdot \langle 4, 2, 1\rangle\,dA \qquad \text{Substitute and convert to a double integral over } R.$$

$$= \int_0^2\int_0^{4-2x}(6 - 4x - 8y)\,dy\,dx \qquad \text{Simplify.}$$

$$= -\frac{88}{3}. \qquad \text{Evaluate integrals.}$$

The circulation around the boundary of R is negative, indicating a net circulation in the clockwise direction on C (looking from above).

Related Exercises 13, 16 ◀

In other situations, Stokes' Theorem may be used to convert a difficult surface integral into a relatively easy line integral, as illustrated in the next example.

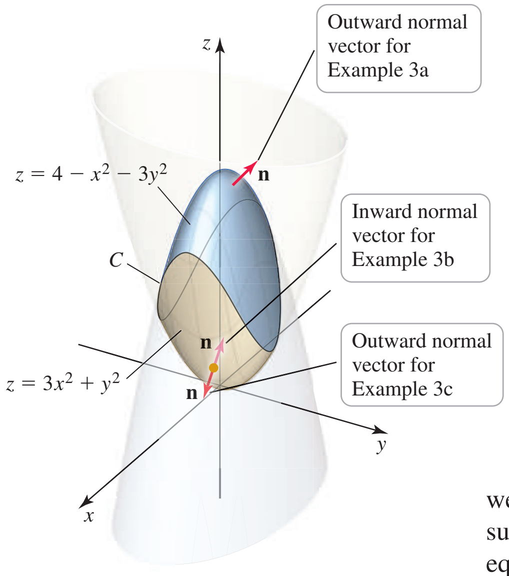


Figure 17.63

EXAMPLE 3 Using Stokes' Theorem to evaluate a surface integral Evaluate $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$, where $\mathbf{F} = -y\,\mathbf{i} + x\,\mathbf{j} + z\,\mathbf{k}$, in the following cases.

a. S is the part of the paraboloid $z = 4 - x^2 - 3y^2$ that lies within the paraboloid $z = 3x^2 + y^2$ (the blue surface in Figure 17.63). Assume $\mathbf{n}$ points in the upward direction on S.

b. S is the part of the paraboloid $z = 3x^2 + y^2$ that lies within the paraboloid $z = 4 - x^2 - 3y^2$, with $\mathbf{n}$ pointing in the upward direction on S.

c. S is the surface in part (b), but $\mathbf{n}$ pointing in the downward direction on S.

SOLUTION

a. Finding a parametric description for S is challenging, so we use Stokes' Theorem to convert the surface integral into a line integral along the curve C that bounds S. Note that C is the intersection between the paraboloids $z = 4 - x^2 - 3y^2$ and $z = 3x^2 + y^2$. Eliminating z from these equations, we find that the projection of C onto the xy-plane is the circle $x^2 + y^2 = 1$, which suggests that we choose $x = \cos t$ and $y = \sin t$ for the x- and y-components of the equations for C. To find the z-component, we substitute x and y into the equation of either paraboloid. Choosing $z = 3x^2 + y^2$, we find that a parametric description of C is $\mathbf{r}(t) = \langle \cos t, \sin t, 3\cos^2 t + \sin^2 t \rangle$; note that C is oriented in the counterclockwise direction, consistent with the orientation of S.

To evaluate the line integral in Stokes' Theorem, it is helpful to first compute $\mathbf{F} \cdot \mathbf{r}'(t)$. Along C, the vector field is $\mathbf{F} = \langle -y, x, z \rangle = \langle -\sin t, \cos t, 3\cos^2 t + \sin^2 t \rangle$. Differentiating $\mathbf{r}$ yields $\mathbf{r}'(t) = \langle -\sin t, \cos t, -4\cos t \sin t \rangle$, which leads to

$$\begin{aligned}\mathbf{F} \cdot \mathbf{r}'(t) &= \langle -\sin t, \cos t, 3\cos^2 t + \sin^2 t \rangle \cdot \langle -\sin t, \cos t, -4\cos t \sin t \rangle \\ &= \underbrace{\sin^2 t + \cos^2 t}_{1} - 12\cos^3 t \sin t - 4\sin^3 t \cos t.\end{aligned}$$

Noting that $\sin^2 t + \cos^2 t = 1$, we are ready to evaluate the integral:

$$\begin{aligned}\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS &= \oint_C \mathbf{F} \cdot d\mathbf{r} && \text{Stokes' Theorem} \\ &= \int_0^{2\pi} \mathbf{F} \cdot \mathbf{r}'(t)\, dt && \text{Definition of line integral} \\ &= \int_0^{2\pi} (1 - 12\cos^3 t \sin t - 4\cos t \sin^3 t)\, dt && \text{Substitute.} \\ &= \int_0^{2\pi} 1\, dt - \underbrace{\int_0^{2\pi} 12\cos^3 t \sin t\, dt}_{0} - \underbrace{\int_0^{2\pi} 4\cos t \sin^3 t\, dt}_{0} && \text{Split integral.} \\ &= 2\pi. && \text{Evaluate integrals.}\end{aligned}$$

A standard substitution in the last two integrals of the final step shows that both integrals equal 0.

b. Because the lower surface $(z = 3x^2 + y^2)$ shares the same boundary C with the upper surface $(z = 4 - x^2 - 3y^2)$, and because both surfaces have an upward-pointing normal vector, the line integral resulting from an application of Stokes' Theorem is identical to the integral in part (a). For this surface S with its associated normal vector, we conclude that $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \oint_C \mathbf{F} \cdot d\mathbf{r} = 2\pi$. In fact, the value of this integral is 2π for *any* surface whose boundary is C and whose normal vectors point in the upward direction.

c. In this case, $\mathbf{n}$ points downward. We use the parameterization $\mathbf{r}(t) = \langle \sin t, \cos t, 3\cos^2 t + \sin^2 t \rangle$ for C so that C is oriented in the clockwise direction, consistent with the orientation of S. You should verify that, when duplicating the calculations in part (a) with a new description for C, we have

$$\mathbf{F} \cdot \mathbf{r}'(t) = \underbrace{-\sin^2 t - \cos^2 t}_{-1} - 12\cos^3 t \sin t - 4\sin^3 t \cos t.$$

➤ Recall that $x = \cos t$, $y = \sin t$ is a standard parameterization for the unit circle centered at the origin with counterclockwise orientation. The parameterization $x = \sin t$, $y = \cos t$ reverses the orientation.

Therefore, the required integral is

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F} \cdot \mathbf{r}'(t)\, dt$$

$$= \int_0^{2\pi} (-1 - 12 \cos^3 t \sin t - 4 \cos t \sin^3 t)\, dt$$

$$= -2\pi.$$

This result is perhaps not surprising when compared to parts (a) and (b): The reversal of the orientation of S requires a reversal of the orientation of C, and we know from Section 17.2 that $\int_C \mathbf{F} \cdot d\mathbf{r} = -\int_{-C} \mathbf{F} \cdot d\mathbf{r}$. As we discuss at the end of this section, it follows that the surface integral over the closed surface enclosed by *both* paraboloids (with normal vectors everywhere outward) has the value $2\pi - 2\pi = 0$.

Related Exercises 21–22 ◄

QUICK CHECK 2 In Example 3a, we used the parameterization $\mathbf{r}(t) = \langle \cos t, \sin t, 3\cos^2 t + \sin^2 t \rangle$ for C. Confirm that the parameterization C: $\mathbf{r}(t) = \langle \cos t, \sin t, 4 - \cos^2 t - 3\sin^2 t \rangle$ also results in an answer of 2π. ◄

Interpreting the Curl

Stokes' Theorem leads to another interpretation of the curl at a point in a vector field. We need the idea of the **average circulation**. If C is the boundary of an oriented surface S, we define the average circulation of $\mathbf{F}$ over S as

$$\frac{1}{\text{area of } S} \oint_C \mathbf{F} \cdot d\mathbf{r} = \frac{1}{\text{area of } S} \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS,$$

where Stokes' Theorem is used to convert the circulation integral to a surface integral.

First consider a general rotation field $\mathbf{F} = \mathbf{a} \times \mathbf{r}$, where $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ is a constant nonzero vector and $\mathbf{r} = \langle x, y, z \rangle$. Recall that $\mathbf{F}$ describes the rotation about an axis in the direction of $\mathbf{a}$ with angular speed $\omega = |\mathbf{a}|$. We also showed that $\mathbf{F}$ has a constant curl, $\nabla \times \mathbf{F} = \nabla \times (\mathbf{a} \times \mathbf{r}) = 2\mathbf{a}$. We now take S to be a small circular disk centered at a point P, whose normal vector $\mathbf{n}$ makes an angle θ with the axis $\mathbf{a}$ (Figure 17.64). Let C be the boundary of S with a counterclockwise orientation.

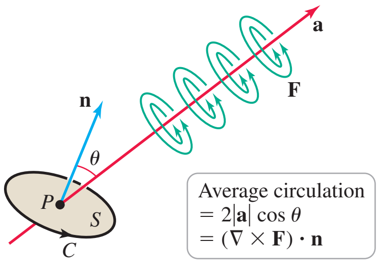


Figure 17.64

The average circulation of this vector field on S is

$$\frac{1}{\text{area of } S} \iint_S \underbrace{(\nabla \times \mathbf{F}) \cdot \mathbf{n}}_{\text{constant}}\, dS \qquad \text{Definition}$$

$$= \frac{1}{\text{area of } S} (\nabla \times \mathbf{F}) \cdot \mathbf{n} \cdot \text{area of } S \qquad \iint_S dS = \text{area of } S$$

$$= \underbrace{(\nabla \times \mathbf{F})}_{2\mathbf{a}} \cdot \mathbf{n} \qquad \text{Simplify.}$$

$$= 2|\mathbf{a}| \cos \theta. \qquad |\mathbf{n}| = 1, |\nabla \times \mathbf{F}| = 2|\mathbf{a}|$$

➤ Recall that $\mathbf{n}$ is a unit normal vector with $|\mathbf{n}| = 1$. By definition, the dot product gives $\mathbf{a} \cdot \mathbf{n} = |\mathbf{a}| \cos \theta$.

If the normal vector $\mathbf{n}$ is aligned with $\nabla \times \mathbf{F}$ (which is parallel to $\mathbf{a}$), then $\theta = 0$ and the average circulation on S has its maximum value of $2|\mathbf{a}|$. However, if the vector normal to the surface S is orthogonal to the axis of rotation ($\theta = \pi/2$), the average circulation is zero.

We see that for a general rotation field $\mathbf{F} = \mathbf{a} \times \mathbf{r}$, the curl of $\mathbf{F}$ has the following interpretations, where S is a small disk centered at a point P with a normal vector **n.**

- The scalar component of $\nabla \times \mathbf{F}$ at P in the direction of $\mathbf{n}$, which is $(\nabla \times \mathbf{F}) \cdot \mathbf{n} = 2|\mathbf{a}| \cos \theta$, is the average circulation of $\mathbf{F}$ on S.
- The direction of $\nabla \times \mathbf{F}$ at P is the direction that maximizes the average circulation of $\mathbf{F}$ on S. Equivalently, it is the direction in which the axis of a paddle wheel should be oriented to obtain the maximum angular speed.

A similar argument may be applied to a general vector field (with a variable curl) to give an analogous interpretation of the curl at a point (Exercise 48).

EXAMPLE 4 **Horizontal channel flow** Consider the velocity field $\mathbf{v} = \langle 0, 1 - x^2, 0 \rangle$, for $|x| \le 1$ and $|z| \le 1$, which represents a horizontal flow in the y-direction (Figure 17.65a).

a. Suppose you place a paddle wheel at the point $P\left(\frac{1}{2}, 0, 0\right)$. Using physical arguments, in which of the coordinate directions should the axis of the wheel point in order for the wheel to spin? In which direction does it spin? What happens if you place the wheel at $Q\left(-\frac{1}{2}, 0, 0\right)$?

b. Compute and graph the curl of $\mathbf{v}$ and provide an interpretation.

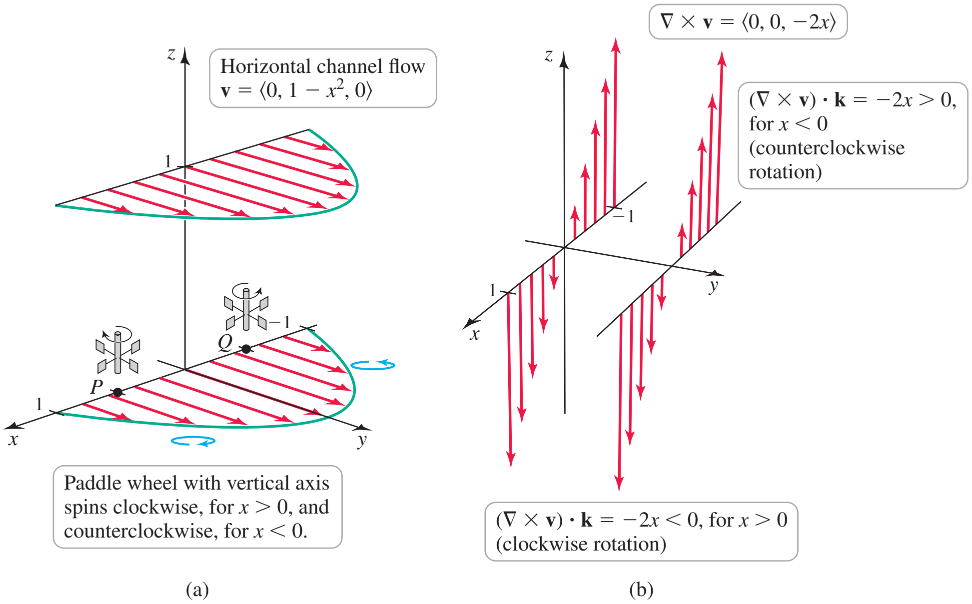


Figure 17.65

SOLUTION

a. If the axis of the wheel is aligned with the x-axis at P, the flow strikes the upper and lower halves of the wheel symmetrically and the wheel does not spin. If the axis of the wheel is aligned with the y-axis, the flow is parallel to the axis of the wheel and the wheel does not spin. If the axis of the wheel is aligned with the z-axis at P, the flow in the y-direction is greater for $x < \frac{1}{2}$ than it is for $x > \frac{1}{2}$. Therefore, a wheel located at $P\left(\frac{1}{2}, 0, 0\right)$ spins in the clockwise direction, looking from above (Figure 17.65a). Using a similar argument, we conclude that a vertically oriented paddle wheel placed at $Q\left(-\frac{1}{2}, 0, 0\right)$ spins in the counterclockwise direction (when viewed from above).

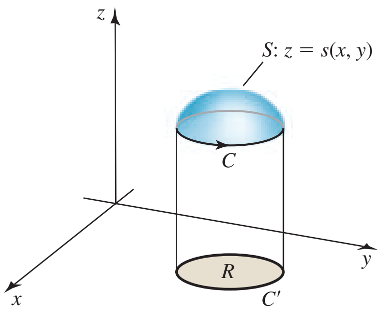
b. A short calculation shows that

$$\nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & 1 - x^2 & 0 \end{vmatrix} = -2x\,\mathbf{k}.$$

QUICK CHECK 3 In Example 4, explain why a paddle wheel with its axis aligned with the z-axis does not spin when placed on the y-axis. ◄

As shown in Figure 17.65b, the curl points in the z-direction, which is the direction of the paddle wheel axis that gives the maximum angular speed of the wheel. Consider the z-component of the curl, which is $(\nabla \times \mathbf{v}) \cdot \mathbf{k} = -2x$. At $x = 0$, this component is zero, meaning the wheel does not spin at any point along the y-axis when its axis is aligned with the z-axis. For $x > 0$, we see that $(\nabla \times \mathbf{v}) \cdot \mathbf{k} < 0$, which corresponds to clockwise rotation of the vector field. For $x < 0$, we have $(\nabla \times \mathbf{v}) \cdot \mathbf{k} > 0$, corresponding to counterclockwise rotation.

Related Exercise 26 ◄



C' is the projection of C in the xy-plane.

Figure 17.66

Proof of Stokes' Theorem

The proof of the most general case of Stokes' Theorem is intricate. However, a proof of a special case is instructive and relies on several previous results.

Consider the case in which the surface S is the graph of the function $z = s(x, y)$, defined on a region in the xy-plane. Let C be the curve that bounds S with a counterclockwise orientation, let R be the projection of S in the xy-plane, and let C' be the projection of C in the xy-plane (**Figure 17.66**).

Letting $\mathbf{F} = \langle f, g, h \rangle$, the line integral in Stokes' Theorem is

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C f\,dx + g\,dy + h\,dz.$$

The key observation for this integral is that along C (which is the boundary of S), $dz = z_x\,dx + z_y\,dy$. Making this substitution, we convert the line integral on C to a line integral on C' in the xy-plane:

$$\begin{aligned}\oint_C \mathbf{F} \cdot d\mathbf{r} &= \oint_{C'} f\,dx + g\,dy + h\underbrace{(z_x\,dx + z_y\,dy)}_{dz}\\ &= \oint_{C'} \underbrace{(f + hz_x)}_{M(x,y)}\,dx + \underbrace{(g + hz_y)}_{N(x,y)}\,dy.\end{aligned}$$

We now apply the circulation form of Green's Theorem to this line integral with $M(x, y) = f + hz_x$ and $N(x, y) = g + hz_y$; the result is

$$\oint_{C'} M\,dx + N\,dy = \iint_R (N_x - M_y)\,dA.$$

A careful application of the Chain Rule (remembering that z is a function of x and y, Exercise 49) reveals that

$$\begin{aligned}M_y &= f_y + f_z z_y + hz_{xy} + z_x(h_y + h_z z_y) \quad \text{and}\\ N_x &= g_x + g_z z_x + hz_{yx} + z_y(h_x + h_z z_x).\end{aligned}$$

Making these substitutions in the line integral and simplifying (note that $z_{xy} = z_{yx}$ is needed), we have

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R (z_x(g_z - h_y) + z_y(h_x - f_z) + (g_x - f_y))\,dA. \tag{1}$$

Now let's look at the surface integral in Stokes' Theorem. The upward vector normal to the surface is $\langle -z_x, -z_y, 1 \rangle$. Substituting the components of $\nabla \times \mathbf{F}$, the surface integral takes the form

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\,dS = \iint_R ((h_y - g_z)(-z_x) + (f_z - h_x)(-z_y) + (g_x - f_y))\,dA,$$

which upon rearrangement becomes the integral in (1).

Two Final Notes on Stokes' Theorem

1. Stokes' Theorem allows a surface integral $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\,dS$ to be evaluated using only the values of the vector field on the boundary C. This means that if a closed curve C is the boundary of two different smooth oriented surfaces S_1 and S_2, which both have an orientation consistent with that of C, then the integrals of $(\nabla \times \mathbf{F}) \cdot \mathbf{n}$ on the two surfaces are equal; that is,

$$\iint_{S_1} (\nabla \times \mathbf{F}) \cdot \mathbf{n}_1\,dS = \iint_{S_2} (\nabla \times \mathbf{F}) \cdot \mathbf{n}_2\,dS,$$

where $\mathbf{n}_1$ and $\mathbf{n}_2$ are the respective unit normal vectors consistent with the orientation of the surfaces (**Figure 17.67a**; see Example 3).

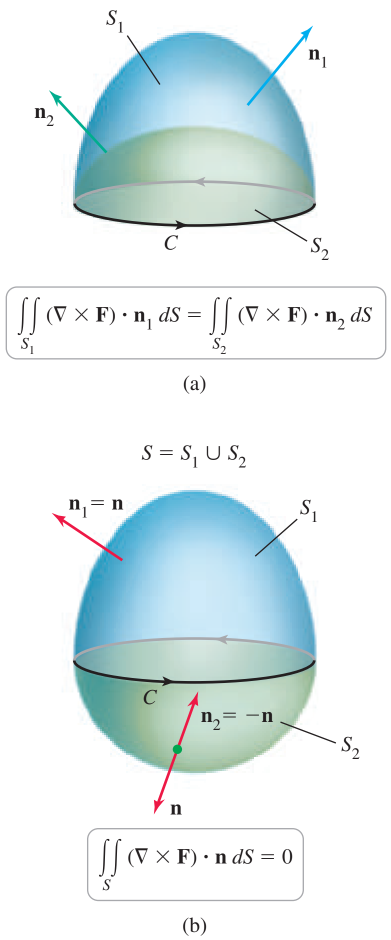


Figure 17.67

Now let's take a different perspective. Suppose S is a *closed* surface consisting of S_1 and S_2 with a common boundary curve C (Figure 17.67b). Let $\mathbf{n}$ represent the outward unit normal vector for the entire surface S. It follows that $\mathbf{n}$ points in the same direction as $\mathbf{n}_1$ and in the direction opposite to that of $\mathbf{n}_2$ (Figure 17.67b). Therefore, $\iint_{S_1} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$ and $\iint_{S_2} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$ are equal in magnitude and of opposite sign, from which we conclude that

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \iint_{S_1} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS + \iint_{S_2} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = 0.$$

This argument can be adapted to show that $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = 0$ over any closed oriented surface S (Exercise 50).

2. We can now resolve an assertion made in Section 17.5. There we proved (Theorem 17.11) that if $\mathbf{F}$ is a conservative vector field, then $\nabla \times \mathbf{F} = \mathbf{0}$; we claimed, but did not prove, that the converse is true. The converse follows directly from Stokes' Theorem.

THEOREM 17.16 Curl $\mathbf{F} = \mathbf{0}$ Implies F Is Conservative
Suppose $\nabla \times \mathbf{F} = \mathbf{0}$ throughout an open simply connected region D of $\mathbb{R}^3$. Then $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ on all closed simple smooth curves C in D, and $\mathbf{F}$ is a conservative vector field on D.

Proof: Given a closed simple smooth curve C, an advanced result states that C is the boundary of at least one smooth oriented surface S in D. By Stokes' Theorem,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \underbrace{(\nabla \times \mathbf{F})}_{0} \cdot \mathbf{n}\, dS = 0.$$

Because the line integral equals zero over all such curves in D, the vector field is conservative on D by Theorem 17.6. ◄

SECTION 17.7 EXERCISES

Getting Started

1. Explain the meaning of the integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ in Stokes' Theorem.
2. Explain the meaning of the integral $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$ in Stokes' Theorem.
3. Explain the meaning of Stokes' Theorem.
4. Why does a conservative vector field produce zero circulation around a closed curve?

Practice Exercises

5–10. Verifying Stokes' Theorem *Verify that the line integral and the surface integral of Stokes' Theorem are equal for the following vector fields, surfaces S, and closed curves C. Assume C has counterclockwise orientation and S has a consistent orientation.*

5. $\mathbf{F} = \langle y, -x, 10 \rangle$; S is the upper half of the sphere $x^2 + y^2 + z^2 = 1$ and C is the circle $x^2 + y^2 = 1$ in the xy-plane.
6. $\mathbf{F} = \langle 0, -x, y \rangle$; S is the upper half of the sphere $x^2 + y^2 + z^2 = 4$ and C is the circle $x^2 + y^2 = 4$ in the xy-plane.
7. $\mathbf{F} = \langle x, y, z \rangle$; S is the paraboloid $z = 8 - x^2 - y^2$, for $0 \le z \le 8$, and C is the circle $x^2 + y^2 = 8$ in the xy-plane.
8. $\mathbf{F} = \langle 2z, -4x, 3y \rangle$; S is the cap of the sphere $x^2 + y^2 + z^2 = 169$ above the plane $z = 12$ and C is the boundary of S.
9. $\mathbf{F} = \langle y - z, z - x, x - y \rangle$; S is the cap of the sphere $x^2 + y^2 + z^2 = 16$ above the plane $z = \sqrt{7}$ and C is the boundary of S.
10. $\mathbf{F} = \langle -y, -x - z, y - x \rangle$; S is the part of the plane $z = 6 - y$ that lies in the cylinder $x^2 + y^2 = 16$ and C is the boundary of S.

11–16. Stokes' Theorem for evaluating line integrals *Evaluate the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ by evaluating the surface integral in Stokes' Theorem with an appropriate choice of S. Assume C has a counterclockwise orientation.*

11. $\mathbf{F} = \langle 2y, -z, x \rangle$; C is the circle $x^2 + y^2 = 12$ in the plane $z = 0$.
12. $\mathbf{F} = \langle y, xz, -y \rangle$; C is the ellipse $x^2 + y^2/4 = 1$ in the plane $z = 1$.
13. $\mathbf{F} = \langle x^2 - z^2, y, 2xz \rangle$; C is the boundary of the plane $z = 4 - x - y$ in the first octant.
14. $\mathbf{F} = \langle x^2 - y^2, z^2 - x^2, y^2 - z^2 \rangle$; C is the boundary of the square $|x| \le 1$, $|y| \le 1$ in the plane $z = 0$.

15. $\mathbf{F} = \langle y^2, -z^2, x \rangle$; C is the circle $\mathbf{r}(t) = \langle 3\cos t, 4\cos t, 5\sin t \rangle$, for $0 \le t \le 2\pi$.

16. $\mathbf{F} = \langle 2xy\sin z, x^2\sin z, x^2y\cos z \rangle$; C is the boundary of the plane $z = 8 - 2x - 4y$ in the first octant.

17–24. Stokes' Theorem for evaluating surface integrals *Evaluate the line integral in Stokes' Theorem to determine the value of the surface integral* $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$. *Assume* $\mathbf{n}$ *points in an upward direction.*

17. $\mathbf{F} = \langle x, y, z \rangle$; S is the upper half of the ellipsoid $x^2/4 + y^2/9 + z^2 = 1$.

18. $\mathbf{F} = \mathbf{r}/|\mathbf{r}|$; S is the paraboloid $x = 9 - y^2 - z^2$, for $0 \le x \le 9$ (excluding its base), and $\mathbf{r} = \langle x, y, z \rangle$.

19. $\mathbf{F} = \langle 2y, -z, x - y - z \rangle$; S is the cap of the sphere $x^2 + y^2 + z^2 = 25$, for $3 \le x \le 5$ (excluding its base).

20. $\mathbf{F} = \langle x + y, y + z, z + x \rangle$; S is the tilted disk enclosed by $\mathbf{r}(t) = \langle \cos t, 2\sin t, \sqrt{3}\cos t \rangle$.

21. $\mathbf{F} = \langle y, z - x, -y \rangle$; S is the part of the paraboloid $z = 2 - x^2 - 2y^2$ that lies within the cylinder $x^2 + y^2 = 1$.

22. $\mathbf{F} = \langle 4x, -8z, 4y \rangle$; S is the part of the paraboloid $z = 1 - 2x^2 - 3y^2$ that lies within the paraboloid $z = 2x^2 + y^2$.

23. $\mathbf{F} = \langle y, 1, z \rangle$; S is the part of the surface $z = 2\sqrt{x}$ that lies within the cone $z = \sqrt{x^2 + y^2}$.

24. $\mathbf{F} = \langle e^x, 1/z, y \rangle$; S is the part of the surface $z = 4 - 3y^2$ that lies within the paraboloid $z = x^2 + y^2$.

25–28. Interpreting and graphing the curl *For the following velocity fields, compute the curl, make a sketch of the curl, and interpret the curl.*

25. $\mathbf{v} = \langle 0, 0, y \rangle$

26. $\mathbf{v} = \langle 1 - z^2, 0, 0 \rangle$

27. $\mathbf{v} = \langle -2z, 0, 1 \rangle$

28. $\mathbf{v} = \langle 0, -z, y \rangle$

29. **Explain why or why not** Determine whether the following statements are true and give an explanation or counterexample.
   a. A paddle wheel with its axis in the direction $\langle 0, 1, -1 \rangle$ would not spin when put in the vector field $\mathbf{F} = \langle 1, 1, 2 \rangle \times \langle x, y, z \rangle$.
   b. Stokes' Theorem relates the flux of a vector field $\mathbf{F}$ across a surface to values of $\mathbf{F}$ on the boundary of the surface.
   c. A vector field of the form $\mathbf{F} = \langle a + f(x), b + g(y), c + h(z) \rangle$, where a, b, and c are constants, has zero circulation on a closed curve.
   d. If a vector field has zero circulation on all simple closed smooth curves C in a region D, then $\mathbf{F}$ is conservative on D.

30–33. Conservative fields *Use Stokes' Theorem to find the circulation of the following vector fields around any simple closed smooth curve C.*

30. $\mathbf{F} = \langle 2x, -2y, 2z \rangle$

31. $\mathbf{F} = \nabla(x\sin ye^z)$

32. $\mathbf{F} = \langle 3x^2y, x^3 + 2yz^2, 2y^2z \rangle$

33. $\mathbf{F} = \langle y^2z^3, 2xyz^3, 3xy^2z^2 \rangle$

34–38. Tilted disks *Let S be the disk enclosed by the curve* C: $\mathbf{r}(t) = \langle \cos\varphi\cos t, \sin t, \sin\varphi\cos t \rangle$, *for* $0 \le t \le 2\pi$, *where* $0 \le \varphi \le \pi/2$ *is a fixed angle.*

34. What is the area of S? Find a vector normal to S.

35. What is the length of C?

36. Use Stokes' Theorem and a surface integral to find the circulation on C of the vector field $\mathbf{F} = \langle -y, x, 0 \rangle$ as a function of φ. For what value of φ is the circulation a maximum?

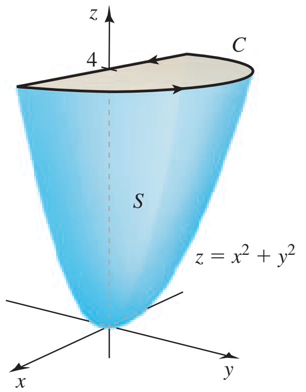
37. What is the circulation on C of the vector field $\mathbf{F} = \langle -y, -z, x \rangle$ as a function of φ? For what value of φ is the circulation a maximum?

38. Consider the vector field $\mathbf{F} = \mathbf{a} \times \mathbf{r}$, where $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ is a constant nonzero vector and $\mathbf{r} = \langle x, y, z \rangle$. Show that the circulation is a maximum when $\mathbf{a}$ points in the direction of the normal to S.

39. **Circulation in a plane** A circle C in the plane $x + y + z = 8$ has a radius of 4 and center $(2, 3, 3)$. Evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$ for $\mathbf{F} = \langle 0, -z, 2y \rangle$, where C has a counterclockwise orientation when viewed from above. Does the circulation depend on the radius of the circle? Does it depend on the location of the center of the circle?

40. **No integrals** Let $\mathbf{F} = \langle 2z, z, 2y + x \rangle$, and let S be the hemisphere of radius a with its base in the xy-plane and center at the origin.
   a. Evaluate $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$ by computing $\nabla \times \mathbf{F}$ and appealing to symmetry.
   b. Evaluate the line integral using Stokes' Theorem to check part (a).

41. **Compound surface and boundary** Begin with the paraboloid $z = x^2 + y^2$, for $0 \le z \le 4$, and slice it with the plane $y = 0$. Let S be the surface that remains for $y \ge 0$ (including the planar surface in the xz-plane) (see figure). Let C be the semicircle and line segment that bound the cap of S in the plane $z = 4$ with counterclockwise orientation. Let $\mathbf{F} = \langle 2z + y, 2x + z, 2y + x \rangle$.
   a. Describe the direction of the vectors normal to the surface that are consistent with the orientation of C.
   b. Evaluate $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$.
   c. Evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$ and check for agreement with part (b).



42. **Ampère's Law** The French physicist André-Marie Ampère (1775–1836) discovered that an electrical current I in a wire produces a magnetic field $\mathbf{B}$. A special case of Ampère's Law relates the current to the magnetic field through the equation $\oint_C \mathbf{B} \cdot d\mathbf{r} = \mu I$, where C is any closed curve through which the wire passes and μ is a physical constant. Assume the current I is given in terms of the current density $\mathbf{J}$ as $I = \iint_S \mathbf{J} \cdot \mathbf{n}\, dS$, where S is an oriented surface with C as a boundary. Use Stokes' Theorem to show that an equivalent form of Ampère's Law is $\nabla \times \mathbf{B} = \mu\mathbf{J}$.

43. **Maximum surface integral** Let S be the paraboloid $z = a(1 - x^2 - y^2)$, for $z \ge 0$, where $a > 0$ is a real number. Let $\mathbf{F} = \langle x - y, y + z, z - x \rangle$. For what value(s) of a (if any) does $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$ have its maximum value?

Explorations and Challenges

44. Area of a region in a plane Let R be a region in a plane that has a unit normal vector $\mathbf{n} = \langle a, b, c \rangle$ and boundary C. Let $\mathbf{F} = \langle bz, cx, ay \rangle$.

a. Show that $\nabla \times \mathbf{F} = \mathbf{n}$.

b. Use Stokes' Theorem to show that

$$\text{area of } R = \oint_C \mathbf{F} \cdot d\mathbf{r}.$$

c. Consider the curve C given by $\mathbf{r} = \langle 5 \sin t, 13 \cos t, 12 \sin t \rangle$, for $0 \le t \le 2\pi$. Prove that C lies in a plane by showing that $\mathbf{r} \times \mathbf{r}'$ is constant for all t.

d. Use part (b) to find the area of the region enclosed by C in part (c). (*Hint:* Find the unit normal vector that is consistent with the orientation of C.)

45. Choosing a more convenient surface The goal is to evaluate $A = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$, where $\mathbf{F} = \langle yz, -xz, xy \rangle$ and S is the surface of the upper half of the ellipsoid $x^2 + y^2 + 8z^2 = 1$ $(z \ge 0)$.

a. Evaluate a surface integral over a more convenient surface to find the value of A.

b. Evaluate A using a line integral.

46. Radial fields and zero circulation Consider the radial vector fields $\mathbf{F} = \mathbf{r}/|\mathbf{r}|^p$, where p is a real number and $\mathbf{r} = \langle x, y, z \rangle$. Let C be any circle in the xy-plane centered at the origin.

a. Evaluate a line integral to show that the field has zero circulation on C.

b. For what values of p does Stokes' Theorem apply? For those values of p, use the surface integral in Stokes' Theorem to show that the field has zero circulation on C.

47. Zero curl Consider the vector field

$$\mathbf{F} = -\frac{y}{x^2 + y^2}\mathbf{i} + \frac{x}{x^2 + y^2}\mathbf{j} + z\,\mathbf{k}.$$

a. Show that $\nabla \times \mathbf{F} = \mathbf{0}$.

b. Show that $\oint_C \mathbf{F} \cdot d\mathbf{r}$ is not zero on a circle C in the xy-plane enclosing the origin.

c. Explain why Stokes' Theorem does not apply in this case.

48. Average circulation Let S be a small circular disk of radius R centered at the point P with a unit normal vector $\mathbf{n}$. Let C be the boundary of S.

a. Express the average circulation of the vector field $\mathbf{F}$ on S as a surface integral of $\nabla \times \mathbf{F}$.

b. Argue that for small R, the average circulation approaches $(\nabla \times \mathbf{F})|_P \cdot \mathbf{n}$ (the component of $\nabla \times \mathbf{F}$ in the direction of $\mathbf{n}$ evaluated at P) with the approximation improving as $R \to 0$.

49. Proof of Stokes' Theorem Confirm the following step in the proof of Stokes' Theorem. If $z = s(x, y)$ and f, g, and h are functions of x, y, and z, with $M = f + hz_x$ and $N = g + hz_y$, then

$$M_y = f_y + f_z z_y + hz_{xy} + z_x(h_y + h_z z_y) \quad \text{and}$$
$$N_x = g_x + g_z z_x + hz_{yx} + z_y(h_x + h_z z_x).$$

50. Stokes' Theorem on closed surfaces Prove that if $\mathbf{F}$ satisfies the conditions of Stokes' Theorem, then $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = 0$, where S is a smooth surface that encloses a region.

51. Rotated Green's Theorem Use Stokes' Theorem to write the circulation form of Green's Theorem in the yz-plane.

QUICK CHECK ANSWERS

1. If S is a region in the xy-plane, $\mathbf{n} = \mathbf{k}$ and $(\nabla \times \mathbf{F}) \cdot \mathbf{n}$ becomes $g_x - f_y$. **3.** The vector field is symmetric about the y-axis. ◄

17.8 Divergence Theorem

Vector fields can represent electric or magnetic fields, air velocities in hurricanes, or blood flow in an artery. These and other vector phenomena suggest movement of a "substance." A frequent question concerns the amount of a substance that flows across a surface—for example, the amount of water that passes across the membrane of a cell per unit time. Such flux calculations may be done using flux integrals as in Section 17.6. The Divergence Theorem offers an alternative method. In effect, it says that instead of integrating the flow into and out of a region across its boundary, you may also add up all the sources (or sinks) of the flow throughout the region.

Divergence Theorem

➤ Circulation form of Green's Theorem $\rightarrow$ Stokes' Theorem

Flux form of Green's Theorem $\rightarrow$ Divergence Theorem

The Divergence Theorem is the three-dimensional version of the flux form of Green's Theorem. Recall that if R is a region in the xy-plane, C is the simple closed piecewise-smooth oriented boundary of R, and $\mathbf{F} = \langle f, g \rangle$ is a vector field, then Green's Theorem says that

$$\underbrace{\oint_C \mathbf{F} \cdot \mathbf{n}\, ds}_{\text{flux across } C} = \iint_R \underbrace{(f_x + g_y)}_{\text{divergence}}\, dA.$$

The line integral on the left gives the flux across the boundary of R. The double integral on the right measures the net expansion or contraction of the vector field within R. If $\mathbf{F}$ represents a fluid flow or the transport of a material, the theorem says that the cumulative effect of the sources (or sinks) of the flow within R equals the net flow across its boundary.

The Divergence Theorem is a direct extension of Green's Theorem. The plane region in Green's Theorem becomes a solid region D in $\mathbb{R}^3$, and the closed curve in Green's Theorem becomes the oriented surface S that encloses D. The flux integral in Green's Theorem becomes a surface integral over S, and the double integral in Green's Theorem becomes a triple integral over D of the three-dimensional divergence (Figure 17.68).

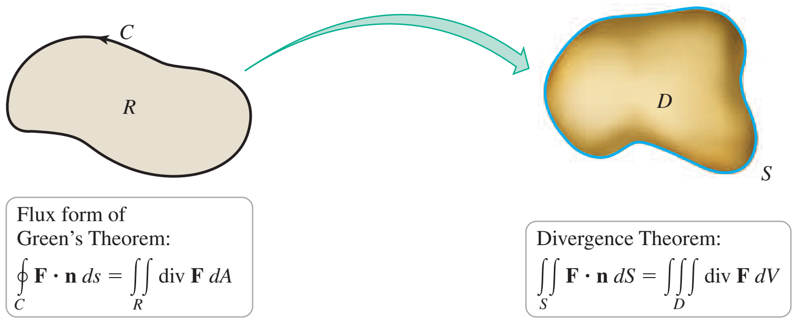


Figure 17.68

> **THEOREM 17.17 Divergence Theorem**
> Let $\mathbf{F}$ be a vector field whose components have continuous first partial derivatives in a connected and simply connected region D in $\mathbb{R}^3$ enclosed by an oriented surface S. Then
> $$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_D \nabla\cdot\mathbf{F}\,dV,$$
> where $\mathbf{n}$ is the outward unit normal vector on S.

QUICK CHECK 1 Interpret the Divergence Theorem in the case that $\mathbf{F} = \langle a, b, c\rangle$ is a constant vector field and D is a ball. ◄

The surface integral on the left gives the flux of the vector field across the boundary; a positive flux integral means there is a net flow of the field out of the region. The triple integral on the right is the cumulative expansion or contraction of the field over the region D. The proof of a special case of the theorem is given later in this section.

EXAMPLE 1 Verifying the Divergence Theorem Consider the radial field $\mathbf{F} = \langle x, y, z\rangle$ and let S be the sphere $x^2 + y^2 + z^2 = a^2$ that encloses the region D. Assume $\mathbf{n}$ is the outward unit normal vector on the sphere. Evaluate both integrals of the Divergence Theorem.

SOLUTION The divergence of $\mathbf{F}$ is

$$\nabla\cdot\mathbf{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3.$$

Integrating over D, we have

$$\iiint_D \nabla\cdot\mathbf{F}\,dV = \iiint_D 3\,dV = 3 \times \text{volume of } D = 3\cdot\frac{4}{3}\pi a^3 = 4\pi a^3.$$

To evaluate the surface integral, we parameterize the sphere (Section 17.6, Table 17.3) in the form

$$\mathbf{r} = \langle x, y, z\rangle = \langle a\sin u\cos v, a\sin u\sin v, a\cos u\rangle,$$

where $R = \{(u, v)\colon 0 \le u \le \pi, 0 \le v \le 2\pi\}$ (u and v are the spherical coordinates φ and θ, respectively). The surface integral is

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_R \mathbf{F}\cdot(\mathbf{t}_u \times \mathbf{t}_v)\,dA,$$

where the required vector normal to the surface is

$$\mathbf{t}_u \times \mathbf{t}_v = \langle a^2 \sin^2 u \cos v, a^2 \sin^2 u \sin v, a^2 \sin u \cos u \rangle.$$

Substituting for $\mathbf{F} = \langle x, y, z \rangle$ and $\mathbf{t}_u \times \mathbf{t}_v$, we find after simplifying that $\mathbf{F} \cdot (\mathbf{t}_u \times \mathbf{t}_v) = a^3 \sin u$. Therefore, the surface integral becomes

➤ See Exercise 32 for an alternative evaluation of the surface integral.

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_R \underbrace{\mathbf{F} \cdot (\mathbf{t}_u \times \mathbf{t}_v)}_{a^3 \sin u}\, dA$$

$$= \int_0^{2\pi} \int_0^{\pi} a^3 \sin u \, du\, dv \quad \text{Substitute for } \mathbf{F} \text{ and } \mathbf{t}_u \times \mathbf{t}_v.$$

$$= 4\pi a^3. \quad \text{Evaluate integrals.}$$

The two integrals of the Divergence Theorem are equal.

Related Exercise 9 ◄

EXAMPLE 2 **Divergence Theorem with a rotation field** Consider the rotation field

$$\mathbf{F} = \mathbf{a} \times \mathbf{r} = \langle 1, 0, 1 \rangle \times \langle x, y, z \rangle = \langle -y, x - z, y \rangle.$$

Let S be the hemisphere $x^2 + y^2 + z^2 = a^2$, for $z \geq 0$, together with its base in the xy-plane. Find the net outward flux across S.

SOLUTION To find the flux using surface integrals, two surfaces must be considered (the hemisphere and its base). The Divergence Theorem gives a simpler solution. Note that

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x - z) + \frac{\partial}{\partial z}(y) = 0.$$

We see that the flux across the hemisphere is zero.

Related Exercise 13 ◄

With Stokes' Theorem, rotation fields are noteworthy because they have a nonzero curl. With the Divergence Theorem, the situation is reversed. As suggested by Example 2, pure rotation fields of the form $\mathbf{F} = \mathbf{a} \times \mathbf{r}$ have zero divergence (Exercise 16). However, with the Divergence Theorem, radial fields are interesting and have many physical applications.

EXAMPLE 3 **Computing flux with the Divergence Theorem** Find the net outward flux of the field $\mathbf{F} = xyz\langle 1, 1, 1 \rangle$ across the boundaries of the cube $D = \{(x, y, z): 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1\}$.

SOLUTION Computing a surface integral involves the six faces of the cube. The Divergence Theorem gives the outward flux with a single integral over D. The divergence of the field is

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(xyz) + \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}(xyz) = yz + xz + xy.$$

The integral over D is a standard triple integral:

$$\iiint_D \nabla \cdot \mathbf{F}\, dV = \iiint_D (yz + xz + xy)\, dV$$

$$= \int_0^1 \int_0^1 \int_0^1 (yz + xz + xy)\, dx\, dy\, dz \quad \text{Convert to a triple integral.}$$

$$= \frac{3}{4}. \quad \text{Evaluate integrals.}$$

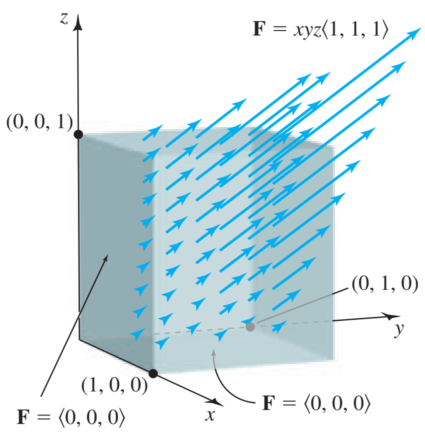


Figure 17.69

On three faces of the cube (those that lie in the coordinate planes), we see that $\mathbf{F}(0, y, z) = \mathbf{F}(x, 0, z) = \mathbf{F}(x, y, 0) = \mathbf{0}$, so there is no contribution to the flux on these faces (**Figure 17.69**). On the other three faces, the vector field has components out of the cube. Therefore, the net outward flux is positive, as calculated.

Related Exercises 18–19 ◄

QUICK CHECK 2 In Example 3, does the vector field have negative components anywhere in the cube D? Is the divergence negative anywhere in D? ◄

➤ The mass transport is also called the *flux density*; when multiplied by an area, it gives the flux. We use the convention that flux has units of mass per unit time.

➤ Check the units: If $\mathbf{F}$ has units of mass/(area · time), then the flux has units of mass/time ($\mathbf{n}$ has no units).

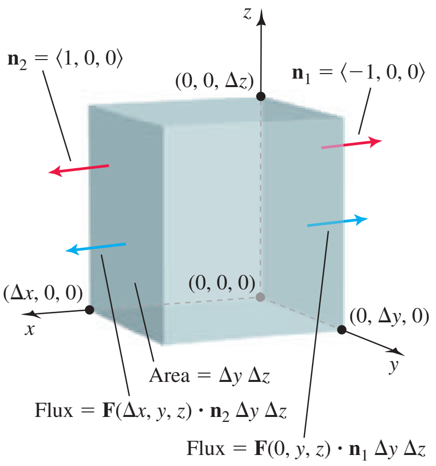


Figure 17.70

Interpretation of the Divergence Theorem Using Mass Transport Suppose $\mathbf{v}$ is the velocity field of a material, such as water or molasses, and ρ is its constant density. The vector field $\mathbf{F} = \rho\mathbf{v} = \langle f, g, h \rangle$ describes the **mass transport** of the material, with units of (mass/vol) × (length/time) = mass/(area · time); typical units of mass transport are $\mathrm{g/m^2/s}$. This means that $\mathbf{F}$ gives the mass of material flowing past a point (in each of the three coordinate directions) per unit of surface area per unit of time. When $\mathbf{F}$ is multiplied by an area, the result is the **flux**, with units of mass/unit time.

Now consider a small cube located in the vector field with its faces parallel to the coordinate planes. One vertex is located at $(0, 0, 0)$, the opposite vertex is at $(\Delta x, \Delta y, \Delta z)$, and (x, y, z) is an arbitrary point in the cube (Figure 17.70). The goal is to compute the approximate flux of material across the faces of the cube. We begin with the flux across the two parallel faces $x = 0$ and $x = \Delta x$.

The outward unit vectors normal to the faces $x = 0$ and $x = \Delta x$ are $\mathbf{n}_1 = \langle -1, 0, 0 \rangle$ and $\mathbf{n}_2 = \langle 1, 0, 0 \rangle$, respectively. Each face has area $\Delta y\, \Delta z$, so the approximate net flux across these faces is

$$\underbrace{\mathbf{F}(\Delta x, y, z)}_{x = \Delta x \text{ face}} \cdot \underbrace{\mathbf{n}_2}_{\langle 1, 0, 0 \rangle} \Delta y\, \Delta z + \underbrace{\mathbf{F}(0, y, z)}_{x = 0 \text{ face}} \cdot \underbrace{\mathbf{n}_1}_{\langle -1, 0, 0 \rangle} \Delta y\, \Delta z$$
$$= (f(\Delta x, y, z) - f(0, y, z))\, \Delta y\, \Delta z.$$

Note that if $f(\Delta x, y, z) > f(0, y, z)$, the net flux across these two faces of the cube is positive, which means the net flow is *out* of the cube. Letting $\Delta V = \Delta x\, \Delta y\, \Delta z$ be the volume of the cube, we rewrite the net flux as

$$\begin{aligned}
&(f(\Delta x, y, z) - f(0, y, z))\, \Delta y\, \Delta z \\
&\quad = \frac{f(\Delta x, y, z) - f(0, y, z)}{\Delta x}\, \Delta x\, \Delta y\, \Delta z \qquad \text{Multiply by } \frac{\Delta x}{\Delta x}. \\
&\quad = \frac{f(\Delta x, y, z) - f(0, y, z)}{\Delta x}\, \Delta V. \qquad \Delta V = \Delta x\, \Delta y\, \Delta z
\end{aligned}$$

A similar argument can be applied to the other two pairs of faces. The approximate net flux across the faces $y = 0$ and $y = \Delta y$ is

$$\frac{g(x, \Delta y, z) - g(x, 0, z)}{\Delta y}\, \Delta V,$$

and the approximate net flux across the faces $z = 0$ and $z = \Delta z$ is

$$\frac{h(x, y, \Delta z) - h(x, y, 0)}{\Delta z}\, \Delta V.$$

Adding these three individual fluxes gives the approximate net flux out of the cube:

$$\begin{aligned}
\text{net flux out of cube} &\approx \Bigg(\underbrace{\frac{f(\Delta x, y, z) - f(0, y, z)}{\Delta x}}_{\approx \frac{\partial f}{\partial x}(0, 0, 0)} + \underbrace{\frac{g(x, \Delta y, z) - g(x, 0, z)}{\Delta y}}_{\approx \frac{\partial g}{\partial y}(0, 0, 0)} \\
&\qquad + \underbrace{\frac{h(x, y, \Delta z) - h(x, y, 0)}{\Delta z}}_{\approx \frac{\partial h}{\partial z}(0, 0, 0)} \Bigg) \Delta V \\
&\approx \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} \right)\Bigg|_{(0, 0, 0)} \Delta V \\
&= (\nabla \cdot \mathbf{F})(0, 0, 0)\, \Delta V.
\end{aligned}$$

Notice how the three quotients approximate partial derivatives when Δx, Δy, and Δz are small. A similar argument may be made at any point in the region.

Taking one more step, we show informally how the Divergence Theorem arises. Suppose the small cube we just analyzed is one of many small cubes of volume ΔV that fill a region D. We label the cubes $k = 1, \ldots, n$ and apply the preceding argument to each cube, letting $(\nabla \cdot \mathbf{F})_k$ be the divergence evaluated at a point in the kth cube. Adding the individual contributions to the net flux from each cube, we obtain the approximate net flux across the boundary of D:

> In making this argument, notice that for two adjacent cubes, the flux into one cube equals the flux out of the other cube across the common face. Therefore, there is a cancellation of fluxes throughout the interior of D.

$$\text{net flux out of } D \approx \sum_{k=1}^{n} (\nabla \cdot \mathbf{F})_k \, \Delta V.$$

Letting the volume of the cubes ΔV approach 0, and letting the number of cubes n increase, we obtain an integral over D:

$$\text{net flux out of } D = \lim_{n\to\infty} \sum_{k=1}^{n} (\nabla \cdot \mathbf{F})_k \, \Delta V = \iiint_D \nabla \cdot \mathbf{F} \, dV.$$

The net flux across the boundary of D is also given by $\iint_S \mathbf{F} \cdot \mathbf{n} \, dS$. Equating the surface integral and the volume integral gives the Divergence Theorem. Now we look at a formal proof.

QUICK CHECK 3 Draw the unit cube $D = \{(x, y, z): 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1\}$ and sketch the vector field $\mathbf{F} = \langle x, -y, 2z \rangle$ on the six faces of the cube. Compute and interpret div $\mathbf{F}$.

Proof of the Divergence Theorem

We prove the Divergence Theorem under special conditions on the region D. Let R be the projection of D in the xy-plane (Figure 17.71); that is,

$$R = \{(x, y): (x, y, z) \text{ is in } D\}.$$

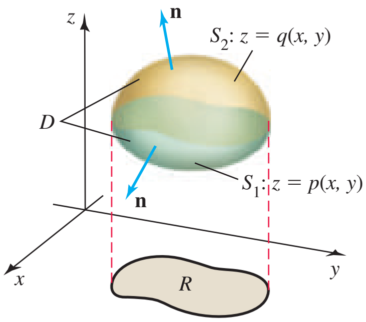


Figure 17.71

Assume the boundary of D is S and let $\mathbf{n}$ be the unit vector normal to S that points outward.

Letting $\mathbf{F} = \langle f, g, h \rangle = f\,\mathbf{i} + g\,\mathbf{j} + h\,\mathbf{k}$, the surface integral in the Divergence Theorem is

$$\begin{aligned}\iint_S \mathbf{F} \cdot \mathbf{n} \, dS &= \iint_S (f\,\mathbf{i} + g\,\mathbf{j} + h\,\mathbf{k}) \cdot \mathbf{n} \, dS \\ &= \iint_S f\,\mathbf{i} \cdot \mathbf{n} \, dS + \iint_S g\,\mathbf{j} \cdot \mathbf{n} \, dS + \iint_S h\,\mathbf{k} \cdot \mathbf{n} \, dS.\end{aligned}$$

The volume integral in the Divergence Theorem is

$$\iiint_D \nabla \cdot \mathbf{F} \, dV = \iiint_D \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} \right) dV.$$

Matching terms of the surface and volume integrals, the theorem is proved by showing that

$$\iint_S f\,\mathbf{i} \cdot \mathbf{n} \, dS = \iiint_D \frac{\partial f}{\partial x} \, dV, \qquad (1)$$

$$\iint_S g\,\mathbf{j} \cdot \mathbf{n} \, dS = \iiint_D \frac{\partial g}{\partial y} \, dV, \text{ and} \qquad (2)$$

$$\iint_S h\,\mathbf{k} \cdot \mathbf{n} \, dS = \iiint_D \frac{\partial h}{\partial z} \, dV. \qquad (3)$$

We work on equation (3) assuming special properties for D. Suppose D is bounded by two surfaces S_1: $z = p(x, y)$ and S_2: $z = q(x, y)$, where $p(x, y) \le q(x, y)$ on R (Figure 17.71). The Fundamental Theorem of Calculus is used in the triple integral to show that

$$\begin{aligned}\iiint_D \frac{\partial h}{\partial z} \, dV &= \iint_R \int_{p(x,y)}^{q(x,y)} \frac{\partial h}{\partial z} \, dz \, dx \, dy \\ &= \iint_R \big(h(x, y, q(x, y)) - h(x, y, p(x, y)) \big) \, dx \, dy. \quad \text{Evaluate inner integral.}\end{aligned}$$

Now let's turn to the surface integral in equation (3), $\iint_S h\,\mathbf{k} \cdot \mathbf{n} \, dS$, and note that S consists of three pieces: the lower surface S_1, the upper surface S_2, and the vertical sides S_3 of the surface (if they exist). The normal to S_3 is everywhere orthogonal to $\mathbf{k}$, so $\mathbf{k} \cdot \mathbf{n} = 0$

and the S_3 integral makes no contribution. What remains is to compute the surface integrals over S_1 and S_2.

The required outward normal to S_2 (which is the graph of $z = q(x, y)$) is $\langle -q_x, -q_y, 1\rangle$. The outward normal to S_1 (which is the graph of $z = p(x, y)$) points *downward*, so it is given by $\langle p_x, p_y, -1\rangle$. The surface integral of (3) becomes

$$\begin{aligned}\iint_S h\,\mathbf{k}\cdot\mathbf{n}\,dS &= \iint_{S_2} h(x, y, z)\,\mathbf{k}\cdot\mathbf{n}\,dS + \iint_{S_1} h(x, y, z)\,\mathbf{k}\cdot\mathbf{n}\,dS\\ &= \iint_R h(x, y, q(x, y))\,\underbrace{\mathbf{k}\cdot\langle -q_x, -q_y, 1\rangle}_{1}\,dx\,dy\\ &\quad+\iint_R h(x, y, p(x, y))\,\underbrace{\mathbf{k}\cdot\langle p_x, p_y, -1\rangle}_{-1}\,dx\,dy\\ &= \iint_R h(x, y, q(x, y))\,dx\,dy - \iint_R h(x, y, p(x, y))\,dx\,dy. \quad \text{Simplify.}\end{aligned}$$

Observe that both the volume integral and the surface integral of (3) reduce to the same integral over R. Therefore, $\iint_S h\,\mathbf{k}\cdot\mathbf{n}\,dS = \iiint_D \frac{\partial h}{\partial z}\,dV$.

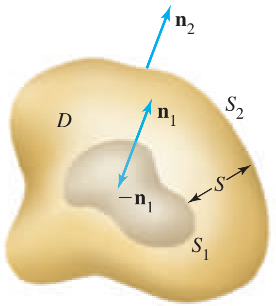
Equations (1) and (2) are handled in a similar way.

- To prove (1), we make the special assumption that D is also bounded by two surfaces, S_1: $x = s(y, z)$ and S_2: $x = t(y, z)$, where $s(y, z) \le t(y, z)$.
- To prove (2), we assume D is bounded by two surfaces, S_1: $y = u(x, z)$ and S_2: $y = v(x, z)$, where $u(x, z) \le v(x, z)$.

When combined, equations (1), (2), and (3) yield the Divergence Theorem. ◄

Divergence Theorem for Hollow Regions

The Divergence Theorem may be extended to more general solid regions. Here we consider the important case of hollow regions. Suppose D is a region consisting of all points inside a closed oriented surface S_2 and outside a closed oriented surface S_1, where S_1 lies within S_2 (**Figure 17.72**). Therefore, the boundary of D consists of S_1 and S_2, which we denote S. (Note that D is simply connected.)



$\mathbf{n}_1$ is the outward unit normal to S_1 and points into D. The outward unit normal to S on S_1 is $-\mathbf{n}_1$.

Figure 17.72

We let $\mathbf{n}_1$ and $\mathbf{n}_2$ be the outward unit normal vectors for S_1 and S_2, respectively. Note that $\mathbf{n}_1$ points into D, so the outward normal to S on S_1 is $-\mathbf{n}_1$. With this observation, the Divergence Theorem takes the following form.

➤ It's important to point out again that $\mathbf{n}_1$ is the unit normal that we would use for S_1 alone, independent of S. It is the outward unit normal to S_1, but it points into D.

THEOREM 17.18 Divergence Theorem for Hollow Regions

Suppose the vector field $\mathbf{F}$ satisfies the conditions of the Divergence Theorem on a region D bounded by two oriented surfaces S_1 and S_2, where S_1 lies within S_2. Let S be the entire boundary of D ($S = S_1 \cup S_2$) and let $\mathbf{n}_1$ and $\mathbf{n}_2$ be the outward unit normal vectors for S_1 and S_2, respectively. Then

$$\iiint_D \nabla\cdot\mathbf{F}\,dV = \iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_{S_2} \mathbf{F}\cdot\mathbf{n}_2\,dS - \iint_{S_1} \mathbf{F}\cdot\mathbf{n}_1\,dS.$$

This form of the Divergence Theorem is applicable to vector fields that are not differentiable at the origin, as is the case with some important radial vector fields.

EXAMPLE 4 **Flux for an inverse square field** Consider the inverse square vector field

$$\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|^3} = \frac{\langle x, y, z\rangle}{(x^2 + y^2 + z^2)^{3/2}}.$$

➤ Recall that an inverse square force is proportional to $1/|\mathbf{r}|^2$ multiplied by a unit vector in the radial direction, which is $\mathbf{r}/|\mathbf{r}|$. Combining these two factors gives $\mathbf{F} = \mathbf{r}/|\mathbf{r}|^3$.

a. Find the net outward flux of $\mathbf{F}$ across the surface of the region $D = \{(x, y, z): a^2 \leq x^2 + y^2 + z^2 \leq b^2\}$ that lies between concentric spheres with radii a and b.

b. Find the outward flux of $\mathbf{F}$ across any sphere that encloses the origin.

SOLUTION

a. Although the vector field is undefined at the origin, it is defined and differentiable in D, which excludes the origin. In Section 17.5 (Exercise 73) it was shown that the divergence of the radial field $\mathbf{F} = \dfrac{\mathbf{r}}{|\mathbf{r}|^p}$ with $p = 3$ is 0. We let S be the union of S_2, the larger sphere of radius b, and S_1, the smaller sphere of radius a. Because $\iiint_D \nabla \cdot \mathbf{F}\, dV = 0$, the Divergence Theorem implies that

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_{S_2} \mathbf{F} \cdot \mathbf{n}_2\, dS - \iint_{S_1} \mathbf{F} \cdot \mathbf{n}_1\, dS = 0.$$

Therefore, the next flux across S is zero.

b. Part (a) implies that

$$\underbrace{\iint_{S_2} \mathbf{F} \cdot \mathbf{n}_2\, dS}_{\text{out of } D} = \underbrace{\iint_{S_1} \mathbf{F} \cdot \mathbf{n}_1\, dS}_{\text{into } D}.$$

We see that the flux out of D across S_2 equals the flux into D across S_1. To find that flux, we evaluate the surface integral over S_1 on which $|\mathbf{r}| = a$. (Because the fluxes are equal, S_2 could also be used.)

The easiest way to evaluate the surface integral is to note that on the sphere S_1, the unit outward normal vector is $\mathbf{n}_1 = \mathbf{r}/|\mathbf{r}|$. Therefore, the surface integral is

$$\begin{aligned}
\iint_{S_1} \mathbf{F} \cdot \mathbf{n}_1\, dS &= \iint_{S_1} \frac{\mathbf{r}}{|\mathbf{r}|^3} \cdot \frac{\mathbf{r}}{|\mathbf{r}|}\, dS && \text{Substitute for } \mathbf{F} \text{ and } \mathbf{n}_1.\\
&= \iint_{S_1} \frac{|\mathbf{r}|^2}{|\mathbf{r}|^4}\, dS && \mathbf{r} \cdot \mathbf{r} = |\mathbf{r}|^2\\
&= \iint_{S_1} \frac{1}{a^2}\, dS && |\mathbf{r}| = a\\
&= \frac{4\pi a^2}{a^2} && \text{Surface area} = 4\pi a^2\\
&= 4\pi.
\end{aligned}$$

The same result is obtained using S_2 or any smooth surface enclosing the origin. The flux of the inverse square field across *any* surface enclosing the origin is 4π. As shown in Exercise 46, among radial fields, this property holds only for the inverse square field $(p = 3)$.

Related Exercises 26–27 ◄

Gauss' Law

Applying the Divergence Theorem to electric fields leads to one of the fundamental laws of physics. The electric field due to a point charge Q located at the origin is given by the inverse square law,

$$\mathbf{E}(x, y, z) = \frac{Q}{4\pi\varepsilon_0} \frac{\mathbf{r}}{|\mathbf{r}|^3},$$

where $\mathbf{r} = \langle x, y, z \rangle$ and ε_0 is a physical constant called the *permittivity of free space*.

According to the calculation of Example 4, the flux of the field $\dfrac{\mathbf{r}}{|\mathbf{r}|^3}$ across any surface that encloses the origin is 4π. Therefore, the flux of the electric field across any surface enclosing the origin is $\dfrac{Q}{4\pi\varepsilon_0} \cdot 4\pi = \dfrac{Q}{\varepsilon_0}$ (Figure 17.73a). This is one statement of Gauss' Law: If S is a surface that encloses a point charge Q, then the flux of the electric field across S is

$$\iint\limits_S \mathbf{E} \cdot \mathbf{n}\, dS = \frac{Q}{\varepsilon_0}.$$

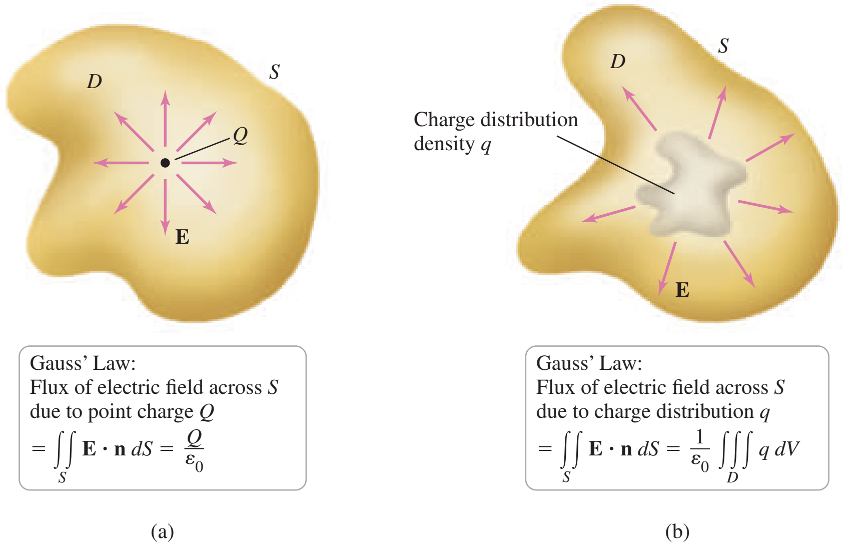


Figure 17.73

In fact, Gauss' Law applies to more general charge distributions (Exercise 39). If $q(x, y, z)$ is a charge density (charge per unit volume) defined on a region D enclosed by S, then the total charge within D is $Q = \iiint_D q(x, y, z)\, dV$ (Figure 17.73b). Replacing Q with this triple integral, Gauss' Law takes the form

$$\iint\limits_S \mathbf{E} \cdot \mathbf{n}\, dS = \frac{1}{\varepsilon_0} \underbrace{\iiint\limits_D q(x, y, z)\, dV}_{Q}.$$

Gauss' Law applies to other inverse square fields. In a slightly different form, it also governs heat transfer. If T is the temperature distribution in a solid body D, then the heat flow vector field is $\mathbf{F} = -k\nabla T$. (Heat flows down the temperature gradient.) If $q(x, y, z)$ represents the sources of heat within D, Gauss' Law says

$$\iint\limits_S \mathbf{F} \cdot \mathbf{n}\, dS = -k \iint\limits_S \nabla T \cdot \mathbf{n}\, dS = \iiint\limits_D q(x, y, z)\, dV.$$

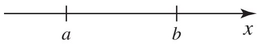
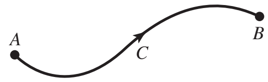
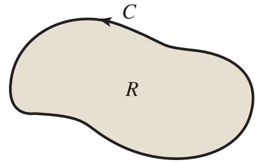
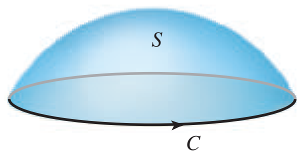
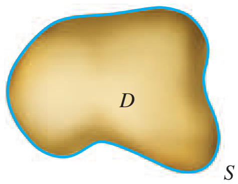
We see that, in general, the flux of material (fluid, heat, electric field lines) across the boundary of a region is the cumulative effect of the sources within the region.

A Final Perspective

Table 17.4 offers a look at the progression of fundamental theorems of calculus that have appeared throughout this text. Each theorem builds on its predecessors, extending the same basic idea to a different situation or to higher dimensions.

In all cases, the statement is effectively the same: The cumulative (integrated) effect of the *derivatives* of a function throughout a region is determined by the values of the function on the boundary of that region. This principle underlies much of our understanding of the world around us.

Table 17.4

Fundamental Theorem of Calculus	$\int_a^b f'(x)\,dx = f(b) - f(a)$	
Fundamental Theorem for Line Integrals	$\int_C \nabla f \cdot d\mathbf{r} = f(B) - f(A)$	
Green's Theorem (Circulation form)	$\iint_R (g_x - f_y)\,dA = \oint_C f\,dx + g\,dy$	
Stokes' Theorem	$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\,dS = \oint_C \mathbf{F} \cdot d\mathbf{r}$	
Divergence Theorem	$\iiint_D \nabla \cdot \mathbf{F}\,dV = \iint_S \mathbf{F} \cdot \mathbf{n}\,dS$	

SECTION 17.8 EXERCISES

Getting Started

1. Explain the meaning of the surface integral in the Divergence Theorem.

2. Interpret the volume integral in the Divergence Theorem.

3. Explain the meaning of the Divergence Theorem.

4. What is the net outward flux of the rotation field $\mathbf{F} = \langle 2z + y, -x, -2x \rangle$ across the surface that encloses any region?

5. What is the net outward flux of the radial field $\mathbf{F} = \langle x, y, z \rangle$ across the sphere of radius 2 centered at the origin?

6. What is the divergence of an inverse square vector field?

7. Suppose div $\mathbf{F} = 0$ in a region enclosed by two concentric spheres. What is the relationship between the outward fluxes across the two spheres?

8. If div $\mathbf{F} > 0$ in a region enclosed by a small cube, is the net flux of the field into or out of the cube?

Practice Exercises

9–12. Verifying the Divergence Theorem *Evaluate both integrals of the Divergence Theorem for the following vector fields and regions. Check for agreement.*

9. $\mathbf{F} = \langle 2x, 3y, 4z \rangle$; $D = \{(x, y, z): x^2 + y^2 + z^2 \le 4\}$

10. $\mathbf{F} = \langle -x, -y, -z \rangle$;
$D = \{(x, y, z): |x| \le 1, |y| \le 1, |z| \le 1\}$

11. $\mathbf{F} = \langle z - y, x, -x \rangle$;
$D = \{(x, y, z): x^2/4 + y^2/8 + z^2/12 \le 1\}$

12. $\mathbf{F} = \langle x^2, y^2, z^2 \rangle$; $D = \{(x, y, z): |x| \le 1, |y| \le 2, |z| \le 3\}$

13–16. Rotation fields

13. Find the net outward flux of the field $\mathbf{F} = \langle 2z - y, x, -2x \rangle$ across the sphere of radius 1 centered at the origin.

14. Find the net outward flux of the field $\mathbf{F} = \langle z - y, x - z, y - x \rangle$ across the boundary of the cube $\{(x, y, z): |x| \le 1, |y| \le 1, |z| \le 1\}$.

15. Find the net outward flux of the field $\mathbf{F} = \langle bz - cy, cx - az, ay - bx \rangle$ across any smooth closed surface in $\mathbb{R}^3$, where a, b, and c are constants.

16. Find the net outward flux of $\mathbf{F} = \mathbf{a} \times \mathbf{r}$ across any smooth closed surface in $\mathbb{R}^3$, where $\mathbf{a}$ is a constant nonzero vector and $\mathbf{r} = \langle x, y, z \rangle$.

17–24. Computing flux *Use the Divergence Theorem to compute the net outward flux of the following fields across the given surface S.*

17. $\mathbf{F} = \langle x, -2y, 3z \rangle$; S is the sphere $\{(x, y, z): x^2 + y^2 + z^2 = 6\}$.

18. $\mathbf{F} = \langle x^2, 2xz, y^2 \rangle$; S is the surface of the cube cut from the first octant by the planes $x = 1$, $y = 1$, and $z = 1$.

19. $\mathbf{F} = \langle x, 2y, z \rangle$; S is the boundary of the tetrahedron in the first octant formed by the plane $x + y + z = 1$.

20. $\mathbf{F} = \langle x^2, y^2, z^2 \rangle$; S is the sphere $\{(x, y, z): x^2 + y^2 + z^2 = 25\}$.

21. $\mathbf{F} = \langle y - 2x, x^3 - y, y^2 - z \rangle$; S is the sphere $\{(x, y, z): x^2 + y^2 + z^2 = 4\}$.

22. $\mathbf{F} = \langle y + z, x + z, x + y \rangle$; S consists of the faces of the cube $\{(x, y, z): |x| \le 1, |y| \le 1, |z| \le 1\}$.

23. $\mathbf{F} = \langle x, y, z \rangle$; S is the surface of the paraboloid $z = 4 - x^2 - y^2$, for $z \ge 0$, plus its base in the xy-plane.

24. $\mathbf{F} = \langle x, y, z \rangle$; S is the surface of the cone $z^2 = x^2 + y^2$, for $0 \le z \le 4$, plus its top surface in the plane $z = 4$.

25–30. Divergence Theorem for more general regions *Use the Divergence Theorem to compute the net outward flux of the following vector fields across the boundary of the given regions D.*

25. $\mathbf{F} = \langle z - x, x - y, 2y - z \rangle$; D is the region between the spheres of radius 2 and 4 centered at the origin.

26. $\mathbf{F} = \mathbf{r}|\mathbf{r}| = \langle x, y, z \rangle \sqrt{x^2 + y^2 + z^2}$; D is the region between the spheres of radius 1 and 2 centered at the origin.

27. $\mathbf{F} = \dfrac{\mathbf{r}}{|\mathbf{r}|} = \dfrac{\langle x, y, z \rangle}{\sqrt{x^2 + y^2 + z^2}}$; D is the region between the spheres of radius 1 and 2 centered at the origin.

28. $\mathbf{F} = \langle z - y, x - z, 2y - x \rangle$; D is the region between two cubes: $\{(x, y, z): 1 \le |x| \le 3, 1 \le |y| \le 3, 1 \le |z| \le 3\}$.

29. $\mathbf{F} = \langle x^2, -y^2, z^2 \rangle$; D is the region in the first octant between the planes $z = 4 - x - y$ and $z = 2 - x - y$.

30. $\mathbf{F} = \langle x, 2y, 3z \rangle$; D is the region between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$, for $0 \le z \le 8$.

31. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. If $\nabla \cdot \mathbf{F} = 0$ at all points of a region D, then $\mathbf{F} \cdot \mathbf{n} = 0$ at all points of the boundary of D.

b. If $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = 0$ on all closed surfaces in $\mathbb{R}^3$, then $\mathbf{F}$ is constant.

c. If $|\mathbf{F}| < 1$, then $\left|\iiint_D \nabla \cdot \mathbf{F}\, dV\right|$ is less than the area of the surface of D.

32. Flux across a sphere Consider the radial field $\mathbf{F} = \langle x, y, z \rangle$ and let S be the sphere of radius a centered at the origin. Compute the outward flux of $\mathbf{F}$ across S using the representation $z = \pm\sqrt{a^2 - x^2 - y^2}$ for the sphere (either symmetry or two surfaces must be used).

33–35. Flux integrals *Compute the outward flux of the following vector fields across the given surfaces S. You should decide which integral of the Divergence Theorem to use.*

33. $\mathbf{F} = \langle x^2 e^y \cos z, -4xe^y \cos z, 2xe^y \sin z \rangle$; S is the boundary of the ellipsoid $x^2/4 + y^2 + z^2 = 1$.

34. $\mathbf{F} = \langle -yz, xz, 1 \rangle$; S is the boundary of the ellipsoid $x^2/4 + y^2/4 + z^2 = 1$.

35. $\mathbf{F} = \langle x \sin y, -\cos y, z \sin y \rangle$; S is the boundary of the region bounded by the planes $x = 1$, $y = 0$, $y = \pi/2$, $z = 0$, and $z = x$.

36. Radial fields Consider the radial vector field $\mathbf{F} = \dfrac{\mathbf{r}}{|\mathbf{r}|^p} = \dfrac{\langle x, y, z \rangle}{(x^2 + y^2 + z^2)^{p/2}}$. Let S be the sphere of radius a centered at the origin.

a. Use a surface integral to show that the outward flux of $\mathbf{F}$ across S is $4\pi a^{3-p}$. Recall that the unit normal to the sphere is $\mathbf{r}/|\mathbf{r}|$.

b. For what values of p does $\mathbf{F}$ satisfy the conditions of the Divergence Theorem? For these values of p, use the fact (Theorem 17.10) that $\nabla \cdot \mathbf{F} = \dfrac{3 - p}{|\mathbf{r}|^p}$ to compute the flux across S using the Divergence Theorem.

37. Singular radial field Consider the radial field $\mathbf{F} = \dfrac{\mathbf{r}}{|\mathbf{r}|} = \dfrac{\langle x, y, z \rangle}{(x^2 + y^2 + z^2)^{1/2}}$.

a. Evaluate a surface integral to show that $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = 4\pi a^2$, where S is the surface of a sphere of radius a centered at the origin.

b. Note that the first partial derivatives of the components of $\mathbf{F}$ are undefined at the origin, so the Divergence Theorem does not apply directly. Nevertheless, the flux across the sphere as computed in part (a) is finite. Evaluate the triple integral of the Divergence Theorem as an improper integral as follows. Integrate div $\mathbf{F}$ over the region between two spheres of radius a and $0 < \varepsilon < a$. Then let $\varepsilon \to 0^+$ to obtain the flux computed in part (a).

38. Logarithmic potential Consider the potential function $\varphi(x, y, z) = \dfrac{1}{2} \ln (x^2 + y^2 + z^2) = \ln |\mathbf{r}|$, where $\mathbf{r} = \langle x, y, z \rangle$.

a. Show that the gradient field associated with φ is $\mathbf{F} = \dfrac{\mathbf{r}}{|\mathbf{r}|^2} = \dfrac{\langle x, y, z \rangle}{x^2 + y^2 + z^2}$.

b. Show that $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = 4\pi a$, where S is the surface of a sphere of radius a centered at the origin.

c. Compute div $\mathbf{F}$.

d. Note that $\mathbf{F}$ is undefined at the origin, so the Divergence Theorem does not apply directly. Evaluate the volume integral as described in Exercise 37.

39. Gauss' Law for electric fields The electric field due to a point charge Q is $\mathbf{E} = \dfrac{Q}{4\pi\varepsilon_0} \dfrac{\mathbf{r}}{|\mathbf{r}|^3}$, where $\mathbf{r} = \langle x, y, z \rangle$, and ε_0 is a constant.

a. Show that the flux of the field across a sphere of radius a centered at the origin is $\iint_S \mathbf{E} \cdot \mathbf{n}\, dS = \dfrac{Q}{\varepsilon_0}$.

b. Let S be the boundary of the region between two spheres centered at the origin of radius a and b, respectively, with $a < b$. Use the Divergence Theorem to show that the net outward flux across S is zero.

c. Suppose there is a distribution of charge within a region D. Let $q(x, y, z)$ be the charge density (charge per unit volume). Interpret the statement that

$$\iint_S \mathbf{E} \cdot \mathbf{n}\, dS = \frac{1}{\varepsilon_0} \iiint_D q(x, y, z)\, dV.$$

d. Assuming $\mathbf{E}$ satisfies the conditions of the Divergence Theorem on D, conclude from part (c) that $\nabla \cdot \mathbf{E} = \dfrac{q}{\varepsilon_0}$.

e. Because the electric force is conservative, it has a potential function φ. From part (d), conclude that $\nabla^2 \varphi = \nabla \cdot \nabla \varphi = \dfrac{q}{\varepsilon_0}$.

40. Gauss' Law for gravitation The gravitational force due to a point mass M at the origin is proportional to $\mathbf{F} = GM\mathbf{r}/|\mathbf{r}|^3$, where $\mathbf{r} = \langle x, y, z\rangle$ and G is the gravitational constant.

a. Show that the flux of the force field across a sphere of radius a centered at the origin is $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = 4\pi GM$.

b. Let S be the boundary of the region between two spheres centered at the origin of radius a and b, respectively, with $a < b$. Use the Divergence Theorem to show that the net outward flux across S is zero.

c. Suppose there is a distribution of mass within a region D. Let $\rho(x, y, z)$ be the mass density (mass per unit volume). Interpret the statement that

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = 4\pi G\iiint_D \rho(x, y, z)\,dV.$$

d. Assuming $\mathbf{F}$ satisfies the conditions of the Divergence Theorem on D, conclude from part (c) that $\nabla\cdot\mathbf{F} = 4\pi G\rho$.

e. Because the gravitational force is conservative, it has a potential function φ. From part (d), conclude that $\nabla^2\varphi = 4\pi G\rho$.

41–45. Heat transfer *Fourier's Law of heat transfer (or heat conduction) states that the heat flow vector* $\mathbf{F}$ *at a point is proportional to the negative gradient of the temperature; that is,* $\mathbf{F} = -k\nabla T$*, which means that heat energy flows from hot regions to cold regions. The constant* $k > 0$ *is called the conductivity, which has metric units of* $J/(m\text{-}s\text{-}K)$*. A temperature function for a region D is given. Find the net outward heat flux* $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = -k\iint_S \nabla T\cdot\mathbf{n}\,dS$ *across the boundary S of D. In some cases, it may be easier to use the Divergence Theorem and evaluate a triple integral. Assume* $k = 1$.

41. $T(x, y, z) = 100 + x + 2y + z$;
$D = \{(x, y, z): 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1\}$

42. $T(x, y, z) = 100 + x^2 + y^2 + z^2$;
$D = \{(x, y, z): 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1\}$

43. $T(x, y, z) = 100 + e^{-z}$;
$D = \{(x, y, z): 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1\}$

44. $T(x, y, z) = 100 + x^2 + y^2 + z^2$; D is the unit sphere centered at the origin.

T **45.** $T(x, y, z) = 100e^{-x^2-y^2-z^2}$; D is the sphere of radius a centered at the origin.

Explorations and Challenges

46. Inverse square fields are special Let $\mathbf{F}$ be a radial field $\mathbf{F} = \mathbf{r}/|\mathbf{r}|^p$, where p is a real number and $\mathbf{r} = \langle x, y, z\rangle$. With $p = 3$, $\mathbf{F}$ is an inverse square field.

a. Show that the net flux across a sphere centered at the origin is independent of the radius of the sphere only for $p = 3$.

b. Explain the observation in part (a) by finding the flux of $\mathbf{F} = \mathbf{r}/|\mathbf{r}|^p$ across the boundaries of a spherical box $\{(\rho, \varphi, \theta): a \le \rho \le b, \varphi_1 \le \varphi \le \varphi_2, \theta_1 \le \theta \le \theta_2\}$ for various values of p.

47. A beautiful flux integral Consider the potential function $\varphi(x, y, z) = G(\rho)$, where G is any twice differentiable function and $\rho = \sqrt{x^2 + y^2 + z^2}$; therefore, G depends only on the distance from the origin.

a. Show that the gradient vector field associated with φ is

$$\mathbf{F} = \nabla\varphi = G'(\rho)\frac{\mathbf{r}}{\rho}, \text{ where } \mathbf{r} = \langle x, y, z\rangle \text{ and } \rho = |\mathbf{r}|.$$

b. Let S be the sphere of radius a centered at the origin and let D be the region enclosed by S. Show that the flux of $\mathbf{F}$ across S is $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = 4\pi a^2 G'(a)$.

c. Show that $\nabla\cdot\mathbf{F} = \nabla\cdot\nabla\varphi = \dfrac{2G'(\rho)}{\rho} + G''(\rho)$.

d. Use part (c) to show that the flux across S (as given in part (b)) is also obtained by the volume integral $\iiint_D \nabla\cdot\mathbf{F}\,dV$. (*Hint:* Use spherical coordinates and integrate by parts.)

48. Integration by parts (Gauss' Formula) Recall the Product Rule of Theorem 17.13: $\nabla\cdot(u\mathbf{F}) = \nabla u\cdot\mathbf{F} + u(\nabla\cdot\mathbf{F})$.

a. Integrate both sides of this identity over a solid region D with a closed boundary S, and use the Divergence Theorem to prove an integration by parts rule:

$$\iiint_D u(\nabla\cdot\mathbf{F})\,dV = \iint_S u\mathbf{F}\cdot\mathbf{n}\,dS - \iiint_D \nabla u\cdot\mathbf{F}\,dV.$$

b. Explain the correspondence between this rule and the integration by parts rule for single-variable functions.

c. Use integration by parts to evaluate $\iiint_D (x^2y + y^2z + z^2x)\,dV$, where D is the cube in the first octant cut by the planes $x = 1$, $y = 1$, and $z = 1$.

49. Green's Formula Write Gauss' Formula of Exercise 48 in two dimensions—that is, where $\mathbf{F} = \langle f, g\rangle$, D is a plane region R and C is the boundary of R. Show that the result is Green's Formula:

$$\iint_R u(f_x + g_y)\,dA = \oint_C u(\mathbf{F}\cdot\mathbf{n})\,ds - \iint_R (fu_x + gu_y)\,dA.$$

Show that with $u = 1$, one form of Green's Theorem appears. Which form of Green's Theorem is it?

50. Green's First Identity Prove Green's First Identity for twice differentiable scalar-valued functions u and v defined on a region D:

$$\iiint_D (u\nabla^2 v + \nabla u\cdot\nabla v)\,dV = \iint_S u\nabla v\cdot\mathbf{n}\,dS,$$

where $\nabla^2 v = \nabla\cdot\nabla v$. You may apply Gauss' Formula in Exercise 48 to $\mathbf{F} = \nabla v$ or apply the Divergence Theorem to $\mathbf{F} = u\nabla v$.

51. Green's Second Identity Prove Green's Second Identity for scalar-valued functions u and v defined on a region D:

$$\iiint_D (u\nabla^2 v - v\nabla^2 u)\,dV = \iint_S (u\nabla v - v\nabla u)\cdot\mathbf{n}\,dS.$$

(*Hint:* Reverse the roles of u and v in Green's First Identity.)

52–54. Harmonic functions *A scalar-valued function* φ *is harmonic on a region D if* $\nabla^2\varphi = \nabla\cdot\nabla\varphi = 0$ *at all points of D.*

52. Show that the potential function $\varphi(x, y, z) = |\mathbf{r}|^{-p}$ is harmonic provided $p = 0$ or $p = 1$, where $\mathbf{r} = \langle x, y, z\rangle$. To what vector fields do these potentials correspond?

53. Show that if φ is harmonic on a region D enclosed by a surface S, then $\iint_S \nabla\varphi\cdot\mathbf{n}\,dS = 0$.

54. Show that if u is harmonic on a region D enclosed by a surface S, then $\iint_S u\nabla u\cdot\mathbf{n}\,dS = \iiint_D |\nabla u|^2\,dV$.

55. Miscellaneous integral identities Prove the following identities.

a. $\iiint_D \nabla\times\mathbf{F}\,dV = \iint_S (\mathbf{n}\times\mathbf{F})\,dS$ (*Hint:* Apply the Divergence Theorem to each component of the identity.)

b. $\iint_S (\mathbf{n}\times\nabla\varphi)\,dS = \oint_C \varphi\,d\mathbf{r}$ (*Hint:* Apply Stokes' Theorem to each component of the identity.)

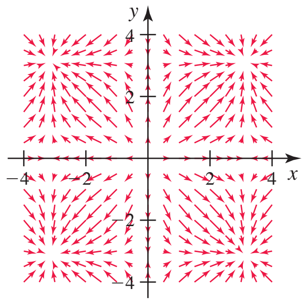
QUICK CHECK ANSWERS

1. If **F** is constant, then div **F** $= 0$, so $\iiint_D \nabla \cdot \mathbf{F}\, dV = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = 0$. This means that all the "material" that flows into one side of D flows out of the other side of D. **2.** The vector field and the divergence are positive throughout D. **3.** The vector field has no flow into or out of the cube on the faces $x = 0$, $y = 0$, and $z = 0$ because the vectors of **F** on these faces are parallel to the faces. The vector field points out of the cube on the $x = 1$ and $z = 1$ faces and into the cube on the $y = 1$ face. div **F** $= 2$, so there is a net flow out of the cube. ◄

CHAPTER 17 REVIEW EXERCISES

1. **Explain why or why not** Determine whether the following statements are true and give an explanation or counterexample.
   a. The rotation field $\mathbf{F} = \langle -y, x \rangle$ has zero curl and zero divergence.
   b. $\nabla \times \nabla\varphi = \mathbf{0}$
   c. Two vector fields with the same curl differ by a constant vector field.
   d. Two vector fields with the same divergence differ by a constant vector field.
   e. If $\mathbf{F} = \langle x, y, z \rangle$ and S encloses a region D, then $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$ is three times the volume of D.

2. **Matching vector fields** Match vector fields a–f with the graphs A–F. Let $\mathbf{r} = \langle x, y \rangle$.
   a. $\mathbf{F} = \langle x, y \rangle$
   b. $\mathbf{F} = \langle -2y, 2x \rangle$
   c. $\mathbf{F} = \mathbf{r}/|\mathbf{r}|$
   d. $\mathbf{F} = \langle y - x, x \rangle$
   e. $\mathbf{F} = \langle e^{-y}, e^{-x} \rangle$
   f. $\mathbf{F} = \langle \sin \pi x, \sin \pi y \rangle$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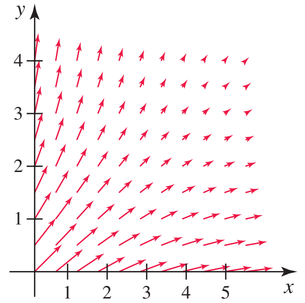
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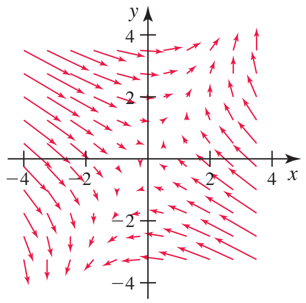
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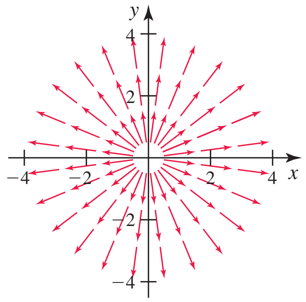
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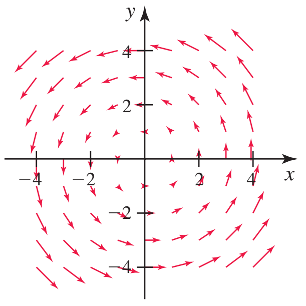
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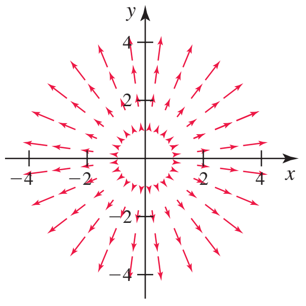
(F)

3–4. Gradient fields in $\mathbb{R}^2$ *Find the vector field $\mathbf{F} = \nabla\varphi$ for the following potential functions. Sketch a few level curves of φ and a few vectors of* **F** *along the level curves.*

3. $\varphi(x, y) = x^2 + 4y^2$, for $|x| \le 5$, $|y| \le 5$

4. $\varphi(x, y) = (x^2 - y^2)/2$, for $|x| \le 2$, $|y| \le 2$

5–6. Gradient fields in $\mathbb{R}^3$ *Find the vector field $\mathbf{F} = \nabla\varphi$ for the following potential functions.*

5. $\varphi(x, y, z) = 1/|\mathbf{r}|$, where $\mathbf{r} = \langle x, y, z \rangle$

6. $\varphi(x, y, z) = \frac{1}{2} e^{-x^2 - y^2 - z^2}$

7. **Normal component** Let C be the circle of radius 2 centered at the origin with counterclockwise orientation. Give the unit outward normal vector at any point (x, y) on C.

8–10. Line integrals *Evaluate the following line integrals.*

8. $\int_C (x^2 - 2xy + y^2)\, ds$; C is the upper half of a circle $\mathbf{r}(t) = \langle 5 \cos t, 5 \sin t \rangle$, for $0 \le t \le \pi$.

9. $\int_C ye^{-xz}\, ds$; C is the path $\mathbf{r}(t) = \langle 2t, 3t, -6t \rangle$, for $0 \le t \le 2$.

10. $\int_C (xz - y^2)\, ds$; C is the line segment from $(0, 1, 2)$ to $(-3, 7, -1)$.

11. **Two parameterizations** Verify that $\oint_C (x - 2y + 3z)\, ds$ has the same value when C is given by $\mathbf{r}(t) = \langle 2 \cos t, 2 \sin t, 0 \rangle$, for $0 \le t \le 2\pi$, and by $\mathbf{r}(t) = \langle 2 \cos t^2, 2 \sin t^2, 0 \rangle$, for $0 \le t \le \sqrt{2\pi}$.

12. **Work integral** Find the work done in moving an object from $P(1, 0, 0)$ to $Q(0, 1, 0)$ in the presence of the force $\mathbf{F} = \langle 1, 2y, -4z \rangle$ along the following paths.
    a. The line segment from P to Q
    b. The line segment from P to $O(0, 0, 0)$ followed by the line segment from O to Q
    c. The arc of the quarter circle from P to Q
    d. Is the work independent of the path?

13–14. Work integrals in $\mathbb{R}^3$ *Given the force field* **F**, *find the work required to move an object on the given curve.*

13. $\mathbf{F} = \langle -y, z, x \rangle$ on the path consisting of the line segment from $(0, 0, 0)$ to $(0, 1, 0)$ followed by the line segment from $(0, 1, 0)$ to $(0, 1, 4)$

14. $\mathbf{F} = \dfrac{\langle x, y, z \rangle}{(x^2 + y^2 + z^2)^{3/2}}$ on the path $\mathbf{r}(t) = \langle t^2, 3t^2, -t^2 \rangle$, for $1 \le t \le 2$

15–18. Circulation and flux *Find the circulation and the outward flux of the following vector fields for the curve* $\mathbf{r}(t) = \langle 2\cos t, 2\sin t\rangle$, *for* $0 \le t \le 2\pi$.

15. $\mathbf{F} = \langle y - x, y\rangle$ **16.** $\mathbf{F} = \langle x, y\rangle$

17. $\mathbf{F} = \mathbf{r}/|\mathbf{r}|^2$, where $\mathbf{r} = \langle x, y\rangle$

18. $\mathbf{F} = \langle x - y, x\rangle$

19. Flux in channel flow Consider the flow of water in a channel whose boundaries are the planes $y = \pm L$ and $z = \pm 1/2$. The velocity field in the channel is $\mathbf{v} = \langle v_0(L^2 - y^2), 0, 0\rangle$. Find the flux across the cross section of the channel at $x = 0$ in terms of v_0 and L.

20–23. Conservative vector fields and potentials *Determine whether the following vector fields are conservative on their domains. If so, find a potential function.*

20. $\mathbf{F} = \langle y^2, 2xy\rangle$ **21.** $\mathbf{F} = \langle y, x + z^2, 2yz\rangle$

22. $\mathbf{F} = \langle e^x\cos y, -e^x\sin y\rangle$ **23.** $\mathbf{F} = e^z\langle y, x, xy\rangle$

24–27. Evaluating line integrals *Evaluate the line integral* $\int_C \mathbf{F}\cdot d\mathbf{r}$ *for the following vector fields* $\mathbf{F}$ *and curves C in two ways.*

a. By parameterizing C
b. By using the Fundamental Theorem for line integrals, if possible

24. $\mathbf{F} = \nabla(x^2y)$; C: $\mathbf{r}(t) = \langle 9 - t^2, t\rangle$, for $0 \le t \le 3$

25. $\mathbf{F} = \nabla(xyz)$; C: $\mathbf{r}(t) = \langle \cos t, \sin t, t/\pi\rangle$, for $0 \le t \le \pi$

26. $\mathbf{F} = \langle x, -y\rangle$; C is the square with vertices $(\pm 1, \pm 1)$ with counterclockwise orientation.

27. $\mathbf{F} = \langle y, z, -x\rangle$; C: $\mathbf{r}(t) = \langle \cos t, \sin t, 4\rangle$, for $0 \le t \le 2\pi$

28. Radial fields in $\mathbb{R}^2$ are conservative Prove that the radial field $\mathbf{F} = \mathbf{r}/|\mathbf{r}|^p$, where $\mathbf{r} = \langle x, y\rangle$ and p is a real number, is conservative on $\mathbb{R}^2$ with the origin removed. For what value of p is $\mathbf{F}$ conservative on $\mathbb{R}^2$ (including the origin)?

29–32. Green's Theorem for line integrals *Use either form of Green's Theorem to evaluate the following line integrals.*

29. $\oint_C xy^2\,dx + x^2y\,dy$; C is the triangle with vertices $(0, 0)$, $(2, 0)$, and $(0, 2)$ with counterclockwise orientation.

30. $\oint_C (-3y + x^{3/2})\,dx + (x - y^{2/3})\,dy$; C is the boundary of the half-disk $\{(x, y): x^2 + y^2 \le 2, y \ge 0\}$ with counterclockwise orientation.

31. $\oint_C (x^3 + xy)\,dy + (2y^2 - 2x^2y)\,dx$; C is the square with vertices $(\pm 1, \pm 1)$ with counterclockwise orientation.

32. $\oint_C 3x^3\,dy - 3y^3\,dx$; C is the circle of radius 4 centered at the origin with *clockwise* orientation.

33–34. Areas of plane regions *Find the area of the following regions using a line integral.*

33. The region enclosed by the ellipse $x^2 + 4y^2 = 16$

34. The region bounded by the hypocycloid $\mathbf{r}(t) = \langle \cos^3 t, \sin^3 t\rangle$, for $0 \le t \le 2\pi$

35–36. Circulation and flux *Consider the following vector fields.*

a. Compute the circulation on the boundary of the region R (with counterclockwise orientation).
b. Compute the outward flux across the boundary of R.

35. $\mathbf{F} = \mathbf{r}/|\mathbf{r}|$, where $\mathbf{r} = \langle x, y\rangle$ and R is the half-annulus $\{(r, \theta): 1 \le r \le 3, 0 \le \theta \le \pi\}$

36. $\mathbf{F} = \langle -\sin y, x\cos y\rangle$, where R is the square $\{(x, y): 0 \le x \le \pi/2, 0 \le y \le \pi/2\}$

37. Parameters Let $\mathbf{F} = \langle ax + by, cx + dy\rangle$, where a, b, c, and d are constants.

a. For what values of a, b, c, and d is $\mathbf{F}$ conservative?
b. For what values of a, b, c, and d is $\mathbf{F}$ source free?
c. For what values of a, b, c, and d is $\mathbf{F}$ conservative and source free?

38–41. Divergence and curl *Compute the divergence and curl of the following vector fields. State whether the field is source free or irrotational.*

38. $\mathbf{F} = \langle yz, xz, xy\rangle$

39. $\mathbf{F} = \mathbf{r}|\mathbf{r}| = \langle x, y, z\rangle\sqrt{x^2 + y^2 + z^2}$

40. $\mathbf{F} = \langle \sin xy, \cos yz, \sin xz\rangle$

41. $\mathbf{F} = \langle 2xy + z^4, x^2, 4xz^3\rangle$

42. Identities Prove that $\nabla\left(\dfrac{1}{|\mathbf{r}|^4}\right) = -\dfrac{4\mathbf{r}}{|\mathbf{r}|^6}$ and use the result to prove that $\nabla\cdot\nabla\left(\dfrac{1}{|\mathbf{r}|^4}\right) = \dfrac{12}{|\mathbf{r}|^6}$.

43. Maximum curl Let $\mathbf{F} = \langle z, x, -y\rangle$.

a. What is the scalar component of curl $\mathbf{F}$ in the direction of $\mathbf{n} = \langle 1, 0, 0\rangle$?
b. What is the scalar component of curl $\mathbf{F}$ in the direction of $\mathbf{n} = \langle 0, -1/\sqrt{2}, 1/\sqrt{2}\rangle$?
c. In the direction of what unit vector $\mathbf{n}$ is the scalar component of curl $\mathbf{F}$ a maximum?

44. Paddle wheel in a vector field Let $\mathbf{F} = \langle 0, 2x, 0\rangle$ and let $\mathbf{n}$ be a unit vector aligned with the axis of a paddle wheel located on the y-axis.

a. If the axis of the paddle wheel is aligned with $\mathbf{n} = \langle 1, 0, 0\rangle$, how fast does it spin?
b. If the axis of the paddle wheel is aligned with $\mathbf{n} = \langle 0, 0, 1\rangle$, how fast does it spin?
c. For what direction $\mathbf{n}$ does the paddle wheel spin fastest?

45–48. Surface areas *Use a surface integral to find the area of the following surfaces.*

45. The hemisphere $x^2 + y^2 + z^2 = 9$, for $z \ge 0$

46. The frustum of the cone $z^2 = x^2 + y^2$, for $2 \le z \le 4$ (excluding the bases)

47. The plane $z = 6 - x - y$ above the square $|x| \le 1$, $|y| \le 1$

48. The surface $f(x, y) = \sqrt{2}\,xy$ above the polar region $\{(r, \theta): 0 \le r \le 2, 0 \le \theta \le 2\pi\}$

49–51. Surface integrals *Evaluate the following surface integrals.*

49. $\iint_S (1 + yz)\,dS$; S is the plane $x + y + z = 2$ in the first octant.

50. $\iint_S \langle 0, y, z \rangle \cdot \mathbf{n}\, dS$; S is the curved surface of the cylinder $y^2 + z^2 = a^2$, for $|x| \le 8$, with outward normal vectors.

51. $\iint_S (x - y + z)\, dS$; S is the entire surface, including the base, of the hemisphere $x^2 + y^2 + z^2 = 4$, for $z \ge 0$.

52–53. Flux integrals *Find the flux of the following vector fields across the given surface. Assume the vectors normal to the surface point outward.*

52. $\mathbf{F} = \langle x, y, z \rangle$ across the curved surface of the cylinder $x^2 + y^2 = 1$, for $|z| \le 8$

53. $\mathbf{F} = \mathbf{r}/|\mathbf{r}|$ across the sphere of radius a centered at the origin, where $\mathbf{r} = \langle x, y, z \rangle$

54. Three methods Find the surface area of the paraboloid $z = x^2 + y^2$, for $0 \le z \le 4$, in three ways.

a. Use an explicit description of the surface.
b. Use the parametric description $\mathbf{r} = \langle v \cos u, v \sin u, v^2 \rangle$.
c. Use the parametric description $\mathbf{r} = \langle \sqrt{v} \cos u, \sqrt{v} \sin u, v \rangle$.

55. Flux across hemispheres and paraboloids Let S be the hemisphere $x^2 + y^2 + z^2 = a^2$, for $z \ge 0$, and let T be the paraboloid $z = a - (x^2 + y^2)/a$, for $z \ge 0$, where $a > 0$. Assume the surfaces have outward normal vectors.

a. Verify that S and T have the same base $(x^2 + y^2 \le a^2)$ and the same high point $(0, 0, a)$.
b. Which surface has the greater area?
c. Show that the flux of the radial field $\mathbf{F} = \langle x, y, z \rangle$ across S is $2\pi a^3$.
d. Show that the flux of the radial field $\mathbf{F} = \langle x, y, z \rangle$ across T is $3\pi a^3/2$.

56. Surface area of an ellipsoid Consider the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, where a, b, and c are positive real numbers.

a. Show that the surface is described by the parametric equations

$$\mathbf{r}(u, v) = \langle a \cos u \sin v, b \sin u \sin v, c \cos v \rangle$$

for $0 \le u \le 2\pi$, $0 \le v \le \pi$.
b. Write an integral for the surface area of the ellipsoid.

57–58. Stokes' Theorem for line integrals *Evaluate the line integral* $\oint_C \mathbf{F} \cdot d\mathbf{r}$ *using Stokes' Theorem. Assume C has counterclockwise orientation.*

57. $\mathbf{F} = \langle xz, yz, xy \rangle$; C is the circle $x^2 + y^2 = 4$ in the xy-plane.

58. $\mathbf{F} = \langle x^2 - y^2, x, 2yz \rangle$; C is the boundary of the plane $z = 6 - 2x - y$ in the first octant.

59–60. Stokes' Theorem for surface integrals *Use Stokes' Theorem to evaluate the surface integral* $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$. *Assume* $\mathbf{n}$ *is the outward normal.*

59. $\mathbf{F} = \langle -z, x, y \rangle$, where S is the hyperboloid $z = 10 - \sqrt{1 + x^2 + y^2}$, for $z \ge 0$

60. $\mathbf{F} = \langle x^2 - z^2, y^2, xz \rangle$, where S is the hemisphere $x^2 + y^2 + z^2 = 4$, for $y \ge 0$

61. Conservative fields Use Stokes' Theorem to find the circulation of the vector field $\mathbf{F} = \nabla(10 - x^2 + y^2 + z^2)$ around any smooth closed curve C with counterclockwise orientation.

62–64. Computing fluxes *Use the Divergence Theorem to compute the outward flux of the following vector fields across the given surfaces S.*

62. $\mathbf{F} = \langle -x, x - y, x - z \rangle$; S is the surface of the cube cut from the first octant by the planes $x = 1$, $y = 1$, and $z = 1$.

63. $\mathbf{F} = \langle x^3, y^3, z^3 \rangle/3$; S is the sphere $\{(x, y, z): x^2 + y^2 + z^2 = 9\}$.

64. $\mathbf{F} = \langle x^2, y^2, z^2 \rangle$; S is the cylinder $\{(x, y, z): x^2 + y^2 = 4, 0 \le z \le 8\}$.

65–66. General regions *Use the Divergence Theorem to compute the outward flux of the following vector fields across the boundary of the given regions D.*

65. $\mathbf{F} = \langle x^3, y^3, 10 \rangle$; D is the region between the hemispheres of radius 1 and 2 centered at the origin with bases in the xy-plane.

66. $\mathbf{F} = \dfrac{\mathbf{r}}{|\mathbf{r}|^3} = \dfrac{\langle x, y, z \rangle}{(x^2 + y^2 + z^2)^{3/2}}$; D is the region between two spheres with radii 1 and 2 centered at $(5, 5, 5)$.

67. Flux integrals Compute the outward flux of the field $\mathbf{F} = \langle x^2 + x \sin y, y^2 + 2 \cos y, z^2 + z \sin y \rangle$ across the surface S that is the boundary of the prism bounded by the planes $y = 1 - x$, $x = 0$, $y = 0$, $z = 0$, and $z = 4$.

68. Stokes' Theorem on a compound surface Consider the surface S consisting of the quarter-sphere $x^2 + y^2 + z^2 = a^2$, for $z \ge 0$ and $x \ge 0$, and the half-disk in the yz-plane $y^2 + z^2 \le a^2$, for $z \ge 0$. The boundary of S in the xy-plane is C, which consists of the semicircle $x^2 + y^2 = a^2$, for $x \ge 0$, and the line segment $[-a, a]$ on the y-axis, with a counterclockwise orientation. Let $\mathbf{F} = \langle 2z - y, x - z, y - 2x \rangle$.

a. Describe the direction in which the normal vectors point on S.
b. Evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$
c. Evaluate $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$ and check for agreement with part (b).

Chapter 17 Guided Projects

Applications of the material in this chapter and related topics can be found in the following Guided Projects. For additional information, see the Preface.

- Ideal fluid flow
- Maxwell's equations
- Planimeters and vector fields
- Vector calculus in other coordinate systems